Non-relativistic fluids

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We present the dynamics of a most generic uncharged dissipative parity, even Galilean fluid, to the first derivative order. The construction is embedded in a symmetry broken phase of one higher dimensional relativistic system, namely the null fluid. Both the null fluid and the Galilean fluid have identical symmetries, thermodynamics and constitutive relations to all order in derivative expansion. Finally, we present the number of transport coefficients for most generic charged Galilean fluid and Galilean Superfluid.

Keywords: Galilean fluids, Galilean superfluid, null fluids.

A generic fluid system

Fluid is a low energy effective description of any field theory and is characterized by low energy or long wavelength fluctuations of fields around thermodynamic equilibrium. Dynamics of fluid is given by conservation of energy-momentum tensor and currents associated with additional symmetries. A fluid configuration is given by fluid variables typically chosen to be normalized velocity, temperature and chemical potentials (conjugate to conserved charges). The length scales over which the fluid variables vary are large compared to the mean path of the system. Thus, fluid energy-momentum tensor and currents admit an expansion, in terms of derivatives of fluid variables, known as constitutive relations. The coefficients appearing in the expansion of energy-momentum tensor are the transport coefficients. These coefficients are properties of individual fluids that depend on their microscopic constitution and motion. Finally, the fluid flow has to obey the physical constraint: the second law of thermodynamics, which states that there should be a local entropy current associated the flow, whose divergence should always be positive definite along the flow. This determines the most generic physical transport of a fluid. For relativistic and non-relativistic fluid dynamics, we refer to previous papers[1,2] and all references therein.

Galilean fluids

This article aims at studying non-relativistic or more generically Galilean fluids and write down complete set of constitutive equations for it. One usual method to get the non-relativistic fluid is to take $c \to \infty$ limit of a generic relativistic fluid[3,4]. There are multiple ways to take this limit, hence, the final system one ends up with is not unique. Here we take a more axiomatic approach[5,6], we construct the most generic fluid whose flow is consistent with Galilean symmetry. Certainly, the fluid flow has to respect the second law of thermodynamics. A $(d+1)$ dimensional Galilean fluid is described by two thermodynamic variables and a spatial velocity vector and these variables characterize the Galilean densities and currents $\{\rho, \rho^i, \varepsilon, \varepsilon^i; t^{ij}\}$ where $(i = 1, 2, \ldots, d)$. The currents and densities are functions of fluid variables and can be written in derivative expansions. Galilean thermodynamics is given by

$$\varepsilon + P = \mu_m \rho + \mathrm{TS}, \quad d\varepsilon = TdS + \mu_m d\rho, \tag{1}$$

where ε is energy density, ρ is mass density, T is temperature and μ_m is mass chemical potential. The other two thermodynamic quantities, pressure P and entropy S are considered as functions of (T, μ_m). The currents satisfy following Galilean ward identities (conservation laws)

$$\partial_t \varepsilon + \partial_i \varepsilon^i = 0, \quad \partial_t \rho + \partial_i \rho^i = 0, \quad \partial_t \rho^j + \partial_i t^{ij} = 0, \tag{2}$$

where the first equation is the energy conservation equation, the second one is the continuity equation and the third one is the Euler equation. We will see below, how we can get back these relations from our construction.

Before we go ahead to present our construction, let us recall why is the study of non-relativistic or Galilean fluid interesting. Non-relativistic system can be thought of as an effective low energy description of an underlying relativistic theory. They are expected to be realized in the low energy physics experiments. In the context of superfluids, for low-energy systems such as liquid helium and ultra-cold atomic gases, a Galilean framework is more accurate. Hence, if we have a Galilean fluid description, it would be easier to understand the properties of realistic systems. In the next section, we present the construction of the most generic first order (in derivative expansion) dissipative parity even Galilean fluid.

Null reduction

Null reduction is a technique to obtain a Galilean system from a given Poincaré system. It is based on a simple fact

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that there is a Galilean subalgebra inside Poincaré algebra. Let us begin with a relativistic system in $(d + 2)$ dimensions defined on a manifold M with coordinates $(x^M, M = 0, 1, \ldots, d + 1)$. The system respects Poincaré symmetry generated by generators P_M and M_{MN}. Next we define null coordinates: $(x^{\pm} = x^0 \pm x^{d+1}, x^i; i = 1, 2, \ldots, d)$. The generators of the Poincaré algebra that commute with the null momentum: $\{P_-, P_+, P_i, M_{ij}, M_{i+}\}$ generate the Galilean algebra in $(d + 1)$ dimensions, with P_- as the Casimir. Thus, when a Poincaré system is evolved at constant x^-, the subsystem has Galilean invariance. Given a relativistic system in $(d + 2)$, reducing it over a null direction x^-, we get a Galilean system in $(d + 1)$ dimensions with $x^+ \equiv t$ as time co-ordinate and x^i as spatial coordinates. We want to construct a fluid compatible with Galilean symmetry (via null reduction). Such a fluid is termed Galilean fluid. A point to be noted is that certain terms compatible with symmetry, might be suppressed in the non-relativistic $c \to \infty$ limit. Hence, the Galilean fluid that we construct shall certainly include all non-relativistic fluids that one may obtain via taking the limit.

It has been shown that performing null reduction of a relativistic fluid[7] does not give most generic non-relativistic fluid, even the thermodynamics of the latter is highly restricted[8,9]. It is not so striking in hindsight as null reduction takes care of symmetry but need not yield the required conserved charges. So, a different relativistic system is needed to get the generic Galilean fluid through reduction, one which will retain the right number of conserved charges (and corresponding chemical potentials) after reduction.

Such a system has been constructed[9,10] and we name it as 'null fluid'. The final proposal is: null fluid is an embedding of a Galilean fluid into a spacetime of one higher dimension. It is a nicer covariant boost-invariant language for Galilean fluids. The mapping between the two is trivial and works exactly to all orders in derivative expansion. They have identical symmetries, thermodynamics, constitutive relations and partition functions.

The construction follows naturally from covariant language of null reduction. First we define a null background as a $(d + 2)$ spacetime manifold that admits metric G_{MN} and a covariantly constant null killing vector V^M. A null theory is a field theory on the null background for fields that are conserved along V^M. Null theories are demanded to be invariant under V-preserving diffeomorphisms and satisfy the corresponding ward identities as field equations of motion. A null fluid is a null theory consistently defined on this background with appropriately chosen field variables. Null reduction of null fluids along V gives the Galilean fluid. One immediate consequence is that the isometry direction V is a background field and has to be taken into account while writing the constitutive relations of the parent null fluid. This introduces effects not captured by reduction of a relativistic fluid. In fact, as mentioned in the proposal, examining the conserved charges, field content, Banerjee[10] demonstrated that null fluids are in exact correspondence with Galilean fluids.

This article will go through the formalism developed earlier[9,10] and use it to obtain uncharged Galilean fluid dynamics to the first derivative order. The construction goes as follows: (i) first we construct a special relativistic system, namely the null fluid and (ii) upon null reduction of null fluid, we get the Galilean fluid of our interest.

Uncharged null fluids

We first construct the dynamics of a uncharged $(d + 2)$ dimensional null fluid on a null background specified by (G_{MN}, V^M). The dynamics is given by $(d + 2)$ conservation of energy–momentum tensor equations: $\nabla_M T^{MN} = 0$. To solve these equations, we need to express T^{MN} in terms of $(d + 2)$ number of variables. A natural choice for these variables is a null velocity u normalized as $u.u = 0$ and $u.V = -1$ and two scalar thermodynamic variables: temperature T and mass potential μ_m.

The most generic constitutive relation in terms of the fluid and background data can be written as

$$T^{MN} = \mathcal{R}u^M u^N + 2\mathcal{E}u^{(M}V^{N)} + \mathcal{P}G^{MN} + 2E^{(M}V^{N)} + 2R^{(M}u^{N)} + \Pi^{MN}. \tag{3}$$

Here, $\mathcal{R}$, $\mathcal{E}$, $\mathcal{P}$ are functions of (T, μ_m). Other quantities E^M, R^M, Π^{MN} contain the higher order corrections in fluid variables and are projected orthogonal to V and u. Terms proportional to $V^M V^N$ in T^{MN} leave the conservation equation invariant and hence are not included. Thermodynamics is obtained by studying a fluid at equilibrium, a fluid configuration independent of time. Equilibrium is characterized by existence of a time-like killing vector, say K. The demand is that there should be a scalar equilibrium partition function W (or free energy), constructed only out of the background data and that dictates dynamics of the system[2]

$$T^{MN} = \frac{2}{\sqrt{G}} \frac{\delta W}{\delta G_{MN}}.$$

In the spirit of fluid dynamics, W admits derivative expansion in the background data. Hence, essentially, the equilibrium fluid configuration is fixed by background to all order in derivative expansion. We will skim through equilibrium analysis without getting into the details that can be found in Banerjee *et al.*[10]. Let us define scalars, $T_0 = 1/K.V$ and $\mu_{m_0} = K.K/2K.V$ as the equilibrium temperature and mass-chemical potential. These are the only two possible background scalars at ideal order (i.e. zero derivative order). Hence, at this order the scalar W can be written as

$$W = \int \mathrm{d}x^d \sqrt{g_d}\, \frac{1}{T_0} P(T_0, \mu_{m_0}), \;\; \mathrm{d}P = S\mathrm{d}T + R d\mu_m.$$

The integration is over spatial section and we have expanded the differential of the function P in terms of two other scalars S and R. By varying this free energy functional, we get the following equilibrium stress tensor

$$T^{MN}_{(0)eq} = R\bar{V}^M\bar{V}^N + 2E\bar{V}^{(M}V^{N)} + PG^{MN}.$$

Here, $\bar{V}^M = (TK^M + \mu_m V^M)$. We also get two thermodynamic relations

$$E + P = R\mu_m + ST, \quad \mathrm{d}E = T\mathrm{d}S + \mu_m \mathrm{d}R.$$

The above relations can be viewed as the first law of thermodynamics and Euler's relation obeyed by the null fluid if we identify P as its pressure, S as entropy density and R as its mass density. It is clear that the thermodynamics of the $(d + 2)$ dimensional null fluid is identical to that of a $(d + 1)$ dimensional Galilean fluid given in equation (1). It is worthwhile to mention that one gets similar relations by demanding that the null fluid carries a local entropy current

$$S^M = \frac{P}{T}u^M - \frac{1}{T}T^{MN}u_N + \left(\frac{\mu_m}{T}\right)T^{MN}V_N + \gamma^N.$$

(γ^N contains generic higher order corrections) that satisfies the second law of thermodynamics, i.e. $\nabla_M S^M \geq 0$ for a physical flow and the equality holds for equilibrium flow. At ideal order, thermodynamics gives $S^M = Su^M$.

Null fluid at first order

Next, we briefly sketch the construction of null fluid to first derivative order. Here, we only present the entropy current analysis. The stress tensor of eq. (3) and entropy current of eq. (3) picks following corrections

$$E^M = \kappa_e P^{MN}\nabla_N T + \kappa_v P^{MN}\nabla_N\left(\frac{\mu_m}{T}\right),$$

$$\Pi^{MN} = -\zeta\Theta P^{MN} - \eta\sigma^{MN},$$

$$\gamma^M = \lambda_e P^{MN}\nabla_N T + \lambda_v{}^{MN}\nabla_N\left(\frac{\mu_m}{T}\right),$$

where $P^{MN} = G^{MN} + V^M u^N + u^M V^N$ is the projector into the space orthogonal to V and u and

$$\Theta = \nabla_M u^M, \; \sigma^{MN} = P^{MR}P^{NS}\left(\nabla_R u_S + \nabla_S u_R - \frac{2}{d}\Theta\right).$$

The divergence of the entropy current at this order takes the following form

$$T\nabla_M S^M_{(1)} = \eta\sigma^2 + \zeta\Theta^2 - \frac{\kappa_e}{T}P^{MN}(\nabla_N T)\nabla_M T$$
$$-\frac{\kappa_v}{T}P^{MN}(\nabla_M T)\nabla_N\left(\frac{\mu_m}{T}\right) + \lambda_e P^{MN}\nabla_M\nabla_N T$$
$$+\lambda_v P^{MN}\nabla_M\nabla_N\left(\frac{\mu_m}{T}\right) + (\nabla_M\lambda_e)P^{MN}\nabla_N T$$
$$+ (\nabla_M\lambda_v)P^{MN}\nabla_N\left(\frac{\mu_m}{T}\right).$$

Imposing non-negativity of the above expressions yields $\lambda_e = \lambda_v = \kappa_v = 0$; $\eta, \zeta \geq 0$; $\kappa_e \leq 0$. Thus the uncharged null fluid is characterized by three dissipative transport coefficients at first derivative order. The inequality constraints are peculiar to Entropy law analysis; the equality constraints are obtained by equilibrium analysis as well. In the next section, we show how we get the Galilean fluid from this system.

Galilean fluids through null reduction

To get the Galilean fluid, we have to reduce the null fluid along the null direction V of the manifold. But as V is null and hence transverse to itself, to decompose the manifold properly, we need to specify another vector T: $\mathcal{M}_{d+2} = S_V \times T \times \mathcal{M}_d$. The split is now well defined as $\mathcal{M}_d$ has vectors orthogonal to both V and T. The choice of T is arbitrary, hence the theory admits freedom under redefinition of T. Next, we define A null vector $\tilde{V}$ is defined using T and V: $\tilde{V}^M = (TT^M + \mu_m V^M)$. The energy–momentum tensor can be decomposed as

$$T^{MN} = \rho\tilde{V}^{(M}\tilde{V}^{N)} + 2\varepsilon\tilde{V}^{(M}V^{N)} + 2j_\varepsilon{}^{(M}V^{N)}$$
$$+2j_\rho{}^{(M}\tilde{V}^{N)} + t^{MN}. \quad (4)$$

Comparing with its null fluid counterpart we see that the null velocity u is mapped to $\tilde{V}$ and R, E are mapped to ρ, ε. The mappings for j^M_ε, j^M_ρ, t^{MN} are also one to one and details can be found in Banerjee *et al.*[10].

Galilean equations of motion

To get the Galilean ward identities in their usual form, we make a specific coordinate choice, $x^M = X^-$, t, x^i: $V = \partial_-$, $T = \partial_t$ and the background metric is

$$\mathrm{d}s^2 = -2e^{-\phi}(\mathrm{d}t + a_i\mathrm{d}x^i)(\mathrm{d}x_- - B_t\mathrm{d}t - B_i\mathrm{d}x^i) + g_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

where, the fields ϕ, a_i, B_i, B_t, g_{ij} depend on $(t, x^i\}$.

This is like a fluid rest frame as $\tilde{V}^i = 0$. In this frame The $(d + 2)$ equations of motion of null fluid reduce to

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}(\rho - e^{-\phi}a_i j_\rho^i) + \tilde{\nabla}_i(e^{-\phi} j_\rho^i)) = 0,$$

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}(\varepsilon - e^{-\phi}a_i j_\varepsilon^i) + \tilde{\nabla}_i(e^{-\phi} j_\varepsilon^i))$$

$$= -\frac{1}{2}t^{ij}\partial_t g_{ij} - e^{-\phi}\alpha_i j_\rho^i,$$

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}(j_\rho^i - e^{-\phi}a_j t^{ij}) + \tilde{\nabla}_i(e^{-\phi} t^{ij}))$$

$$= -a_i t^{kj}\partial_t g_{kj} + e^{-\phi}(-\rho\alpha_i + \omega_{ij} j_\rho^j).$$

Here $\tilde{\nabla}$ is the covariant derivative for metric connection of $\mathcal{M}_d$. These are respectively equations for conservation of mass, energy and momentum for a Galilean fluid given in eq. (2) on generic background g_{ij} seen by a noninertial frame with acceleration α_i and vorticity ω_{ij}. Here $\tilde{V}^M$ is interpreted as the frame velocity, α_i and ω_{ij} are derivatives of $\tilde{V}^M$ and are interpreted as acceleration and vorticity of the frame respectively. Thus we see that the null reduction of null fluids equations of motion gives us the Galilean ward identities.

Galilean boosts

We have already noticed that the above construction has a T-redefinition invariance. This is the same as Galilean boosts invariance of the Galilean theory. To see this, let us consider a T-redefinition parametrized by a vector P^M. Under this transformation, $\tilde{V}^M$ transforms as

$$\tilde{V}^M \to \tilde{V}^M + \bar{P}^M + \frac{1}{2}\bar{P}^2 V^M,$$

where $\bar{P}$ is the projected part of P. This is indeed the right transformation for frame velocity under Galilean boosts. Thus, the T-redefinition freedom translates to invariance under Galilean boosts. As the construction of null fluid does not even introduce a T-vector, it is inherently invariant under Galilean boosts. So, the fluid obtained from it also enjoys this much required invariance.

In the last section, the Galilean fluid that we have presented, is in particular fluid rest frame, where the fluid velocity is zero. For the generic case, a T-redefinition parametrized by $\bar{P} = -u$ is performed. The quantities transform in the following way

$$\rho' = \rho,\ \varepsilon' = \varepsilon + \frac{1}{2}\rho\bar{u}^2,\ j_\rho^i = \rho\bar{u}^i + R^i,$$

$$j_\varepsilon^i = \varepsilon'\bar{u}^i + E^i + (Pg^{ij} + \Pi^{ij})\bar{u}_j + \frac{1}{2}R^i\bar{u}^2,$$

$$t^{ij} = \rho\bar{u}^i\bar{u}^j + Pg^{ij} + 2\bar{u}^{(i}R^{j)} + \Pi^{ij}.$$

$\bar{u}$ is the projected part of u and is readily interpreted as the velocity of Galilean fluid. These equations show how various quantities transform under Galilean boosts.

The entropy current

We can also reduce the second law equation as

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}(s' - e^{-\phi}a_i j_s^i) + \tilde{\nabla}_i(e^{-\phi} j_s^i)) \geq 0,$$

where in a generic fluid frame, we have the Galilean entropy current as $j_s^i = s'\bar{u}^i + 1/T(j_\varepsilon^i) - \mu_m/T(j_\rho^i) + \gamma^i$ and scalar entropy functional as $s' = 1/T(\varepsilon' + P) - (\mu_m/T)\rho + \gamma_-$. It is easy to see that $s' = S$, the thermodynamic entropy, at ideal order.

One can also perform similar reduction to first derivative order. Here, we present the results in so-called 'mass frame': $j_\rho^i = 0$. The various first order quantities of eq. (4) get following form

$$j_\varepsilon^i = \kappa_e g^{ij}(\nabla_j T - a_j\partial_t T),\ t^{ij} = -\zeta\Theta g^{ij} - \eta\sigma^{ij},$$

where, $\eta,\ \zeta \geq 0;\ \kappa_e \leq 0$. This defines the generic first order uncharged Galilean fluid on a generic background. κ_e is the thermal conductivity which is constrained to be negative definite. ζ, η are respectively bulk and shear viscosities, constrained to be positive definite. These are the age-old results of first order non-relativistic fluid dynamics.

Thus, we have seen that starting from the null fluid and following the null reduction procedure we get the thermodynamics and dynamics of the Galilean fluid. We have also seen how various quantities transform under Galilean boosts and also obtained the entropy current. This mapping is essentially trivial.

Summary and discussion

We have presented the total number of transport coefficients for most generic, parity even uncharged Galilean fluid, charged Galilean fluid and Galilean superfluid to first order in derivative expansion. This analysis includes and extends the non-relativistic dissipative hydrodynamics studied in Landau-Lifshitz[1] for uncharged fluids and superfluids. As we see, there are three different kinds of transports: hydrostatic (HS) transport, that vanishes at equilibrium, non-hydrostatic (non-HS) transport, that

Table 1. Classification of transport coefficients for Galilean systems

Transport coefficients	Galilean fluid (uncharged)	Galilean fluid (charged)	Galilean superfluid
HS	0	0	3
Non-HS	0	1	13
DS	3	5	22
Total	3	6	38

does not vanish at equilibrium and dissipative (DS) transport, that vanishes at equilibrium and contributes in entropy production. Both HS and non-HS transports are non-dissipative in nature. The complete structure of the constitutive relations even in the presence of parity odd effects for all these three systems has been reported earlier[10,11]. We summarize the results in the Table 1.

Let us end this article with two important points:

(1) The equilibrium analysis on null fluid already uses a preferred frame, hence T is fixed. At equilibrium, the field contents of null fluids and Galilean fluids are exactly the same and they are in exact correspondence. The partition function analysis is identical. (This is a direct consequence of the fact that variations of W for a null fluid were demanded to preserve V isometry.)

(2) Our analysis gives us the most generic Galilean fluid, i.e. a fluid flow consistent with most generic Galilean isometry. The usual non-relativistic fluid, that comes from $c \to \infty$ limit of a relativistic system is certainly a part of our construction. But, it may not be the most generic system that we have constructed. For example, the number of transport for uncharged Galilean fluid and non-relativistic fluid is the same, whereas for other systems, we do not know the counting for non-relativistic fluids. It would be interesting to understand how this physical non-relativistic systems sit in our construction.

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ACKNOWLEDGEMENTS. N.B. would like to thank Suvankar Dutta, IISER Bhopal, India and Akash Jain, Durham University, UK for collaboration on studies presented in this article. Research of N.B. is partially supported by DST Ramanujan Fellowship.

doi: 10.18520/cs/v112/i07/1385-1389