

Einstein is Newton with space curved

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The two defining features of the Einsteinian gravity are that it is self-interactive and it links universally to all particles, including zero-mass particles. In the process of obtaining the Schwarzschild solution for gravitational field of a mass point, we demonstrate how these features are incorporated? We also show, unlike the Newtonian gravity, why gravitational potential in the Schwarzschild solution can have its zero only at infinity and nowhere else, i.e. it is determined absolutely. Further we consider particle orbits to expose certain insightful and subtle points of concept and principle.

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THE Newtonian gravity links to all massive particles and they attract each other by the inverse square law force. However, massless particle or light remains unaffected. The Einsteinian gravity results from universalization of the Newtonian gravity. That is, to include massless particles as well in gravitational interaction. This requirement uniquely asks for gravity to be described by spacetime curvature¹. Of course, universalization also means that energy distribution in any form, including energy of the gravitational field itself must also participate in gravitational interaction. These are the two properties that drive us from Newton to Einstein. As a matter of fact, incorporation of these two features naturally derives without solving gravitational field equation the Schwarzschild solution describing gravitational field of a mass point. It is therefore pertinent to see how these features are actually included in Einstein's theory of gravitation – general relativity (GR)?

The usual derivation of the Schwarzschild solution describing gravitational field of a mass point in textbooks does not bring out explicitly these subtle and important conceptual aspects. Our main aim in this pedagogical discussion is to demonstrate how these features are elegantly incorporated in GR. In the next section, we shall first discuss a simple derivation of the Schwarzschild solution by demanding that the time-like radial geodesic should incorporate the Newtonian gravitational law, whereas photon should experience no acceleration (it is equivalent to choosing radial coordinate as the affine parameter for photon trajectory²) as it cannot be at rest in any instantaneous frame to feel the $-\nabla\Phi$ pull. It is interesting that these two simple considerations determine the Schwarzschild solution exactly. Then we solve the vacuum equation and in the process we expose where and how gravitational interaction of massless particles and self-

interaction are actually incorporated. We also consider particle orbits again to illuminate some subtle and insightful points. We end up with a discussion.

The Schwarzschild field

For gravitational field of a mass point, we begin with the usual spherically symmetric metric

$$ds^2 = Adt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A and B are functions of r .

Geodesics

If GR has to include the Newtonian gravity, the time-like radial geodesic should reduce to $\ddot{r} = -\Phi'$, where dot and prime denote respectively, the derivative relative to proper time s and r . Second, photon experiences no acceleration, $d^2r/d\lambda^2 = 0$, where λ is an affine parameter along the null geodesic. Since the metric is free of t , we immediately write

$$Ai = E, \quad (2)$$

where $i = dt/ds$ for the time-like particle and $\dot{t} = dt/d\lambda$ for the photon. Substituting this in the metric, we readily get for the photon

$$\ddot{r} = \frac{1}{2} \left(\frac{E^2}{AB} \right)'. \quad (3)$$

Now it should experience no acceleration $\ddot{r} = 0$, which means $AB = k = \text{const}$. On the other hand, for the radially falling time-like particle, we similarly write

$$\dot{r}^2 = \frac{E^2}{AB} - \frac{1}{B} = \frac{E^2 - A}{k}. \quad (4)$$

Now constant k can be absorbed by redefining proper time as ds/\sqrt{k} , and hence $AB = 1$ without any loss of generality. Differentiating the above equation we get

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$$\ddot{r} = -\frac{A'}{2}. \tag{5}$$

For inclusion of the Newtonian law, $A = b + 2\Phi$, where $\Phi = -GM/r$ is the Newtonian potential and b is a constant. However spacetime should be asymptotically flat, which determines $b = 1$ and so we have $A = 1/B = 1 - 2GM/r$. This is exactly the Schwarzschild solution obtained by solving nonlinear vacuum equation, $R_{ab} = 0$. This is the simplest derivation of the solution which is purely driven by the physically reasonable guiding considerations of inclusion of the Newtonian law and photon experiencing no acceleration. Note that it is photon motion that requires space to be curved ($B \neq 1$), while the Newtonian law is incorporated entirely by $A = 1 + 2\Phi$ even when space is flat with $B = 1$. It is a reflection of the fact that photon or light can feel gravity only through curvature of space. That is where it freely propagates. It is therefore clear that Einstein is Newton with space curved.

Solving the equation

Now we will explicitly solve the Einstein vacuum equation to demonstrate how the Newtonian potential is determined absolutely as it can vanish only at infinity and nowhere else. We shall now solve the vacuum equation

$$R_{ab} = 0. \tag{6}$$

There are three independent components of the Ricci curvature and two of which read as

$$R_t^t = \frac{1}{2AB} \left[A'' - \frac{A'}{2} \left(\frac{A'}{A} + \frac{B'}{B} \right) + \frac{2A'}{r} \right], \tag{7}$$

$$R_r^r = R_t^t + \frac{1}{rB} \left(\frac{A'}{A} + \frac{B'}{B} \right). \tag{8}$$

Clearly $R_t^t = R_r^r$ implies^{1,2} $AB = \text{const} = k$. Note that it is the same condition which followed from the photon experiencing no acceleration. This is what is known as the null energy condition given by $R_{ab}k^ak^b = 0$, $k_ak^a = 0$. Then writing $A = 1 + 2\Phi$, $R_t^t = 0$ reduces to the familiar Laplace equation³

$$\nabla^2\Phi = 0, \tag{9}$$

which integrates to give the familiar solution

$$\Phi = k_1 - M/r. \tag{10}$$

Now the remaining equation $R_\theta^\theta = 0$ takes the form

$$(r/B)' = 1, \tag{11}$$

which on integration gives

$$1/B = 1 - 2M/r = A/k = (1 + 2\Phi)/k. \tag{12}$$

First, k can be absorbed⁴ by redefining t and then consistency with eq. (10) leads to $k_1 = 0$ and hence $\Phi = -M/r$.

That is, potential is determined absolutely and it can vanish only at infinity and nowhere else. This is in contrast to all classical physics where the potential is determined only up to the addition of a constant. It is important to note that this freedom does not exist for GR. We have thus obtained the Schwarzschild solution by solving the vacuum equation $R_{ab} = 0$.

It however raises a couple of interesting questions. First and foremost, where has the defining property of GR, self-interaction of gravity gone as potential is still determined by the Laplace equation. Second, how is it that potential is determined absolutely, vanishing at infinity and nowhere else? This is what we take up in the next section.

Self-interaction and zero of potential

The new features that Einsteinian gravity brings in are essentially two, self-interaction and gravity linking to the photons. Would it not be the most elegant that the former facilitates the latter leaving the Newtonian inverse square law intact? This is precisely what happens. That is, gravitational effect of self-interaction is such that it makes the photon feel gravity. For the photon to feel gravity space has to be curved, such that it does not have to change its velocity¹. This means gravitational field energy or self-interaction gravitates by curving space and not by making any contribution to Φ . That is why the Laplace equation giving Newton's inverse square law remains undisturbed. This is how self-interaction is incorporated through curvature of space, while potential is still given by the good old Laplace equation.

The condition for photon to feel no acceleration like ordinary time-like particles is $A'/A + B'/B = 0$. It is this condition that reduces $R_t^t = 0$ to the Laplace equation. If space were flat, which means $B = 1$, then it would have taken the form

$$\nabla^2\Phi \approx \Phi'^2, \tag{13}$$

indicating self-interaction contribution as energy density of gravitational field going as square of field strength. What really happens is the most remarkable. Self-interaction contribution goes into curving space by making $B \neq 1$ (however, $B = 1/A$ by the photon experiencing no acceleration) and then makes the photon feel gravity as it simply follows geometry of curved space¹. Inclusion of self-interaction as above would have been relative to flat spacetime, which could in no way have incorporated the photon feeling gravity. Therefore, the only way out is gravitational field energy should only gravitate by curving space and leaving the Laplace equation undisturbed.

The important point to realize is that gravitational field energy gravitates in a subtler way than matter density (which produces $\nabla\Phi$) by curving space and not by producing acceleration. This is how it should be because gravitational field energy is not the primary source of gravity like matter density. It is produced by matter fields

and has no independent existence of its own. It is thus a secondary source and hence it should not do what matter does. It should therefore not sit alongside energy-momentum distribution on the right in the Einstein equation. This is an important point of principle that a secondary source should therefore always gravitate more subtly than a primary source by enlarging framework like gravitational field energy curving space. We shall invoke this principle later in the context of how vacuum energy should gravitate. Thus gravitational field gravitates by curving space without making any contribution to disturb the inverse square law. This is how self-interaction is incorporated in GR by enlarging the spacetime framework and not by modifying the gravitational law⁵.

The next question is why is potential determined absolutely, it can vanish only at infinity and nowhere else. In the Newtonian gravity, potential is determined only up to addition of a constant which can be chosen arbitrarily. In contrast, as we have seen above, the equation corresponding to $R_{\theta}^{\theta} = 0$ determines this constant to be zero leaving no option for choosing zero of potential. That is, constant potential attains non-trivial physical meaning here as it produces Ricci curvature $R_{\theta}^{\theta} = -2k/r^2$. This is strange because in all classical physics constant potential is dynamically trivial and has no physical significance. Let us then ask what is it that is different for the Einsteinian gravity? It is universal and hence it makes an unusual demand on spacetime that it has to curve to describe its dynamics. No other force makes such a demand on spacetime. For the rest of physics, spacetime background is fixed and it is not affected by physical phenomena happening in it. In contrast, the Einsteinian gravitational dynamics can only be described by spacetime curvature and hence it cannot remain inert and fixed as for rest of physics.

As we have seen above, it is gravitational self-interaction that curves space and it is given by $B \neq 1$. That means even if B is constant so long as it is different from 1, space will remain curved. Since gravitational field is spread all over space, hence its gravitational effect should also be spread all over space. When potential is constant, the Newtonian gravity is completely annulled, while this is not the case for the Einsteinian gravity. This is because constant potential does not make space at $B = 1$, and for that potential has to vanish. That is, for Einsteinian gravity to completely switch-off what is required is that $A = \text{const}$ and $B = 1$, which can happen only at infinity where $\Phi = 0$. This is why potential in this situation is completely determined in contrast to classical physics. Thus potential in the standard Schwarzschild coordinates gets determined absolutely.

Particle orbits and self-interaction

As we saw in an earlier section, space curvature has no effect on radial motion as eq. (5) entirely agrees with the Newtonian law, except for the derivative here being w.r.t.

proper time. It easily integrates to give the finite proper time of fall from radius r_0 to $r = 0$ as $\sqrt{2r_0^3/9M}$, where we have set $G = 1$. This is because the inverse square law remains intact and the space curvature does not affect radial motion. It would, however, make contribution for non-radial motion.

Since the field is radially symmetric, there is no loss of generality in setting $\theta = \pi/2$ and like energy, there is also conservation of angular momentum

$$r^2 \dot{\phi} = l. \tag{14}$$

Substituting the two constants of motion in the metric, we write the standard expression

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right) \left(\frac{l^2}{r^2} + \mu^2\right), \tag{15}$$

where $\mu^2 = 1, 0$ refers respectively, to time-like and null particle. The effective potential is then defined as

$$V^2 = \mu^2 - \frac{2M}{r} \left(\frac{l^2}{r^2} + \mu^2\right) + \frac{l^2}{r^2}. \tag{16}$$

Note that the gravitational potential, which also represents transverse spatial curvature, $R_{\phi\theta\phi}^{\theta} = 2M/r$, links to transverse kinetic energy. This linkage is what brings about the effect of space curvature in particle orbits causing perihelion shift as well as light bending. For photon $\mu^2 = 0$, let us note that $V^2 = -2Ml^2/r^3 + l^2/r^2$, which clearly shows that photon feels no Φ , but instead feels only transverse spatial curvature. It may be noted that it is space bending, which we measure by means of light, because light truthfully follows curved (bent) space. Thus, light bending is actually space bending.

By differentiating the above equation, we write the condition for circular orbit as

$$\frac{M}{r^2} \mu^2 + \frac{3Ml^2}{r^4} - \frac{l^2}{r^3} = \frac{M}{r^2} \left(\mu^2 + \frac{3l^2}{r^2}\right) - \frac{l^2}{r^3} = 0. \tag{17}$$

Here the first and last terms are the familiar inverse square attraction and the centrifugal repulsion, while the middle term is due to coupling of space curvature with transverse motion. For photon, $\mu^2 = 0$, we have $(3M - r)l^2/r^4 = 0$, giving radius of photon circular orbit, $r = 3M$. That is why there cannot exist any circular orbit below this radius. Clearly, no particle can have circular orbit below the photon orbit. On the other hand, by writing $M\mu^2/r^2 - (3M/r - 1)l^2/r^3$, it has also been argued⁷ that centrifugal force changes sign at $r = 3M$. In this interpretation, for the photon there is only centrifugal force, $(3M/r - 1)l^2/r^3$, which changes from repulsion to attraction at the photon circular orbit. It all has come about due to space curvature which is produced by gravitational interaction of gravitational field energy. It is therefore gravitational in character rather than kinematic, and the

photon feels gravity through space curvature. It should not therefore be clubbed with centrifugal force.

We can in the standard way write the orbit equation for time-like particle

$$u'' + u = \frac{M\mu^2}{l^2} + 3Mu^2, \quad (18)$$

which for the photon reduces to

$$u'' + u = 3Mu^2, \quad (19)$$

where $u = 1/r$, $u' = du/d\phi$. Note that it is $3Mu^2$, which is the non-Newtonian contribution due to self-interaction and it manifests in curving the space. It is clear that photons only feel space curvature. For time-like particles like planets, the orbit would be elliptical in the first approximation because the gravitational attraction law is the same as the inverse square law. Since the force law is not changed, the nature of the orbit has essentially to remain undisturbed. It could then accommodate the effect of space curvature by suffering precession of perihelion. Why perihelion? It is because that is where the force is strongest. Thus self-interaction through space curvature makes perihelion of the orbit precess. The orbits in the Einsteinian gravity are therefore precessing ellipses. Further, note that gravitational field energy which is negative for positive mass curves space in such a way that it is in consonance with the attraction due to mass. It has been argued elsewhere⁸ that positive energy condition for gravitational field energy is negative. It defines the norm of positivity for non-localizable energy distribution. For example, the electric field energy of a charged source is positive, which is opposite of the norm set by negative gravitational field energy. It is therefore gravitationally 'negative' and that is why it contributes a repulsive effect opposing attraction due to mass for the field of a charged black hole. Let us consider the potential at some r due to a charged particle of mass M and charge Q which would be given by

$$\Phi = -\frac{M - Q^2/2r}{r}. \quad (20)$$

This is because the electric field energy, $Q^2/2r$, lying outside the radius r does not contribute and hence has to be subtracted out. It would give rise to the acceleration $-M/r^2 + Q^2/r^3$, which shows the repulsive effect of the electric field energy⁹. Since electric field energy is positive, it is therefore 'gravitationally negative' and hence repulsive.

Discussion

The main aim of this article is to bring out transparently inclusion of gravitational self-interaction and its role in particle orbits, and why potential in the Schwarzschild solution cannot vanish anywhere but at infinity. This is

interesting and insightful for appreciating the remarkable features of the Einsteinian gravity over the Newtonian gravity. Note that ultimately the equation we need to solve is the first-order linear differential equation, which is the first integral of the Laplace equation. It is this that determines the potential absolutely. The Newtonian gravity is included through $A = 1+2\Phi$, which gives $\nabla\Phi$. It can be squared out by redefining t when $\Phi = k = \text{const}$, hence $A = \text{const}$ is innocuous. However, $\Phi = \text{const}$ in B has non-trivial effect because it refers to curvature of space which is sourced by self interaction and it does not vanish when $B = \text{const}$. It may be noted that the constant potential generates the following stresses

$$T_t^t = T_r^r = \frac{k}{r^2}, \quad T_\theta^\theta = T_\phi^\phi = 0, \quad (21)$$

and they asymptotically agree with that of a global monopole^{10,11}. It is remarkable that constant potential therefore describes a global monopole gravitationally.

The important point to be noted is that gravitational field energy – self-interaction gravitates more subtly than matter energy. It is the latter that determines the inverse square law, while the former does not disturb the law at all; instead it curves space. That is, gravitational field energy gravitates by enlarging the framework – curving space and not by contributing a source on the right of the Laplace equation. This is because it is a secondary source created by matter and therefore has no independent existence of its own. Hence it cannot sit alongside energy density in the equation; yet gravitate it must because energy in any form must gravitate. It then does that by curving space, which is the only available avenue. From this emanates a general principle that any other secondary source created by matter fields must also do the same. That is, it must gravitate only by enlarging framework rather than contributing a stress tensor in the equation. For example, vacuum energy created by quantum fluctuations of vacuum due to matter fields is a secondary source on the same footing as gravitational field energy. It must not therefore gravitate via a stress tensor, though it is possible to write a stress tensor for it relative to flat spacetime and it has the same form as the cosmological constant Λg_{ab} . It is a matter of principle that it cannot gravitate through a stress tensor⁶ and has therefore nothing to do with the cosmological constant, which is now free to have the value required to explain the observed acceleration of expansion of the Universe¹³.

It is well known that energy in GR is an ill-defined and ambiguous concept and it has no covariant expression. Beginning with Einstein himself, several authors have attempted to define it and there exist as many, if not more expressions for it, and they all do not agree^{14,15}. The reason for ambiguity for its measure is precisely due to its character, that it resides in curvature of space. The question we are addressing here is not of its measure, instead on how it gravitates? To the best of my knowledge, this

question was for the first time asked by the present author⁶ for liberating the cosmological constant from the Planck length. Here we have given an explicit exposition of how gravitational field energy gravitates³.

We have derived here the static black hole solution simply by appealing to general physical considerations that the Newtonian gravity be included and the photon experiences no acceleration. The question arises, could this method work for obtaining Kerr solution for a rotating black hole or for a black hole with a charge or a black hole in cosmological de Sitter background? The answer is yes. Following the first appearance of this article¹², we have similarly obtained the Kerr solution¹⁸. The trick is in choosing the appropriate metric having ellipsoidal symmetry for accommodating rotation and the rest followed in the same manner as discussed earlier. There are two considerations; one inclusion of the Newtonian law and the other the photon feeling no acceleration. The latter would always be true so long as the photon can freely propagate. That should happen in any spacetime free of all ponderable matter (the only non-ponderable energy distribution is the electric field energy for a charged black hole). This means, as mentioned earlier in the text, even for a charged black hole we would have $AB = 1$. Now $A = 1 + 2\Phi$ has to be determined where electric field energy also contributes to gravitational potential, which is in fact given by eq. (20). Thus follows the metric for the Reissner–Nordstrom solution for a charged black hole. Further, we have elsewhere argued⁶ that spacetime free of all forces is maximally symmetric spacetime of constant curvature and not necessarily of zero curvature – flat. This is how the cosmological constant naturally arises as a constant of spacetime structure. It does not represent an energy distribution, instead it defines the geometry of spacetime free of all matter and forces. Again $AB = 1$, and for potential we should include its contribution and it should go as $M/r + \Lambda r^2$. Thus our physically insightful method works for obtaining black hole solutions in general.

The Einsteinian gravity is essentially driven from the Newtonian gravity by the two new properties of self-interaction and the photon feeling gravity without experiencing acceleration for radial motion. The former contributes through curving space which also facilitates interaction of the photon with gravity. It is remarkable that these two properties take good care of each other, leaving essentially the Newtonian gravity intact. This is indeed the most elegant and satisfying feature of the Einsteinian gravity. The standard derivation and discussion of the Schwarzschild solution do not expose these interesting aspects of the Einsteinian gravity in such a trans-

parent and explicit manner. That is precisely what we had set out to do.

We can therefore rightly say that Einstein is Newton with space curved. Subsequent to its first appearance on the arxiv¹², this view has been further strengthened by the recent calculations of the Mercury perihelion advance and light bending in Newtonian gravity with space curved^{19,20}.

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