Mathematics: the language of science

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The purpose of this article is to bring out the unique role of mathematics in providing a base to the diverse sciences which conform to its rigid structure. Of these, the physical and economic sciences are so intimately linked with mathematics, that they have become almost a part of its structure under the generic title of 'applied mathematics'. But with the progress of time, more and more branches of science are getting quantified and coming under its ambit. And once a branch of science gets articulated into a mathematical structure, the process goes beyond mere classification and arrangement, and becomes eligible as a candidate for enjoying its predictive powers. Indeed it is this single property of mathematics which gives it the capacity to predict the nature of evolution in time of the said branch of science. This has been well verified in the domain of physical sciences, but now even biological sciences are slowly feeling its strength and the list is expanding.

Keywords: Interdependence, mathematics, predictive power, sciences, tree of knowledge.

'I think, therefore I am.' – Rene Descartes.

MATHEMATICS has been so much ingrained in the very thinking of mankind since the days of Plato, Aristotle and Ptolemy, that it is hard to offer a formal definition for this unique creation of nature. Nevertheless, some great thinkers have attempted approximate descriptions to capture its essence. Thus, according to Bertrand Russell, 'Mathematics is the chief source of the belief in eternal and exact truth, as well as a sensible intelligible world'. But such an omnipotent view of mathematics is not shared by all thinkers. For another giant (Goethe) felt otherwise: 'Mathematics has the completely false reputation of yielding infallible conclusions'. Eugine Wigner was dumbfounded by 'the unreasonable effectiveness of mathematics', yet felt an 'eternal gratitude' for the same. Despite such disparate views on the unusual powers of mathematics, there is almost universal agreement on its unique role in shaping the thinking of mankind on diverse phenomena of nature. Therefore the universal appeal of mathematics as the language of science – the subject of this article – will probably strike a concordant note with anyone interested in exploring its dimensions.

To give a broad analogy, the position of mathematics vis-à-vis the sciences has been likened to that of the main trunk constituting the vast tree of knowledge, while the sciences occupy positions corresponding to the different branches sprouting successively outwards in decreasing order of theoretical basis. Thus the physical sciences – especially physics, correspond to the main branches adjoining the trunk, whereas the various applied sciences and their derivatives, biological sciences, social sciences (especially economics), and so on, branch further and further out on this tree. This looks like a working model for putting in perspective the role of mathematics, not only in its own right, but also in shaping the various branches of science which – once put under its ambit – must automatically share its logical basis. And once you have succeeded in putting your physical premises within a mathematical framework, you may be rest assured that its huge dynamical powers are freely available for predicting the outcome of your studies in more directions than one, something your physical intuition alone was utterly incapable of anticipating. On the other hand, not all aspects of the model – and even a formidable one like mathematics is no exception – can be taken literally, lest the oversimplified conclusion of a model being a substitute for reality, should obscure our thoughts.

Some aspects of this simple model are convincing enough. For example, it is a fair statement that, just as the trunk is a more rigid structure than the branches, so is the fabric of mathematical reasoning stronger (and tougher) than the flexible format of reasoning in physics. Indeed if mathematics is structured on the strong and short-range forces of purely deductive logic, physics may be thought to be held together by the (weaker) long-range forces of analogy, intuition as well as observable evidence. But the quest for a 'mathematical proof' of a successful physical theory which is concerned with 'deciphering the secrets of nature' – often by unorthodox means – is not a properly defined exercise. After all, such 'proofs' cannot be more convincing than the inputs on which the physical theory is based in the first place, and the latter derive their support from various indirect evidences which have no place in a formal mathematical theorem. In 'pure' mathematics on the other hand, there is no place for any

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hypothesis/hypotheses other than those that are present in the statement of a particular theorem. This is just as true of a simple Euclid's theorem as of the more complex Yang–Baxter theorem[1]. On these premises it is not difficult to imagine that a fool-proof mathematical theorem is not necessarily a good physical theory, especially if its 'hypotheses' do not have adequate physical support, or vice versa. A famous example of this apparent paradox is Heisenberg's theory of turbulence which was 'proved' by a mathematical theorem to be 'wrong', and yet was found to be in excellent agreement with experiment. This story was told by Werner Heisenberg in a lecture arranged at the International Centre for Theoretical Physics, Trieste[2] in 1968, which was presided over by Paul Dirac. And this work represented the content of Heisenberg's Ph D thesis carried out under the direction of his teacher, Arnold Sommerfeld, who had insisted that his student should rather do some 'solid' work for a Ph D, than indulge in some 'airy' ideas like matrix mechanics which was apparently too 'speculative' to risk for a doctorate!

Pure versus applied mathematics

Nevertheless, most physical sciences have fairly well-defined domains of jurisdiction characterized by definite procedures for formulation of problems, as well as elaborate techniques for solution, a scenario in which mathematics is both an indispensable tool for procedure as well as an essential language of description. Indeed, many of the physical sciences, especially mechanics, elasticity, fluid dynamics, magnetohydrodynamics – and even the general theory of relativity for that matter – have grown out of a deep involvement of mathematicians in these fields, which by usage and tradition were once regarded as different domains of applied mathematics. In contrast, the more traditional branches of physics – theoretical physics, astrophysics and quantum mechanics – have generally been regarded as belonging to the physical sciences, despite deep involvement of mathematicians in these fields. These anomalies reveal an artificial kind of barrier between the domains of mathematics and physics, which has more to do with the history of usage than any serious logical reasoning. In particular, as the physical sciences have evolved together with their associated experimental programmes, those topics which were once thought to belong to applied mathematics have inevitably shifted to well-defined areas of physics and physical sciences. Perhaps the only two subjects which are still thought to belong to mathematics proper – albeit in applied form – are statistics as well as its thermodynamic counterpart in statistical mechanics. They generated their own momentum, thanks to the seminal contributions of Boltzmann and Planck, and Einstein and Smoluchowski, and have stayed active ingredients of mathematics. Apart from 'owning' these subjects, applied mathematics has largely stayed content with providing a 'temporary shelter' for many branches of science which were once found to be amenable to the logic of mathematics, but eventually developed into well-defined disciplines on their own, albeit with a strong mathematical orientation. This is particularly true of subjects like mechanics, elasticity and fluid dynamics to name some, which have long remained under the ambit of applied mathematics – perhaps for lack of enough observational motivations – but there are now distinct signs of at least some of them (especially general relativity) branching out into independent disciplines on the strength of observational motivations.

Hardy's 'apology'

Concerning the place of applied mathematics vis-à-vis pure mathematics, within the framework of mathematics as a whole, the famous mathematician G. H. Hardy, in his book *A Mathematician's Apology*[3] has helped greatly in putting this issue in a clearer perspective. Hardy makes the following points (in his own words): (1) I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the only criteria by which his patterns should be judged. (2) It is not possible to justify the life of any genuine mathematician on the ground of the utility of his work. (3) One rather curious conclusion emerges, that pure mathematics is on the whole distinctly more useful than applied.

Hardy's perception of pure mathematics is quite unambiguous insofar as the definition of the 'core domain' is fairly absolute and does not depend on possible interactions with other fields of knowledge. His justification of the life of a genuine mathematician is strangely reminiscent of Michael Faraday's remark on the 'use of a newborn baby', in response to a query on the possible utility of his discovery of the law of electromagnetic induction. His definition is also not inconsistent with the 'tree-trunk' analogy, which merely emphasizes the feeder role of mathematics for the development of the other sciences. Perhaps the biggest asset of (pure) mathematics is its capacity to predict an outcome by virtue of its closely knit logic. For, once a branch of science gets articulated into a mathematical structure, the process goes beyond mere classification and arrangement, all the way to the fruits of its predictive powers. Indeed it is this single property of mathematics which gives it the capacity to predict the nature of evolution in time of the said branch of science. And since this power stems from its logical structure, the issue centres around the very process of mathematical thinking, which in turn presupposes the existence of order as the very basis of mathematical logic. In this respect it has been argued[4] that the boundary between order and disorder is the realm of reason, the playfield of creativity. As to the precise relation of mathematics with creativity, however, a formal consensus seems to be lacking.

Relation to the sciences: role of 'identity'

To come back to the role of mathematics in shaping the sciences, Hardy's reluctance to give a precise status to applied mathematics perhaps stems from the absence of an 'identity' (a core definition) akin to that of pure mathematics, which he had himself provided. This is probably because of the mere 'umbrella' role of applied mathematics in providing temporary shelter to newly emerging disciplines with a strong mathematical flavour, but whose pace of development was not significant enough to let them claim an identity (independent status) of their own. And because of its mere umbrella status, applied mathematics has missed a formal identity which most other sciences (physical, biological, economic) enjoy by virtue of their independent sources of inspiration. The former has stayed content with merely providing a 'jacket' for the developing sciences needing its language and tools. In the process new mathematics has often got created, even though the science concerned has enjoyed an identity independent of the mother science (mathematics) whose job is only to create fresh mathematics for its own sake. This has frequently happened in the domain of theoretical physics where historically, 'new' mathematical patterns have often got created, although the source of inspiration had stemmed entirely from within. In other words, while the task of the mathematician is to create new mathematics per se, the task of the physicist is to determine in which domains of physics these new creations of mathematics should apply. But there is nothing in the books to prevent any interaction between the two disciplines, since each provides inspiration and motivation to the other, often blurring their mutual dividing line. Indeed this interaction has sometimes been so strong as to give rise to the term 'mathematical physics' to represent this mutuality. And the race for new mathematics for its own sake has been particularly noticeable in the further development of theoretical physics – quantum field theory and string theory – where the original physical motivation for understanding new phenomena got completely lost in the enthusiasm for mathematical self-consistency per se.

Mathematics versus physics: a special relationship

In view of the close historical link of mathematics with physics, it is tempting to dwell further on the extent of this relationship through some leisurely examples, the very first one being Newton's laws of motion. Indeed it was to give mathematical shape to these laws, especially the second, that Newton had to invoke a most vital branch of mathematics, viz. differential calculus, first discovered by the German mathematician Liebnitz. This language proved so elegant and versatile that it became amenable to elaborate formulations at the hands of great thinkers like Laplace, Lagrange, Gauss, Fourier, Hamilton and Maxwell, leading to successively deeper foundations of the very same laws. In particular, the Lagrangian and Hamiltonian formulations, which offered fresh insights into the hidden richness of the original Newtonian premises, paved the way to still greater depths of knowledge, which could not possibly have been anticipated by its founder. Thus the Lagrangian formulation gave birth to the concept of action (as the time integral of the Lagrangian), a new kind of invariant which for the first time put all the four degrees of freedom – three space dimensions and one time dimension – under one roof. [This last concept was also to play a key role later in the formulation of the theory of relativity – both special and general – at the hands of Albert Einstein, who was able to integrate the two independent dimensions of space and time into an organic whole with the help of a universal constant known as the velocity of light.]

Action: a new territory

To come back to the virtues of 'action', this quantity, which possesses the dimensions of angular momentum (another key concept which was to prove vital for the feature of discreteness in quantum theory – see below) in turn gave rise to a more universal yet highly compact law, called the principle of least action, from which would naturally – and more compactly – emerge not only the laws of motion from a variational principle, but also that the latter would show far greater predictive powers than those realizable from the original Newtonian premises. For example, the Hamiltonian equations of motion – a by-product of the same principle – which, though identical in physical content with the original Newtonian form, nevertheless were to show the directions towards new territories which had hitherto remained inaccessible to the Newtonian world.

As to the 'new territory' that had remained invisible to the original Newtonian world, it needed the genius of Dirac[5] – inspired by Heisenberg's intuitive idea of a matrix structure[6] for the concerned dynamical variables – to replace the classical Poisson brackets for any two dynamical variables by the corresponding operator commutator brackets, obtained simply by dividing with the Planck's constant called $\hbar$ as the basic unit of angular momentum, together with the mysterious factor i. Perhaps a word about the mysterious factor i is in order at this stage. While its numerical value is merely a 'square root of minus one', this 'static' quantity, got transformed at the hands of Dirac to the status of a dynamical variable with great potential for fresh adventures. Indeed Dirac demonstrated that this strange quantity called 'commutator bracket divided by $i\hbar$' happened to possess identical algebraic properties to the classical Poisson brackets[5].

And this 'fresh adventure' carried precisely the seeds of discreteness that characterize quantum theory of today. Thus was born the quantum theory, a new paradigm emerging from the original premises of Newton's continuum theory that would have been impossible to guess from the Newtonian equations of motion. Indeed, Herbert Goldstein, in his famous book on classical mechanics[7], termed Hamilton's canonical equations as providing 'the golden road to quantization'. An alternative, albeit equally revolutionary, formulation of the same paradigm of discreteness by Erwin Schrödinger[8] – using Louis de Broglie's concept of wave–particle duality[9] – gave rise to still another, equally vibrant, form of dynamics in the shape of a wave equation with identical physical content to the Heisenberg–Dirac form – the celebrated Schrödinger equation. And it took a new mathematical vehicle – the theory of unitary transformations – to prove the equivalence of the two.

Dirac equation: a synthesis of matter and radiation

Further incursions into the rapidly developing territory of physics with the help of mathematical machinery became possible via the realm of interaction of (Newtonian) matter with (Maxwellian) radiation. Thus the special force F which characterizes Newton's law for the motion of a charged particle acquires the form of the Lorentz force which expresses the resultant of the electric and magnetic forces on the charged particle concerned. Conversely, the laws that determine the influence of matter on the evolution of the electromagnetic field could not be left far behind. The latter have been termed Maxwell's equations, after the man who first gave a unified description of the piecemeal influence of matter on radiation, discovered individually by several giants (Coulomb, Gauss, Faraday, Biot–Savert, Lenz), into one organic whole, leading to the emergence of light as a (universal) form of wave motion with an electromagnetic origin. This mutual relationship between the two basic entities of nature also follows from the master action principle defined above, as a single source of their mutual relationship. The remarkable thing about the Lorentz-cum-Maxwell equations is that they are already compatible with Einstein's special theory of relativity as they stand, without the need for further physical assumptions. To see Dirac's unifying role in bringing about the synthesis of relativity with quantum theory, his first step was to provide the quantum version of the classical Hamilton equations through the formal equivalence of the Poisson and commutator brackets (see above). His second step provided the next crucial element[10]: a self-consistent mathematical equation – the Dirac equation – for the interaction of a relativistic electron with the electromagnetic field, bringing out in the process its spin property together with the associated magnetic moment.

Emergence of quantum field theory

Problems with relativity plus quantum theory

The Dirac equation has had a profound impact on the very direction of physics through its diverse ramifications born out of certain consistency problems inherent in its formulation. The most important one concerns the impossibility of a self-consistent quantum description of a single relativistic particle. This may sound paradoxical, since a 'non-relativistic' (slow-moving) small particle is perfectly capable of quantization, just as Einstein had encountered no difficulty in obtaining his 'classical' relativistic equations for a macroscopic, fast-moving particle. It is only when both conditions (relativity and quantum theory) are simultaneously imposed that strange consistency problems arise. Now Dirac was already aware of the problem of non-positive probability inherent in a naive application of the second-order Klein–Gordon equation, which has two time derivatives to go with two space derivatives so as to preserve the structure of special relativity. So he tried his luck with first-order differential equations, viz. single time derivative which must go only with single space-derivatives to maintain a relativistic balance. Such a structure had necessarily to be at the cost of a multi-component (no longer single component) wave function. [To give a useful analogy, the Maxwell equations are also coupled first-order differential equations in the electromagnetic field components E and H.] In doing so, Dirac succeeded in obtaining non-negative probability densities to be sure, but he could not avoid the problem of negative energy states. He could resolve this vexing problem only after postulating that the vacuum (the state of lowest energy) is already full of negative energy states, so that a positive energy electron cannot directly make a transition to a negative energy state – thanks to the Pauli exclusive principle characteristic of spin-one-half particles. It is only when a 'hole' is created in this sea of negative energy states by one of these negative energy particles acquiring enough positive energy (given from outside) to jump out of the vacuum, that a transition to a negative energy state by another positive energy particle becomes possible.

Cost of unification: concept of field

In making such a hypothesis therefore, Dirac was dealing effectively with not one but an infinite number of particles (a field) at the same time. And he was promptly vindicated by Anderson's cosmic ray discovery of a positively charged particle with the electron's mass, playing precisely the part of this 'hole'. Pauli immediately saw through the 'message' of Dirac's negative energy sea interpretation, and proceeded to show that this property of spin-half electrons (known as fermions) being accompanied by an infinite number of its kind is also shared by the spin-zero particles (known as bosons) which obey the second order Klein–Gordon equation. All that was needed was that the parameter of energy be replaced by the parameter of probability density[11]. Thus the non-positive probability implicit in a Klein–Gordon wave function should now be reinterpreted as an average charge density of an infinite number of spin-zero particles – a field again – of both positive and negative charges. Thus the common message from both cases is that of the existence of an infinite number of particles – be it spin-half fermions or spin-zero bosons – so as to be consistent with relativistic quantum theory. A single particle just will not do when quantum theory and relativity are sought to be put together. It was perhaps Freeman Dyson who, through his Cornell lectures of 1951 (ref. 12), clearly brought out the field role for both fermions and bosons, in preference to their single particle interpretation.

Mathematical language of quantum field theory

This in short is the story of the genesis of quantum field theory, or QFT for short, whose applications in physics extend all the way from the theory of elementary particles to condensed matter physics, and are now spreading even to the biological sciences. Its mathematical language being that of the harmonic oscillator (HO), QFT may be regarded as a collection of HOs which must be systematically classified and put to different jobs. Now for a single HO, the total energy is half kinetic and half potential, so it is convenient to define its basic variables as 50–50 admixture of momentum (p) and coordinate (x) variables whose basic commutator $[x, p]$ has the value $i\hbar$ by virtue of their Poisson bracket structure (see Dirac above). These 50–50 admixtures, termed a and $a^\dagger$, expressed in dimensionless units, may be easily shown to obey the commutation relations:

$$[a, a^\dagger] = 1;\ [a, a] = [a^\dagger, a^\dagger] = 0.$$

And the energy operator H which equals $\omega\hbar\ (a^\dagger a + 1/2)$, in units of the spring constant ω, incorporates the essential dynamics in terms of the 'number operator' N, which equals $a^\dagger a$, and whose integer eigenvalues n are called occupation numbers; the corresponding 'states' (wave functions) are called the eigenstates $|n\rangle$ of N. By virtue of the equalities $a|n\rangle = \sqrt{n}\ |n - 1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\ |n + 1\rangle$, a and $a^\dagger$ are called destruction/creation operators respectively, since they reduce/increase the occupation number in a given state $|n\rangle$ by one unit each. Hence by successive applications of the $a^\dagger$ operators, the eigenstates $|n\rangle$ can be built up from the ground state ('vacuum') $|0\rangle$. In this language the successive eigenvalues of the energy operator become E_n, which equals $\omega\hbar(n + 1/2)$. The QFT generalization for an infinite collection of harmonic oscillators a_k, $a_k^\dagger$ indexed by an integer k (or a collection of such integers thereof), is now only a matter of systematic construction of the energy contents as well as of the successive states which are all expressible in terms of a 'master' ground state $|0\rangle$. This 'master' ground state, or simply the vacuum state is a central theme around which the entire concept and methodology of QFT devolve.

Applications of QFT: success of quantum electrodynamics

The earliest application of the QFT formalism has been in the area of quantum electrodynamics (QED), the theory of interaction of charged fermions (electrons, free or bound) with the bosonic electromagnetic field, which lends itself easily to an HO formulation. And the success of QED in unravelling the mysteries of the 'fine-structure constant' for understanding atomic and molecular spectroscopy (upwards of the Lamb shift) to the accuracy of 'one in a trillion' is but too well known for further elaboration[13]. Even Dirac was impressed by this achievement, in answer to Dyson's query, but he did not feel happy at this development, and wished the formalism were not so 'ugly'[14].

Conclusion: feedback effects

This long story of the strong interaction of mathematics with physics, which suggests a sort of mutual interdependence, need not give the impression that the influence of mathematics on the other physical sciences is any less profound, so as not to invalidate in any manner the basic theme of this article as a whole. Indeed mathematical techniques are finding increasing applications into most other sciences (from the physical to the biological); only the aspect of mutual interdependence that has characterized its relationship with theoretical physics, is perhaps absent.

Apart from the influence that mathematics exerts on the sciences, its own frontiers are expanding daily in newer and newer directions. Perhaps the most significant development in this regard is the expansion of the techniques of mathematics to the domain of information theory as well as the related field of computer technology. In this respect, a vital ingredient of physics, namely quantum theory has had a crucial role, inasmuch as 'quantum computation' – still under active development – has a great potential for a significantly faster action than its classical counterpart.

Before ending this narrative on the role of mathematics in shaping the different sciences, it is perhaps in order to express some thoughts on the nature of the feedback from the latter to the former. Namely, how does science as a whole react to the language of mathematics? To address this question, one may wish to inquire into the methodology of science in actual practice, namely the scientific method. In this respect the vital role of checking or verification is crucial. To support this view, it is of interest to quote from Bridgeman[15], taken from his Nobel lecture (1955): 'Scientific method is something talked about by people standing on the outside and wondering how the scientist manages to do it. These people have been able to uncover various generalities applicable to at least most of what the scientist does, but it seems to me that these generalities are not very profound, and could have been anticipated by anyone who knows enough about scientists to know what is their primary objective. I think that the objectives of all scientists have this in common – that they are all trying to get the correct answer to the particular problem in hand. This may be expressed in more pretentious language as the pursuit of truth. Now if the answer to the problem is correct there must be some way of knowing and proving that it is correct – the very meaning of truth implies the possibility of checking or verification. Hence the necessity for checking his results always inheres in what the scientist does.'

This article is dedicated to the memory of my father, Jatindranath Mitra, who had been the main inspiration in my pursuit of mathematics.

1. Chari, V. and Pressley, A., *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, UK, 1994.
2. *Theoretical Physics*, IAEA Publication, Vienna, 1968.
3. Hardy, G. H., *A Mathematician's Apology*, Cambridge University Press, Cambridge, UK, 1940.
4. Follendore III, R. D., Copyright, 2005.
5. Dirac, P. A. M., The fundamental equations of quantum mechanics. *Proc. R. Soc. London, Ser. A*, 1925, **109**, 642–653.
6. Heisenberg, W., Über quantentheoretishe Umdeutung kinematischer und mechanisher Beziehungen. *Phys. Zeits*, 1925, **33**, 879–893.
7. Goldstein, H., *Classical Mechanics*, Addison Wesley Publ Co, Reading, MA, USA, 1950.
8. Schrödinger, E., Quantisierung als Eigenwertproblem. *Annu. Phys.*, 1926, **79**, 489–527.
9. de Broglie, L., Researcher on the quantum theory. *Ann. Phys.* (*Paris*), 1924, **3**, 22; A tentative theory of light quanta. *Philos. Mag.*, 1924, **47**, 446.
10. Dirac, P. A. M., The quantum theory of the electron. *Proc. R. Soc., Ser. A*, 1928, **117**, 610–624.
11. Pauli, W. and Weisskopf, V. F., *Helv. Phys. Acta*, 1934, **7**, 709.
12. Dyson, F. J., *Advanced Quantum Mechanics*, Lithoprinted notes, Cornell University, 1951, reprinted by L'cole d't de Physique Thorique, Les Houches, Haute-Savoie, and University of Grenoble, France, 1954.
13. Schweber, S., Bethe, H. A. and de Hoffmann, F., *Mesons and Fields*, *Vol. I* (*Fields*), Rho Peterson and Co Inc, Evanston Il, 1955.
14. Farmelo, G., *The Strangest Man*, Faber and Faber, London, 2009.
15. Bridgeman, P. W., *On Scientific Method*, 1955.

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