Existence of stationary equilibrium for mixtures of discounted stochastic games[†]

Nagarajan Krishnamurthy[1], T. Parthasarathy[2] and Sujatha Babu[2,]*

[1]Indian Institute of Management, Indore 453 556, India
[2]Indian Statistical Institute, 37 Nelson Manickam Road, Aminjikarai, Chennai 600 029, India

We consider non-zero sum discounted uncountable state space stochastic games that contain mixtures of game classes. We prove the existence of a stationary equilibrium pair for these mixture games. In particular, we consider mixture class games that contain a mix of countable/finite and uncountable state space subsets, Separable Reward - State Independent Transition subgame, State Independent Transition subgame and certain types of Additive Reward Additive Transition subgames. The technique used in this article involves defining a new stochastic game, and defining its reward and transition functions in terms of the original game's reward and transition functions. We then show that the new game, and hence the original game, each contain a stationary equilibrium pair. Though this article considers two-player non-zero sum discounted uncountable state space stochastic games, the proof of existence of a stationary equilibrium pair can be extended to *n*-player games also.

Keywords: Discounted stochastic games, mixture class, stationary equilibrium, uncountable state space.

Introduction

SHAPLEY'S[1] fundamental paper discusses zero-sum discounted finite stochastic games consisting of a finite number of states, finite sets of actions for the players, a payoff matrix in each state, a discount factor between 0 and 1, and transition probabilities from each state to every other state for every pair of actions of the players. In a two-player game, the row chooser and column chooser can be considered as the maximizer and minimizer respectively, without any loss of generality. For a given starting state of the game, each player simultaneously chooses actions that result in an immediate payoff. The game then moves to a new state depending on the transition probabilities defined for the game and this continues over the infinite horizon. As the game progresses, the payoffs are discounted by a discount factor $\beta \in (0, 1)$. Further, Shapley[1] proved that every zero-sum discounted finite stochastic game has an optimal value vector and an optimal stationary strategy for each player where the state space can be finite or countable. We sometimes refer to optimal strategies and optimal values as equilibrium strategies and equilibrium values respectively, in the zero-sum case. In the non-zero sum case, equilibrium refers to Nash equilibrium[2].

Maitra and Parthasarathy[3,4] proved the existence of equilibrium for stochastic games with uncountable state space and infinite action space. Maitra and Sudderth[5] proposed an alternative proof for the existence of the optimal value in the undiscounted, zero-sum games for finite, countable and uncountable state space scenarios. These existence results have also been extended to non-zero-sum stochastic games[6–8].

Gillette[9,10] introduced undiscounted stochastic games. For both the zero-sum and non-zero-sum undiscounted stochastic games, stationary equilibrium points may not exist in general. For example, Blackwell and Ferguson[11] showed that the 'Big Match' game (a zero-sum undiscounted game) does not have a stationary equilibrium point. For discounted zero-sum games with uncountable state space, it is not known if there exists a stationary equilibrium point (we recently came to know that this problem has been answered negatively by Levy[12,13]). Mertens and Parthasarathy[14] prove the existence of subgame-perfect equilibria for discounted stochastic games with general state and action sets, provided the transition probabilities are norm-continuous functions of the actions. Also refer to Mertens[15], Nowak[16] and related papers in the volume edited by Neyman and Sorin[17].

Special classes of stochastic games were then studied for stationary equilibrium. The existence of p-equilibrium stationary strategies for non-zero sum stochastic games, where the reward functions and transition probability functions satisfy certain separability conditions, was shown by Himmelberg *et al.*[18]. As a strengthening of this result, Parthasarathy[19] showed that if there exists a p-equilibrium stationary pair for the discounted stochastic game, then there exists an equilibrium pair under the assumptions of reward function continuity and strong continuity of transition probability over the state and action space ($S \times A \times B$). For the zero-sum discounted game, the assumptions can be relaxed for the above to hold.

Later, Parthasarathy *et al.*[20] discuss Separable Reward - State Independent Transition (SER-SIT) games

[†]This article is dedicated to Prof. Lloyd Shapley on the occasion of his 90th birthday. *For correspondence. (e-mail: sujathab@gmail.com)

with uncountable state space and show the existence of independent stationary equilibrium strategy for both players. Parthasarathy and Sinha[21] further show the existence of stationary equilibrium strategies for non-zero sum discounted stochastic games with uncountable state space, finite action space, state independent transitions (SITs) and where the transition probabilities are absolutely continuous with respect to a fixed non-atomic measure μ (i.e. $q(.|.) \ll \mu$). Nowak[22] generalized the result shown by Parthasarathy and Sinha[21]. He shows the existence of stationary equilibrium strategies in a new class of non-zero sum discounted games where the transition probability function is a combination of finitely many measures (not necessarily atomless) on the state space, and where the coefficients depend on the state and action variables. For some related references on dynamic programing, refer to Blackwell[23] and Maitra[24,25].

Krishnamurthy *et al.*[26] discuss the orderfield property for mixtures of stochastic games. As a natural extension of this and as it is not known if a stationary equilibrium pair exists for uncountable state space stochastic games, we would like to explore the existence of a stationary equilibrium pair for mixtures of stochastic games with uncountable state space.

A mixture of classes of games refers to games with transitions among states of different classes. For example, we can partition the state space of the game into disjoint subsets of states (say, S_1 and S_2 that are subsets of a complete, separable metric space and S_2 is an uncountable absorbing Borel state space), where each subset can be one of SER-SIT, Additive Reward Additive Transition (ARAT), one-player control, etc. In this article, we consider the following mixture classes for games with uncountable state space.

1. S_1 is finite or countable. S_2 is an uncountable absorbing Borel state space with stationary equilibrium property.
2. The game restricted to S_1 is a SER-SIT game. The game restricted to S_2 has stationary equilibrium property.
3. The game restricted to S_1 is a ARAT game. The game restricted to S_2 has stationary equilibrium property.
4. The game restricted to S_1 is a SIT game, as defined by Nowak[22]. The game restricted to S_2 has stationary equilibrium property.
5. The games restricted to S_1 and S_2 are both SER-SIT games. Also, the subgame restricted to S_2 has stationary equilibrium property, and S_2 may not necessarily be an absorbing state.
6. The game is a mixture of more than two classes of games.

We show the existence of a stationary equilibrium pair for the above mixture class of two-player non-zero sum discounted stochastic game. We also consider the game where the state space is a union of disjoint cycle-free subsets. The above results can be extended to the n-player game.

A few remarks on the computational aspects of finding a stationary equilibrium pair in stochastic games have been added following the suggestions of one of the referees. Pollatschek and Avi-Itzhak[27] propose a Newton–Raphson type iterative algorithm to solve two-person zero-sum discounted games. In general, this algorithm can be used to find only approximate optima as there is no guarantee on the number of steps such iterative algorithms take to converge. In fact, there are stochastic games (even in the finite state space discounted zero-sum case) for which no (finite arithmetic-step) exact algorithms exist. The reason for this is that there are stochastic games which do not possess the orderfield property.

For known classes of finite state space stochastic games (such as SER-SIT and ARAT games), algorithms do exist and complexity results are known too. Some of these results may be found in the literature[20,28–37]. Mohan *et al.*[38] provide an algorithm for a class of multi-player stochastic games. Algorithms have also been proposed for mixtures of classes of stochastic games. For example, Neogy *et al.*[39] propose an algorithm to solve a mixture of switching control (SC) and ARAT states. Krishnamurthy *et al.*[40] propose polynomial time algorithms for some subclasses.

For certain classes of stochastic games with uncountable state space, a value iteration algorithm has been provided by Altman *et al.*[41] and further by Borkar *et al.*[42]. In general, a stationary equilibrium pair may not exist for stochastic games with uncountable state space. Even for classes of these stochastic games with stationary equilibria, efficient algorithms may not exist.

Background and preliminaries

Definition 1. *Two-player stochastic game: A* 2*-player stochastic game consists of:*

1. *Two players – players* 1 *and* 2 (*often denoted as* P_1 *and* P_2).
2. *The state space S can be countable, finite or an uncountable Borel set.*
3. *For each state* $s \in S$, *the actions available to player* P_k ($k = 1, 2$) *are denoted by the finite, non-empty sets* $A_k(s) = \{1, 2, \ldots, m_k(s)\}$. *Without loss of generality, we may assume* $A_k(s) = A_k$, $\forall s \in S$.
4. *The immediate rewards for the players when the game is in state s and the players choose actions i and j respectively, are given by* $r_1(s, i, j)$ *for player* P_1 *and* $r_2(s, i, j)$ *for player* P_2. *The payoff matrices* (*i.e. the matrices of immediate rewards) in state s for players* P_1 *and* P_2 *are denoted by* $R_1(s)$ *and* $R_2(s)$ *respectively.*
5. *The probability of transition from state s to state s' given that players* P_1 *and* P_2 *choose actions* $i \in A_1$ *and* $j \in A_2$ *is given by* $q(s' \mid s, i, j)$.

Remark 1. For finite and countable state space, the transition probabilities can be represented as a matrix (for example, as a $N \times N$ matrix where $|S| = N$ in case of finite state space). The probability transition will be a measure in the case of uncountable state space.

Definition 2. *Stationary strategy: Let S be the state space and P_{A_1} be the set of probability distributions on player 1's action set A_1. A stationary strategy for player P_1 is a Borel measurable mapping $f: S \to P_{A_1}$ that is independent of the history that led to the state $s \in S$. Similarly, we define a stationary strategy for player P_2 as a Borel measurable mapping $g: S \to P_{A_2}$ that is independent of the history that led to the state $s \in S$.*

Definition 3. *β-Discounted payoffs: Given the initial state s_0, a pair of stationary strategies (f, g) for the players, and a discount factor $\beta \in (0, 1)$, the β-discounted payoff for player P_k, $k = 1, 2$ is as follows:*

$$I_\beta^{(k)}(f, g)(s_0) = \sum_{t=0}^{\infty} \beta^t r_k^{(t)}(s_0, f, g). \tag{1}$$

Here $r_k^{(t)}(s_0, f, g)$ is the expected immediate reward at the t-th stage to player P_k.

Definition 4. *Optimal strategies and optimal value: A pair of stationary strategies (f^o, g^o) is optimal in the zero-sum discounted case, if $\forall s \in S$,*

$$I_\beta(f, g^o)(s) \le I_\beta(f^o, g^o)(s) \le I_\beta(f^o, g)(s),$$
$$\forall f \in P_{A_1}, \forall g \in P_{A_2}. \tag{2}$$

In other words,

$$I_\beta(f^o, g^o)(s) = \inf_g [I_\beta(f^o, g)(s)]$$
$$= \sup_f [I_\beta(f, g^o)(s)], \forall s \in S. \tag{3}$$

For finite state space, Shapley[1] proved the existence and uniqueness of the optimal value $I_\beta(f^o, g^o)$ across all pairs of optimal strategies (f^o, g^o), and denoted the optimal value by v_β. That is,

$$v_\beta(s) = I_\beta(f^o, g^o)(s) = \sup_f \inf_g [I_\beta(f, g)(s)]$$
$$= \inf_g \sup_f [I_\beta(f, g)(s)], \quad \forall s \in S. \tag{4}$$

While the optimal value is unique, optimal strategy pair may not be unique.

Definition 5. *Nash equilibrium: A pair of stationary strategies (f^o, g^o) constitutes a Nash equilibrium in the discounted case if $\forall s \in S$,*

$$I_\beta^{(1)}(f^o, g^o)(s) \ge I_\beta^{(1)}(f, g^o)(s), \ \forall f \in P_{A_1},$$

and

$$I_\beta^{(2)}(f^o, g^o)(s) \ge I_\beta^{(2)}(f^o, g)(s), \ \forall g \in P_{A_2}, \tag{5}$$

assuming that both players want to maximize their payoffs.

Definition 6. *Stationary equilibrium property of a game: A stochastic game $(S, A_1, A_2, r_1, r_2, q, \beta)$ has the stationary equilibrium property if the game has a stationary equilibrium pair (f^o, g^o) satisfying the following conditions $\forall s \in S$.*

1. *For player P_1,*

$$u_1(s) = \max_{\mu \in A_1}[r_1(s, \mu, g^o(s)) + \beta \int u_1(s') \mathrm{d}q(s', \mu, g^o(s))]$$
$$= r_1(s, f^o(s), g^o(s)) + \beta \int u_1(s') \mathrm{d}q(s', f^o(s), g^o(s)). \tag{6}$$

2. *For player P_2,*

$$u_2(s) = \max_{\lambda \in A_2}[r_2(s, f^o(s), \lambda) + \beta \int u_2(s') \mathrm{d}q(s', f^o(s), \lambda)]$$
$$= r_2(s, f^o(s), g^o(s)) + \beta \int u_2(s') \mathrm{d}q(s', f^o(s), g^o(s)). \tag{7}$$

Unlike the uniqueness of the optimal value in the zero-sum game, Nash equilibrium payoffs may not be unique.

Remark 2. From the above, it is clear that $u_k(s) = I_\beta^{(k)}(f^o, g^o)(s)$, for $k = 1, 2$.

Definition 7. *Class of games: We refer to games having a common structure as a class of games. A state s_0 controlled by player P_1 has property C (that is, $s_0 \in C$), if the structure of C holds (locally) in s_0. For example, s_0 is a one-player control state if $q(s|s_0, i, j) = q(s|s_0, i)$, $\forall s \in S$, $i \in A_1, j \in A_2$.*

The following are definitions of some classes of stochastic games.

Definition 8. *Stochastic games with perfect information: First defined by Shapley[1], these are stochastic games where the action space of (at least) one of the players is a singleton in every state. In other words, we can partition the set of states S into S_1 and S_2 as follows: S_1 is the set of states for player P_1 where player P_2 has just one action and hence no choice, and S_2 is the set of states for player P_2 where player P_1 has just one action.*

Definition 9. *SER-SIT games: As defined in Parthasarathy et al.[20], SER-SIT stochastic games exhibit the following two properties.*

1. *The reward function can be written as the sum of two functions – one function that depends on the state alone, and another function that depends on the actions alone. That is,* $\forall s \in S$, $\forall i \in A_1$, $\forall j \in A_2$,

$$r_k(s, i, j) = c_k(s) + a_{ij}, \quad k = 1, 2, \tag{8}$$

where $c_k(s)$ *is a measurable function.*

2. *The transition probabilities are independent of the state from which the game transitions. That is,* $\forall i \in A_1$, $\forall j \in A_2$ *and* $\forall s, s' \in S$,

$$q(s'|s, i, j) = q(s'|i, j). \tag{9}$$

Definition 10. *One-player control stochastic games: Defined by Parthasarathy and Raghavan*[43]*, these are stochastic games where only one of the players controls the transitions. For example,* $\forall i \in A_1$, $\forall j \in A_2$ *and* $\forall s, s' \in S$, *we have*

$$q(s'|s, i, j) = \begin{cases} q(s'|s, i), \text{ when player } P_1 \text{ controls transitions,} \\ q(s'|s, j), \text{ when player } P_2 \text{ controls transitions.} \end{cases} \tag{10}$$

For more details on this, refer to Filar[44,45].

Definition 11. *ARAT games*[18,46]*: Himmelberg et al.*[18] *define a class of non-zero sum stochastic games where the reward function and transition satisfy certain separability conditions as follows.*

1. *The reward function can be written as the sum of two functions, one depending on the action of player* P_1 *and the other on the action of player* P_2*. That is,* $\forall i \in A_1$, $\forall j \in A_2$ *and* $\forall s \in S$,

$$r_k(s, i, j) = r'_k(s, i) + r''_k(s, j), \quad k = 1, 2, \tag{11}$$

where r_k *is a bounded measurable function in* s.

2. *Similarly, the transition probability can be written as the average of two probability distribution functions, one depending on player* P_1*'s action and the other on player* P_2*'s action. That is,* $\forall i \in A_1$, $\forall j \in A_2$ *and* $\forall s, s' \in S$,

$$q(s'|s, i, j) = \frac{1}{2}[q_1(s'|s, i) + q_2(s'|s, j)], \tag{12}$$

where q_k *is measurable in* s*. Let* p *be a fixed probability distribution on* S*. Assume that the transition probability* q *is absolutely continuous with respect to* p.

Raghavan et al.[46] *prove that the orderfield property holds in finite state space stochastic games with additive rewards and additive transitions (ARATs), and also show the existence of pure optimal strategy in the zero-sum case. They define the separability condition for the reward function similar to the above. They define the transition probabilities as a sum of two functions, one depending on player* P_1*'s action and the other on player* P_2*'s action. That is,* $\forall i \in A_1$, $\forall j \in A_2$ *and* $\forall s, s' \in S$,

$$q(s'|s, i, j) = q_1(s'|s, i) + q_2(s'|s, j). \tag{13}$$

Definition 12. *SIT class of games as defined by Nowak*[22]*: These refer to non-zero sum Borel state space discounted stochastic games as specified in Nowak*[22]*. Assume that* D *is a countable subset of the nonempty Borel state space* S*. Let* $C = S \setminus D$*. Let* X_i *be a non-empty Borel space of actions for player* i*, and* $X = X_1 \times \cdots \times X_m$*. Let* $X_i(s)$ *be a non-empty compact subset of* X_i *and represent the set of actions available to player* i*. Define* $\Delta = \{(s, x): s \in S, x \in X(s)\}$*. The basic assumptions are as follows.*

1. *The Borel measurable transition probability function is a combination of finitely many measures (not necessarily atomless) on the state space.*

   That is, there are atomless non-negative measures μ_j *on* C*, non-negative measures* δ_t *concentrated on subsets of* D*, and Borel measurable functions* $c_j: \Delta \to [0, 1]$ *and* $b_t: \Delta \to [0, 1]$ $(j = 1, \ldots, k$ *and* $t = 1, \ldots, l)$ *such that for all* $(s, x) \in \Delta$*, we have*

$$p(.|s, x) = q(.|s, x) + \delta(.|s, x),$$

where δ *is the 'atomic part' of* p *and is as follows.*

$$\delta(.|s, x) = \sum_{t=1}^{l} b_t(s, x)\delta_t(.).$$

The 'atomless part' q *of* p *is of the form*

$$q(.|s, x) = \sum_{j=1}^{k} c_j(s, x)\mu_j(.).$$

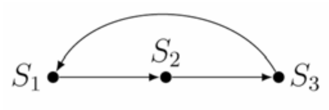
2. *All the functions* $r_i(s, .)$*,* $b_j(s, .)$ *and* $c_j(s, .)$ *are continuous on* $X(s)$ $\forall s \in S$.

For more details on some of these topics, refer to Nowak[22].

The following definitions from Krishnamurthy *et al.*[26] also hold when the state space is uncountable.

Definition 13. *Cycle between subsets: Given a stochastic game* Γ *with set of states* S *and a partition of* S *into* S_1 *and* S_2 *(that is,* $S = S_1 \cup S_2$*,* $S_1 \cap S_2 = \phi$*), we say* S_1 *and* S_2 *are in a cycle if there exist pairs of states* $(s_1, s_2) \in S_1 \times S_2$ *and* $(s'_1, s'_2) \in S_1 \times S_2$*, and pairs of actions* $(i, j) \in A_1 \times A_2$ *and* $(i', j') \in A_1 \times A_2$ *such that* $q(s_2|s_1, i, j) > 0$ *and* $q(s'_1 | s'_2, i', j') > 0$.

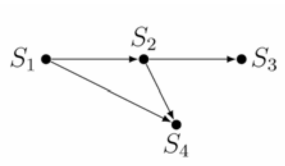
We can also extend this definition to a partition of S into more than two subsets where we can talk of cycles between pairs of subsets as well as cycles involving more than two subsets. Following is an example of a cycle.



Definition 14. *Cycle free* (*acyclic*) *classes: Consider a stochastic game* Γ *with a set of states* $S = S_1 \cup S_2$ *and* $S_1 \cap S_2 = \phi$. S_1 *and* S_2 *are cycle-free if they are not in a cycle*.

Extending this definition to a partition of S into more than two subsets where $S = S_1 \cup S_2 \cup \ldots \cup S_m$ and $S_{m_1} \cap S_{m_2} = \phi\,(m_1 \neq m_2)$, then $S_1, \ldots, S_m$ are cycle-free if there is no cycle of any length ≥ 2 among them. In such a case, at least one of them is an absorbing subset with no transitions going out of the subset.

In the following example, S_1–S_4 are cycle-free, and S_3 and S_4 are sinks.



Definition 15. *Mixture class/game: A stochastic game* Γ *with set of states* S *is a mixture of classes* C_1 *and* C_2, *if* $S = S_1 \cup S_2$, $S_1 \cap S_2 = \phi$, *where* $S_1 \in C_1$, $S_2 \in C_2$ *and where* S_1 *and* S_2 *are subsets of a complete, separable metric space. This definition can be extended to games with more than two disjoint cycle-free state spaces.*

Definition 16. *Absorbing state space: A state space* S_2 *is called absorbing if there are no transitions from the states in* S_2 *to states in another state space* S_1. *That is,*

$$q(s' \mid s, i, j) = 0,\ \forall s \in S_2,\ \forall s' \in S_1,\ \forall i \in A_1,\ \forall j \in A_2.$$

Existence of stationary equilibrium for mixture class of games

We will prove the existence of a stationary equilibrium pair for mixtures of two-person non-zero sum discounted stochastic games with uncountable state space. These can be easily generalized for mixtures of n-player non-zero sum discounted stochastic games.

In this article, we assume that the reward functions are bounded and measurable for all players, and all uncountable state spaces are Borel sets. Also for ease of notation, we will denote the expected discount income by I_k instead of $I_\beta^{(k)}$ by dropping the suffix β. Also S_1 and S_2 are considered as subsets of a complete, separable metric space, where $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$.

We consider only games with finite action spaces, since it is known that games with infinite action spaces may not have an equilibrium value.

Mixture class of games where S_1 *is finite or countable*

Theorem 1. *Consider a mixture class of discounted non-zero sum stochastic game* $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ *where*

1. S_1 *and* S_2 *are subsets of a complete, separable metric space* S *such that* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.
2. S_1 *is finite or countable.* S_2 *is an absorbing complete, separable metric space that has the stationary equilibrium property* (*refer to definition* 6). *Let* (f_2, g_2) *be a stationary equilibrium pair of* S_2.

Then the mixture game Γ_β *has a stationary equilibrium pair.*

Proof. Since S_2 is an absorbing state space, we have $\forall s \in S_2$, $\forall i \in A_1$, $\forall j \in A_2$,

$$\int_{s' \in S_2} \mathrm{d}q(s' | s, i, j) = 1.$$

Let (f_2, g_2) be a stationary equilibrium pair for the subgame Γ_β restricted to S_2, and the corresponding discounted expected income/payoff for player k be $I_k(f_2, g_2)(s)$, where $k = 1, 2$.

Define a new game $\Gamma'_\beta = (S \cup \{s^*\}, A_1, A_2, r'_1, r'_2, q', \beta)$, where s^* is a new absorbing state, i.e. $\forall i \in A_1$, $\forall j \in A_2$, we have

$$q'(s^* | s^*, i, j) = 1. \tag{14}$$

In this new game, $\forall i \in A_1$, $\forall j \in A_2$, $k = 1, 2$, let

$$r'_k(s, i, j) = r_k(s, i, j) + \beta \int_{S_2} I_k(f_2, g_2)(s') \mathrm{d}q(s' \mid s, i, j), \quad \forall s \in S_1, \tag{15}$$

$$r'_k(s^*, i, j) = 0, \tag{16}$$

$$r'_k(s, i, j) = r_k(s, i, j), \quad \forall s \in S_2, \tag{17}$$

$$q'(s^* | s, i, j) = 1 - \int_{s' \in S_1} \mathrm{d}q(s' | s, i, j), \quad \forall s \in S_1, \tag{18}$$

$$q'(s' \mid s, i, j) = q(s' \mid s, i, j), \quad \forall s, s' \in S_2, \tag{19}$$

$$q'(s' \mid s, i, j) = 0, \quad \forall s \in S_2, \forall s' \in S_1. \tag{20}$$

It is apparent that the subgame restricted to S_2 is the same independent stochastic game in both Γ_β and Γ'_β. Hence a stationary equilibrium pair for the subgame restricted to S_2 in Γ'_β remains (f_2, g_2).

The subgame restricted to $S_1 \cup \{s^*\}$ is a finite or countable non-zero sum discounted stochastic game, and hence has an equilibrium pair, say (f_1, g_1) (see Fink[6], and Takahashi[7] for finite games; Federgruen[47] for countable state space; and Maitra and Parthasarathy[3] for uncountable state space under suitable assumptions).

Let us define a strategy pair (f', g') for the new game Γ' as follows.

$$f'(s) = \begin{cases} f_1(s), \; s \in S_1 \\ f_2(s), \; s \in S_2 \end{cases} \quad \text{for player } P_1. \tag{21}$$

Note that $f'(s^*)$ can be any arbitrary element in P_{A_1} as s^* is an absorbing state.

$$g'(s) = \begin{cases} g_1(s), \; s \in S_1 \\ g_2(s), \; s \in S_2 \end{cases} \quad \text{for player } P_2. \tag{22}$$

Similarly, $g'(s^*)$ can be any arbitrary element as s^* is an absorbing state.

We will now show that (f', g') is a stationary equilibrium pair for Γ_β.

Let the new game Γ'_β start in state space S_2. Since S_2 is an absorbing state space, the game never moves to S_1 from S_2.

$\Rightarrow \forall s \in S_2, f'(s) = f_2(s)$, and $g'(s) = g_2(s)$.

$\Rightarrow (f', g') = (f_2, g_2)$ is a stationary equilibrium pair trivially.

For player P_1, $\forall s \in S_2$,

$$I_1(f_2, g_2)(s) = \max_\mu [r_1(s, \mu, g_2(s)) + \beta \int_{S_2} I_1(f_2, g_2)(s') \mathrm{d}q(s'|s, \mu, g_2(s))]$$

$$= \max_\mu [r_1(s, \mu, g_2(s)) + \beta \int_{S_1 \cup S_2} I_1(f_2, g_2)(s') \mathrm{d}q(s' | s, \mu, g_2(s))]$$

(as S_2 is an absorbing state)

$$= \max_\mu [r_1(s, \mu, g_2(s)) + \beta \int_{S_1 \cup S_2} I_1(f', g')(s') \mathrm{d}q(s' | s, \mu, g_2(s))]$$

(using eqs (21) and (22))

$$= u_2(s) \text{ (say)}. \tag{23}$$

Now let the new game Γ'_β start in $S_1 \cup \{s^*\}$.

$\forall s \in S_1 \cup \{s^*\}$,

$$u_1(s) = \max_\mu [r'_1(s, \mu, g'(s)) + \beta \int_{S_1 \cup \{s^*\}} I_1(f', g')(s') \mathrm{d}q(s'|s, \mu, g'(s))]$$

$$= \max_\mu [r'_1(s, \mu, g'(s)) + \beta \int_{S_1} I_1(f', g')(s') \mathrm{d}q(s'|s, \mu, g'(s))]$$

$$= \max_\mu [r_1(s, \mu, g_1(s)) + \beta \int_{S_2} I_1(f_2, g_2)(s') \mathrm{d}q(s' | s, \mu, g_1(s)) + \beta \int_{S_1} I_1(f', g')(s') \mathrm{d}q(s'|s, \mu, g'(s))] \quad \text{(using eq. (15))}$$

$$= \max_\mu [r_1(s, \mu, g_1(s)) + \beta \int_{S_1 \cup S_2} I_1(f', g')(s') \mathrm{d}q(s'|s, \mu, g_1(s))] \tag{24}$$

(using eqs (21) and (22)).

Define

$$u^*(s) = \begin{cases} u_1(s), \; \forall s \in S_1 \\ u_2(s), \; \forall s \in S_2 \end{cases} \quad \text{for player } P_1,$$

$$v^*(s) = \begin{cases} v_1(s), \; \forall s \in S_1 \\ v_2(s), \; \forall s \in S_2 \end{cases} \quad \text{for player } P_2. \tag{25}$$

It can be easily seen that $u^*(s^*) = 0$ and $v^*(s^*) = 0$. Using the definition of $u^*(s)$, eqs (23) and (24) can be combined to yield

$$u^*(s) = \max_\mu [r_1(s, \mu, g'(s)) + \beta \int_{S_1 \cup S_2} u^*(s') \mathrm{d}q(s'|s, \mu, g'(s))]$$

$$= r_1(s, f'(s), g'(s)) + \beta \int_{S_1 \cup S_2} u^*(s') \mathrm{d}q(s'|s, f'(s), g'(s)). \tag{26}$$

By Banach fixed point theorem, the value of $u^*(s)$ is unique.

Proceeding along the same lines for player P_2, we get

$$v^*(s) = \max_\lambda [r_2(s, f'(s), \lambda) + \beta \int_{S_1 \cup S_2} v^*(s') \mathrm{d}q(s'|s, f'(s), \lambda)]$$

$$= r_2(s, f'(s), g'(s)) + \beta \int_{S_1 \cup S_2} v^*(s') \mathrm{d}q(s'|s, f'(s), g'(s)), \tag{27}$$

where $v^*(s)$ is unique by Banach fixed point theorem.

Hence (f', g') is a stationary equilibrium pair for the mixture class of game, and (u^*, v^*) is the equilibrium payoff for the mixture class of game. □

Mixture class of games where S_1 is SER-SIT

For a mixture class of games that contains a SER-SIT subgame, we will use the following theorem.

Theorem 2. (*Parthasarathy et al.*[20] (*theorem 4.1*)). *Let Γ be a non-zero sum SER-SIT game. Let $(\hat{p}, \hat{q})$ be an equilibrium point of the $m \times n$ bimatrix game with*

$$\left[a_1(i, j) + \beta \sum_{t \in S} p(t \mid i, j) c_1(t),\ a_2(i, j) + \beta \sum_{t \in S} p(t \mid i, j) c_2(t)\right],\quad i = 1, \ldots, m \text{ and } j = 1, \ldots, n.$$

Then $(\hat{p}, \hat{q})$ is an equilibrium pair for the β-discounted game Γ.

Using the above theorem, the following lemma can be proved.

Lemma 1. *Consider a SER-SIT game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where S is a finite, countable or uncountable Borel set. Then there exists a stationary equilibrium pair (f, g) for the game Γ_β that is independent of the initial state of the game.*

Theorem 3. *Consider a mixture class of discounted non-zero sum stochastic game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where*:

1. *S_1 and S_2 are subsets of a complete, separable metric space. Also, $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.*
2. *S_1 and/or S_2 are uncountable Borel sets, and the reward function is totally bounded in S_1 and S_2.*
3. *The subgame restricted to S_1 is a SER-SIT game.*
4. *S_2 is an absorbing state space that has the stationary equilibrium property. Let (f_2, g_2) be a stationary equilibrium pair for S_2.*

Then the mixture game Γ_β has a stationary equilibrium pair.

Proof. Since the game Γ_β restricted to S_1 is a SER-SIT game, we have $\forall s \in S_1$, $\forall i \in A_1$, $\forall j \in A_2$ and $k = 1, 2$

$$r_k(s, i, j) = c_k(s) + a_{ij}, \tag{28}$$

where $c_k(s)$ is a measurable function.

$$q(.|s, i, j) = q(.|i, j). \tag{29}$$

Define a new game $\Gamma'_\beta = (S \cup \{s^*\}, A_1, A_2, r'_1, r'_2, q', \beta)$ as specified in Theorem 1 and proceed along the same lines as Theorem 1.

For player P_1,

$$r'_1(s, i, j) = r_1(s, i, j) + \beta \int_{S_2} I_1(f_2, g_2)(s')\mathrm{d}q(s'|i, j)$$

$= c_1(s) + a_{ij} + \beta d_{ij}$ (from eq. (28) and setting

$$d_{ij} = \int_{S_2} I_1(f_2, g_2)(s')\mathrm{d}q(s'|i, j))$$

$$= c_1(s) + a'_{ij} \quad (\text{setting } a'_{ij} = a_{ij} + \beta d_{ij}). \tag{30}$$

Let q' be the same as q restricted to S_1. That is, $q'(E|i, j) = q(E|i, j)$ for every Borel set E in S_1. Also,

$$q'(s^*|s, i, j) = q'(s^*|i, j)$$

$$= 1 - \int_{s' \in S_1} \mathrm{d}q(s'|i, j) = 1 - q(S_1|i, j). \tag{31}$$

If the state space is finite or countable, we use sum instead of integral in the above equation.

Hence by eqs (30) and (31), the new game restricted to $S_1 \cup \{s^*\}$ is a SER-SIT game and is the same as the game restricted to S_1. We have just shown in Lemma 1 that there exists a stationary equilibrium pair for a SER-SIT game which is independent of the initial state of the game.

Proceeding along the same lines as Theorem 1, we can easily see that the SER-SIT mixture class of game has a stationary equilibrium pair. □

Mixture class of games where S_1 is ARAT

For the ARAT mixture class of games, we use the following theorems proved by Himmelberg *et al.*[18], and later extended by Parthasarathy[19].

Theorem 4. (*Himmelberg et al.*[18] (*theorem* 1)). *Let* $S = [0, 1]$, $A_1 = 1, 2, \ldots, k$ and $A_2 = 1, 2, \ldots, l$. *Let $r_k(s, i, j) = a_k(s, i) + b_k(s, j)$, $k = 1, 2$, where a_k and b_k are bounded measurable functions in s, $\forall i \in A_1$, $\forall j \in A_2$. Let $q(.|s, i, j) = (1/2)[q_1(.|s, i) + q_2(.|s, j)]$, where q_1 and q_2 are probability measures and are further measurable in s, $\forall i \in A_1$, $\forall j \in A_2$. Then for any probability distribution p on* $[0, 1]$ *with $q(.|s, i, j) \ll p$, $\forall s, i, j$ and for any $0 \le \beta < 1$, there exists a p-equilibrium stationary pair (f_1, g_1) for the two players.*

Theorem 5. (*Parthasarathy*[19] (*theorem* 1)). *Let* $S = [0, 1]$, $A_1 = 1, 2, \ldots, k$ and $A_2 = 1, 2, \ldots, l$ *and* $\beta \in (0, 1)$. *Let $r_1(s, i, j)$ and $r_2(s, i, j)$ be continuous over $S \times A_1 \times A_2$ and let $q(.|s, i, j)$ be strongly continuous over $S \times A_1 \times A_2$. Let p be a probability distribution over* $[0, 1]$ *with $q(.|s, i, j)$*

absolutely continuous with respect to p. If there exists a p-equilibrium stationary pair for the discounted stochastic game, then there exists an equilibrium pair.

Theorem 6. (*Parthasarathy*[19] (*theorem* 2)). *Let* $S = [0, 1]$, $A_1 = 1, 2, \ldots, k$ and $A_2 = 1, 2, \ldots, l$ *and* $\beta \in (0, 1)$. *Let* $r_k(s, i, j) = a_k(s, i, j) + b_k(s, i, j)$ for $k = 1, 2, \ldots$, *where* a_1 *and* b_1 *are continuous functions in* s. *Let* $q(.|s, i, j) = (1/2)[q_1(.|s, i) + q_2(.|s, j)]$, *where* q_1 *and* q_2 *are probability measures which are strongly continuous in s for each i, j and are also absolutely continuous with respect to the Lebesgue measure. Then the discounted stochastic game has a pair* (f_1, g_1) *of equilibrium stationary strategies and further* $I_1(f_1, g_1)(s)$ *and* $I_2(f_1, g_1)(s)$ *are Borel measurable in* s.

Remark 3. The results in Theorems 4–6 hold good even when S is a separable metric space.

Theorem 7. *Consider a mixture class of discounted non-zero sum stochastic game* $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ *where*

1. S_1 *and* S_2 *are subsets of a complete, separable metric space. Also,* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.
2. S_1 *and/or* S_2 *are uncountable Borel sets, and the reward function is totally bounded in* S_1 *and* S_2.
3. *The subgame restricted to* S_1 *is an ARAT game. Assume that the reward functions are continuous and the transition function is strongly continuous.*
4. S_2 *is an absorbing state space that has the stationary equilibrium property. Let* (f_2, g_2) *be a stationary equilibrium pair for* S_2.

Then the mixture game Γ_β *has a stationary equilibrium pair.*

Proof. Once again, we proceed along the lines of Theorem 3. Define a new game $\Gamma'_\beta = (S \cup \{s^*\}, A_1, A_2, r'_1, r'_2, q', \beta)$ as specified in Theorem 3.

For the ARAT game, we know that $\forall i \in A_1$, $\forall j \in A_2$ and $\forall s \in S_1$,

$$r_k(s, i, j) = a_k(s, i) + b_k(s, j), \; k = 1, 2, \tag{32}$$

$$q(s' \mid s, i, j) = \frac{1}{2}[q_1(s'|s, i) + q_2(s'|s, j)]. \tag{33}$$

For the new game Γ'_β restricted to $S_1 \cup \{s^*\}$, we have for player P_1,

$$r'_k(s, i, j) = r_k(s, i, j) + \beta \int_{S_2} I_1(f_2, g_2)(s')\mathrm{d}q'(s'|s, i, j)$$

$$= a_k(s, i) + b_k(s, j)$$

$$+\beta \int_{S_2} I_1(f_2, g_2)(s')\frac{1}{2}[\mathrm{d}q(s'|s, i) + \mathrm{d}q(s'|s, j)]$$

(using eqs (32) and (33))

$$= a'_k(s, i) + b'_k(s, j),$$

where

$$a'_k(s, i) = a_k(s, i) + \frac{\beta}{2}\int_{S_2} I_1(f_2, g_2)(s')\mathrm{d}q(s'|s, i)$$

and

$$b'_k(s, j) = b_k(s, j) + \frac{\beta}{2}\int_{S_2} I_1(f_2, g_2)(s')\mathrm{d}q(s'|s, j).$$

Also, $q'(s' \mid s, i, j) = q(s' \mid s, i) + q(s' \mid s, j)$, $\forall s' \in S_1$, $s \in S_2$ and $q'(s^* \mid s, i, j) = 1 - q(S_1 \mid s, i, j)$. Here $q(S_1 \mid s, i, j)$ is the probability measure of the set S_1 given $s \in S_2$.

Hence the new game restricted to $S_1 \cup \{s^*\}$ is an ARAT game and is the same as the game restricted to S_1.

For the game Γ, $q(s'' \mid s, i, j) = 1 - q(s' \mid s, i, j)$, $\forall s'' \in S_2$. Let $q \ll p$, where p is any probability distribution on S, and let $E \subseteq S$. Then, $p(E) = 0 \Rightarrow p(E \cap S_1) = 0 \Rightarrow q(E \cap S_1) = 0$.

For the new game, we have $q'(s' \mid s, i, j) = q(s' \mid s, i, j)$, $\forall i$, $\forall j$, $\forall s' \in S_1$. Also, $q(s^* \mid s, i, j) = 1 - q(s' \mid s, i, j)$. So q' is now a probability measure on S.

$$q \ll p \Rightarrow q'(E \cap S_1) = 0 \Rightarrow q' \ll p \text{ restricted to } S_1.$$

From Theorems 4–6, we can find a stationary equilibrium pair (f_1, g_1) for the game restricted to S_1. Since s^* is absorbing, (f_1, g_1) is also a stationary equilibrium pair for the game restricted to $S_1 \cup \{s^*\}$.

Proceeding along the same lines as Theorem 3, we can easily see that the ARAT mixture class of games has a stationary equilibrium pair. □

Mixture class of games where S_1 is SIT

For SIT mixture class of games, we use the following theorem proved by Nowak[22].

Theorem 8. (*Nowak*[22] (*theorem* 1)). *Every non-zero sum discounted stochastic game satisfying the conditions for SIT games as listed in Definition 12 has a stationary Nash equilibrium.* □

Proceeding along the same lines as Theorem 7 and using Nowak's theorem stated above, we can easily prove the following for the SIT mixture class of game.

Theorem 9. *Consider a mixture class of discounted non-zero sum stochastic game* $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ *where:*

1. S_1 *and* S_2 *are subsets of a complete, separable metric space. Also,* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.

2. *S_1 and/or S_2 are uncountable Borel sets, and the reward function is totally bounded in S_1 and S_2.*
3. *The subgame restricted to S_1 is a SIT game as defined by Nowak.*
4. *S_2 is an absorbing state space that has the stationary equilibrium property. Let (f_2, g_2) be a stationary equilibrium pair for S_2.*

Then the mixture game Γ_β has a stationary equilibrium pair.

Mixture of two SER-SIT class of games

We will use the following lemma and proceed along the same lines as Theorem 3 for mixture games where both subsets of state spaces are SER-SIT and the state space is uncountable.

Lemma 2. *Consider a SER-SIT game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where S is a finite, countable or uncountable Borel set and $\int_S \mathrm{d}q(s|i, j) \le 1$. Then there exists a stationary equilibrium pair (f, g) for the game Γ_β that is independent of the initial state of the game.*

Proof. Define a new game Γ'_β by introducing a new absorbing state $\{s^*\}$ as follows.

$$\text{Let } q(s^*|i, j) = 1 - \int_S \mathrm{d}q(s|i, j) \text{ whenever } \int_S \mathrm{d}q(s|i, j) < 1.$$

For this new game with state space $S \cup \{s^*\}$, let us denote by S_1 the states belonging to the original game, and $S_2 = \{s^*\}$, where S_1 is a SER-SIT game. Then by Theorem 3, the new game, and hence the original game, have a stationary equilibrium pair. □

Theorem 10. *Consider a mixture class of discounted non-zero sum stochastic game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, R_1, R_2, q, \beta)$ where:*

1. *S_1 and S_2 are subsets of a complete, separable metric space. Also, $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.*
2. *S_1 and/or S_2 are uncountable Borel sets, and the reward function is totally bounded in S_1 and S_2.*
3. *The subgame restricted to S_1 is a SER-SIT game. That is, $\forall s \in S_1$, $\forall i \in A_1$, $\forall j \in A_2$ and $k = 1, 2$, we have $r_k(s, i, j) = c_k(s) + a_k(i, j)$, where $c_k(s)$ is a measurable function and a_k: $A_1 \times A_2 \to \Re$,*

   $$q(.\,|\,s, i, j) = q(.\,|\,i, j).$$

4. *The subgame restricted to S_2 is also a SER-SIT game. That is, $\forall s \in S_2$, $\forall l \in A_1$, $\forall m \in A_2$ and $k = 1, 2$, we have $R_k(s, l, m) = d_k(s) + b_k(l, m)$, where $d_k(s)$ is a measurable function and b_k: $A_1 \times A_2 \to \Re$,*

   $$q(.\,|\,s, l, m) = q(.\,|\,l, m).$$

Then the mixture game Γ_β has a stationary equilibrium pair.

Proof. By Lemma 2, the subgame restricted to S_1 has a stationary equilibrium pair (say (f_1, g_1)), and the subgame restricted to S_2 has a stationary equilibrium pair (say (f_2, g_2)).

Define a new game $\Gamma'_\beta = (S_1 \cup \{s^*\}, A_1, A_2, r'_1, r'_2, q, \beta)$, where $\forall s \in S_1$, $\forall i \in A_1$, $\forall j \in A_2$,

$$r'_k(s, i, j) = r_k(s, i, j) + \beta \int_{S_2} I_k(f_2, g_2)(.)\mathrm{d}q(.|i, j), \; k = 1, 2,$$

and s^* is an absorbing state. This new game is a SER-SIT game and hence has a stationary equilibrium pair (f'_1, g'_1) independent of the initial state.

Similarly, we can define a new game $\Gamma''_\beta = (S_2 \cup \{s^{**}\}, A_1, A_2, R'_1, R'_2, q, \beta)$ where $\forall s \in S_2$, $\forall l \in A_1$, $\forall m \in A_2$,

$$R'_k(s, l, m) = R_k(s, l, m)$$
$$+\beta \int_{S_1} I_k(f_1, g_1)(.)\mathrm{d}q(.|l, m), \; k = 1, 2,$$

and s^{**} is an absorbing state. This new game is a SER-SIT game and hence has a stationary equilibrium pair (f'_2, g'_2) independent of the initial state.

Let us define a strategy pair (f^*, g^*) for the combined game as follows

$$f^*(s) = \begin{cases} f'_1(s), & s \in S_1 \\ f'_2(s), & s \in S_2 \end{cases} \quad \text{for player } P_1, \tag{34}$$

$$g^*(s) = \begin{cases} g'_1(s), & s \in S_1 \\ g'_2(s), & s \in S_2 \end{cases} \quad \text{for player } P_2. \tag{35}$$

Then, proceeding along the same lines as Theorem 3, the combined game has a stationary equilibrium pair. □

Mixture of more than two classes of games

We now extend Theorems 1, 3, 7, 9 and 10 to a mixture of more than two classes. The proof involves inductively applying the respective theorem.

Theorem 11. *Consider a mixture class of discounted non-zero sum stochastic game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where:*

1. *$S_1, S_2, \ldots, S_m$ are subsets of a complete, separable metric space. Also, $S = S_1 \cup S_2 \cup \cdots \cup S_m$ and $S_{m_1} \cap S_{m_2} = \phi$, for $m_1 \ne m_2$.*
2. *$S_1, S_2, \ldots, S_m$ are cycle-free. That is, for all m_1, m_2 $(1 \le m_1 < m_2 \le m)$,*

$q(s_{m_1} | s_{m_2}, i, j) = 0, \forall i \in A_1,$

$j \in A_2, s_{m_1} \in S_{m_1}, s_{m_2} \in S_{m_2}.$

3. *At least one of $S_1, \ldots, S_m$ is an uncountable Borel set, and the reward function is totally bounded in S.*
4. *For each l where $1 \leq l \leq m - 1$, S_l is either finite or countable, SER-SIT, ARAT or SIT mixture. S_m is an absorbing state that has the stationary equilibrium property.*

Then the mixture game Γ_β has a stationary equilibrium pair.

Conclusion and future work

We extend known classes of stochastic games with stationary equilibrium property to uncountable state space games with mixtures of some of these classes. We consider the two-player case where the set of states S can be partitioned into cycle-free subsets $S_1, S_2, \ldots, S_m$, where the non-sink subsets are one of the following types of games: finite or countable, SER-SIT, ARAT or SIT. This can also be extended to other classes and also to correlated equilibrium[48,49]. Though we consider only two-player games, the above theorems can be extended to n-player games also with suitable assumptions. It is not known if all discounted stochastic games with uncountable state space have a stationary equilibrium.

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