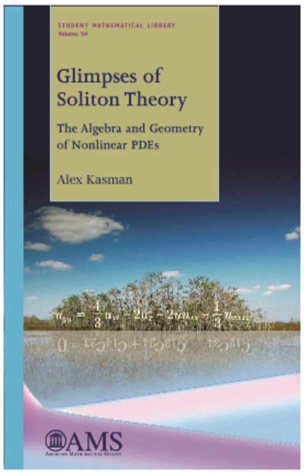

Glimpses of Soliton Theory: The Algebra and Geometry of Nonlinear PDEs, Vol. 54. Alex Kasman. American Mathematical Society, Providence, Rhode Island, USA. 2010. 304 pages. Price: US$ 39.20.

An important revolution took place during the 20th century in two stages in mathematics and physics. This revolution concerns soliton theory.

The first stage of revolution started in 1965 with a numerical experiment of the solution of the KdV equation (D. J. Korteweg and G. de Vries) by N. J. Zabusky and M. D. Kruskal, and then in 1967, by the theoretical discovery of *n*-soliton solution of this equation by C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura. This stage of revolution was more or less completed before the end of 1974. Peter Lax (1968), V. E. Zakharov and A. B. Shabat (1972), and M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur (1974) also played significant roles.

The second stage of revolution concerns with the theory related to the discovery of a large number soliton equations and writing their exact solutions by methods more algebraic–geometric in nature. Important contributions to this were made by R. Hirota (1971–74), David Mumford (1978–79), I. M. Krichever and S. P. Novikov (1980), Mikio Sato (1981), E. Date, M. Jimbo, M. Kashiwara and T. Miwa (1982), and G. Segal and G. Wilson (1985). In the second stage also Peter Lax's work (1968, 1975) played an important role. Kasman's book deals with the material in the second stage of revolution.

In 1834 John Scott Russell made a remarkable observation of a great 'wave of translation' (Russell's description of the observation is well known; see Wikipedia). The first stage of revolution was about a special property (also observed by Russell) that two such 'solitary waves' of different amplitudes interact and re-emerge unchanged. When Zabusky and Kruskal observed this nonlinear interaction property in their numerical experiment, they coined a new word 'soliton' for this solitary wave. But sadly, even after 133 years, Zabusky and Kruskal, and Gardner *et al*. do not refer to the great wave of translation or 'solitary wave' of Russell because this discovery was not given due attention for all these years. Korteweg and de Vries in their much referred paper in 1895 did not mention Russell's observation; instead they referred to G. B. Airy's work of 1845 and G. G. Stokes' work of 1847; both Airy and Stokes had ridiculed Russell. Since more general non-stationary solution of their equation seemed impossible, hardly any attention was paid to Korteweg and de Vries work for about 70 years. In Epilogue, Kasman mentions the failure of Korteweg and de Vries to recognize the importance of their work.

In chapter 3, Kasman describes the neglect and ridicule of Russell's discovery. Nevertheless, it is highly satisfying that Miura (one of the authors of the 1867 paper) pays glowing tribute to Russell in his survey article in *SIAM Review* in 1976. However, today several references are available to Russell's discovery that it has taken almost mythological importance. So at this stage, it becomes important to know 'who was the first to justify Russell's solitary wave mathematically?'. Kasman answers this in the footnote on p. 50. I quote from E. M. de Jager (https://arxiv.org/pdf/math/0602661.pdf): 'As to the credit of the "a priori demonstration a posteriori" of the stable solitary wave, this credit belongs, of course, to M. Boussinesq. On the other hand, Korteweg and De Vries merit to be acknowledged for removing doubts on the existence of the "Great Wave" and for their contribution to the theory of long waves in shallow water.'

I need not write more on this history, since a review article by Kasman, based on chapter 3 of the book, has appeared in *Current Science* (2018, **115**(8), 1486–1496). It provides beautifully the historical context necessary to appreciate the spectacular developments of the first stage of revolution. In this article, Kasman describes the developments in the sequence in which they occurred. It also includes the relevance of the Schrödinger equation, which played an important role in finding the *n*-soliton solution. A mathematician, not familiar with quantum mechanics, can understand the role played by the Schrödinger equation here. But he/she will be left with a feeling of incompleteness in understanding when he reads the brief statement 'quantities we previously thought of numbers (such as "speed") are actually operators' in quantum mechanics. Kasman warns that 'this sounds strange and nonsensical'. But it would have been more meaningful, had he added one or two more sentences of explanation. Hence, I give here (also included in the *Current Science* article) the web-address – http://www.math.ucla.edu/~tao/preprints/schrodinger.pdf – of a six-page article by the Fields Medalist Terence Tao. It is a brief presentation suitable for mathematicians, who need not go through the vast literature and big textbooks on quantum mechanics.

One of the two basic assumptions of quantum mechanics is 'particles themselves are waves' – a dual role. Kasman emphasizes a parallel in soliton theory – 'solitons, which are waves, behave like particles'.

Kasman points out (p. 61) that there are two aspects of soliton theory arising out of the KdV equation: pure and applied.

1. The analysis of nonlinear partial differential equations (PDE) leading to dynamics of waves generally falls on the applied side. In this one tries to solve the initial value problem (IVP) of KdV and other soliton equations through 'inverse scattering transform (IST) method'. Though the analysis involved is difficult, it is simple to understand the general procedure when we recollect the well-known method of solving IVP by Fourier transform (FT). Generally, solving an IVP directly is difficult. One defines FT $\hat{u}$ with respect to spatial variable x of the solution u and uses FT of the equation to derive an equation for the evolution of $\hat{u}$ to find $\hat{u}|_t$ from $\hat{u}|_0$. The final step is to find $u|_t$ from $\hat{u}|_t$ by taking inverse FT. In the IST method, the role of FT is replaced by IST. The important point is that, unlike FT method for linear equations, the IST method is used to solve a nonlinear equation.

2. The algebraic and geometric approach uses purest of pure mathematics. This approach is the outcome of the seminal paper by Lax in 1968. Kasman deals only with this approach and shows that it is far more simple not only to find the n-solitons solution of soliton equations, but also to find a large number of new soliton equations.

Kasman aims his book for an undergraduate one-semester capstone unit (a senior thesis or senior seminar) or reading course. When this book is used in this way, a teacher needs to interact briefly with students on a regular basis. I have personally used this method successfully in the case of many bright students with the main aim: not to teach them mathematics, but to teach them 'how to learn mathematics on their own'. The book is written in an informal style and assumes little background. It has several examples, problem sets and Mathematica code interspersed with the text, making it ideal for a reading course. The subject matter is intriguing, and this book is a great introduction to it. The problems are not simply for routine practice, but to recreate some proofs to understand them deeply (for example, see reference to Problem 2 on p. 153 after the statement of Theorem 8.3). This problem starts with an explanation of the theorem and contains a sequence of six hints.

Let me quote a few lines from the Preface of the book, which also shows the beautiful and persuasive language used by Kasman.

'…Original interest in solitons was just because they behaved a lot more like particles than we would have imagined. But shortly after that, it became clear that there was something about these soliton equations that made them not only interesting, but also ridiculously easy as compared to most other wave equations.

'As you will see, in some ways it is like a magic tricks. …

'In soliton theory, the role of "mirrors" and "hidden pockets" of a (magician) is played by a surprising combination of algebra and geometry. … Now that the tricks have been revealed to us, however, we can do amazing things with soliton equations. …

'Just as solitons have revealed to us secrets about the nature of waves that we did not know (and have benefited science and engineering), the study of these "tricks" of soliton theory have revealed hidden connections between different branches of mathematics that also were hidden before. …'

No doubt the language used by Kasman is persuasive to learn more of the solitons and interaction of some apparently unrelated branches of mathematics, but at many places he is too brief in statements and proofs. Let me point out one such case on p. 129.

Theorem 6.18: Let $V \subset \ker L$ be a k-dimensional subspace of the kernel of the differential operator L and let B be the basis for V. Then the operator L can be factored as

$$L = Q \circ K,$$

where Q is an ordinary differential operator and K is the operator from B in the Theorem 6.16.

Proof: Let B' be a maximal, linearly independent set of functions in the kernel of L, which are not in V. Then $B \cup B'$ is a basis of the kernel of L (I have replaced V by L). Let Q be constructed as the leading coefficient of L times the unique monic operator whose kernel is spanned by the functions one gets by applying K to the elements of B'. Then if …

Comment: The sentence 'Let Q be constructed as the leading coefficient of L times the unique monic operator whose kernel is spanned by the functions one gets by applying K to the elements of B'.' is mathematically correct but it will be easier to understand if broken in three or four simple sentences.

Magician's discovery of new soliton equations: From chapter 8 onwards, new solitons equations are obtained like a magic using Lax equation. Examples are many, say eq. (8.1)

$$u_t = \frac{1}{16}(30n^2 u_x + 20 u_x u_{xx} + 10 u u_{xxx} + u_{xxxx}).$$

Then after a few lines, Kasman mentions 'Like KdV equation, this equation has n-soliton solution …'. Some justification is given in problem 3. But it really looks like magic and leaves one wondering with a question 'How?'.

'So far I have not reviewed the material in the book. Each chapter begins with an abstract – which, if reproduced, will be a good review but instead let me quote (American Mathematical Society and Kasman have permitted the reviewer to quote some material from the book in this review) some material from the preface.

'Use of technology

'This textbook assumes that the reader has access to the computer program Mathematica. For your convenience, an appendix to the book is provided which explains the basic use of this software and offers 'troubleshooting' advice. In addition, at the time of this writing, a file containing the code for many of the commands and examples in the textbook can be downloaded from the publisher's website: www.ams.org/bookpages/stml-54.

'It is partly through this computer assistance that we are able to make the subject of soliton theory accessible to undergraduates. … continued on a full page… .

Book overview

'Chapters 1 and 2 introduce the concepts of and summarize some of the key differences between linear and nonlinear differential equations. For those who have encountered differential equations before, some of this may appear extremely simple. However, it should be noted that the approach is slightly different than what one would encounter in a typical differential equations class. The representation of linear differential equations in terms of differential operators is emphasized, as these will turn out to be important objects in understanding the special nonlinear equations that are the main object of study in later chapters. The equivalence of differential equations under a certain simple type of change of variables is also emphasized… .

'The *story* of solitons is then presented in chapter 3, beginning with the observation of a solitary wave on a canal in Scotland by John Scott Russell in 1834 and proceeding through to the modern use of solitons in optical fibers for telecommunications. In addition, this chapter poses the questions which will motivate the rest of the book: What makes the

KdV Equation (which was derived to explain Russell's observation) so different than most nonlinear PDEs, what other equations have these properties, and what can we do with that information?

'The connection between solitary waves and algebraic geometry is introduced in chapter 4, where the contribution of Korteweg and de Vries is reviewed. They showed that under a simple assumption about the behavior of its solutions, the wave equation bearing their name transforms into a familiar form and hence can be solved using knowledge of elliptic curves and functions. The computer program *Mathematica* here is used to introduce the Weierstrass $\mathcal{P}$-function and its properties without requiring the background in complex analysis which would be necessary to work with this object unassisted.

'The n-soliton solutions of the KdV Equation are generalizations of the solitary wave solutions discovered by Korteweg and de Vries based on Russell's observations. At first glance, they appear to be linear combinations of those solitary wave solutions, although the nonlinearity of the equation and closer inspection reveal this not to be the case. These solutions are introduced and studied in chapter 5.

'Although differential operators were introduced in chapter 1 only in the context of *linear* differential equations, it turns out that their algebraic structure is useful in understanding the KdV equation and other nonlinear equations like it. Rules for multiplying and factoring differential operators are provided in chapter 6.

'Chapter 7 presents a method for making an $n \times n$ matrix M depending on a variable t with two interesting properties: its eigenvalues do not depend on t (the matrix is *isospectral*) and its derivative with respect to t is equal to $AM - MA$ for a certain matrix A (so it satisfies a differential equation). This digression into linear algebra is connected to the main subject of the book in chapter 8. There we rediscover the important observation of Peter Lax that the KdV Equation can be produced by using the 'trick' from chapter 7 applied not to matrices but to a differential operator (like those in chapter 6) of order two. This observation is of fundamental importance not only because it provides an algebraic method for solving the KdV Equation, but also because it can be used to produce and recognize *other* soliton equations. By applying the same idea to other types of operators, we briefly encounter a few other examples of nonlinear partial differential equations which, though different in other ways, share the KdV Equation's remarkable properties of being exactly solvable and supporting soliton solutions.

'Chapter 9 introduces the KP Equation, which is a generalization of the KdV Equation involving one additional spatial dimension (so that it can model shallow water waves on the surface of the ocean rather than just waves in a canal). In addition, the Hirota Bilinear version of the KP Equation and techniques for solving it are presented. Like the discovery of the Lax form for the KdV Equation, the introduction of the Bilinear KP Equation is more important than it may at first appear. It is not simply a method for producing solutions to this one equation, but a key step towards understanding the geometric structure of the solution space of soliton equations.

'The wedge product of a pair of vectors in a 4-dimensional space is introduced in chapter 10 and used to motivate the definition of the Grassmann Cone $\Gamma_{2,4}$. Like elliptic curves, this is an object that was studied by algebraic geometers before the connection to soliton theory was known. This chapter proves a finite dimensional version of the theorem discovered by Mikio Sato who showed that the solution set to the Bilinear KP Equation has the structure of an infinite dimensional Grassmannian. This is used to argue that the KP Equation (and soliton equations in general) can be understood as algebro-geometric equations which are merely *disguised* as differential equations.

'Some readers may choose to stop at chapter 10, as the connection between the Bilinear KP Equation and the Plücker relation for $\Gamma_{2,4}$ makes a suitable 'finale', and because the material covered in the last two chapters necessarily involves a higher level of abstraction.

'Extending the algebra of differential operators to pseudo-differential operators and the KP Equation to the entire KP Hierarchy, as is done in chapter 11, is only possible if the reader is comfortable with the infinite. Pseudo-differential operators are infinite series and the KP Hierarchy involves infinitely many variables. Yet, the reader who persists is rewarded in chapter 12 by the power and beauty of Sato's theory which demonstrates a complete equivalence between the soliton equations of the KP Hierarchy and the infinitely many algebraic equations characterizing all possible Grassmann Cones.

'A concluding chapter reviews what we have covered, which is only a small portion of what is known so far about soliton theory, and also hints at what more there is to discover. The appendices which follow it are a *Mathematica* tutorial, supplementary information on complex numbers, a list of suggestions for independent projects which can be assigned after reading the book, the bibliography, a Glossary of Symbols and an Index.'

There are also four 'Big Questions' in chapter 3 entitled 'Story of solitons'. Kasman answers these questions gradually by the end of the book. I quote them from the book.

'**Big Question I** Why is it that we can write so many exact solutions to the KdV Equation when we cannot do so for most nonlinear equations?

'**Big Question II** The relationship between the n-soliton solutions and the n different 1-soliton solutions that it resembles suggests there is some way in which solutions of the KdV Equation can be combined. We know that they are not actually linear combinations and do not form a vector space. What *is* the method in which solutions are combined and can we give them a geometric structure analogous to the vector space structure for solutions to linear equations?

'**Big Question III** How can we identify *other* equations – either known already to researchers or yet to be discovered – that have these same interesting features?

'**Big Question IV** What can we *do* with this new information?

The briefest possible answer to these questions is to note that the KdV Equation has a hidden underlying algebraic structure that generic nonlinear PDEs do not share, but that by understanding this structure we can find infinitely many other equations that share all of these features and so also deserve the name 'soliton equations'.'

Kasman wrote this book for undergraduate students, but he keeps on reminding us that the subject is an active area of research integrating algebraic geometry, probability and statistics, symplectic geometry, Lie algebra and Lie groups with mysterious links to quantum mechanics. He has provided some references for these on p. 255.

I joined the Indian Institute of Science (IISc), Bengaluru as a research student in 1965, when research papers on the KdV equations started coming (including the paper of Zabusky and Krushkal) to the institute library. The subject interested me deeply but there was no question of getting involved in research in this area by an young research student without any research experience and with no one to discuss even the most elementary aspects. I collected about 30 reprints (including the review article of Miura – 1976). When I went to Mehta Research Institute (now renamed as Harish-Chandra Research Institute) at Allahabad, I handed over all the reprints to P. L. Bhatnagar, who was my research advisor at IISc during 1965–67. He took interest in the KdV equation, tried to promote it in India by holding a month long lecture workshop in 1976 (with only two resource persons, he and me) and wrote a book entitled *Nonlinear Waves in One-dimensional Dispersive Systems* (Oxford University Press (OUP), 1979). This book is available (with permission from OUP) at: https://drive.google.com/file/d/1Ffohxed1I1ZAPYR54rebewpENI-n_cIUu/view.

Much later, Zakharov was my guest at IISc and told me that he was surprised to see the first book (a good book – he emphasized) from a country where no contribution to the subject was made. Soon after, he got it translated into Russian.

I have written the above two paragraphs with a feeling of disappointment since mathematical aspects of the subject were not pursued in India – also because I could not have pursued the subject. One of the finest contributions in mathematics and physics in the last century has been completely neglected by Indian mathematicians. There are some physicists in India who have contributed to the subject but their interest has not been in the development of mathematics associated with solitons. Kasman's book is an excellent one to encourage mathematicians to take note of the subject and start training some students at graduate level. The applied mathematics aspect, 'analysis of nonlinear PDE leading to dynamics of waves' is not covered in this book. Apart from Bhatnagar's book above, I mention one more book *Solitons: An Introduction* (Cambridge University Press, 1989) by P. G. Drazin and R. S. Johnson, which can be used as a textbook.

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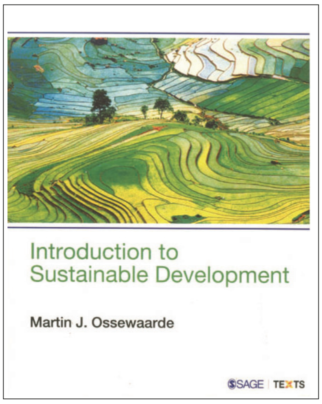

Introduction to Sustainable Development. Martin J. Ossewaarde. SAGE Publications India Pvt. Ltd, B1/I-1 Mohan Cooperative Industrial Area, Mathura Road, New Delhi 110 044. 2018. xxv + 277 pages. Price: Rs 350.

Writing a textbook on 'sustainable development' (SD) especially for the undergraduate students is a formidable challenge. Yet, the author, Martin J. Ossewaarde has made an extremely good beginning and he deserves much appreciation and credit. The challenge lies not only in its multidisciplinary character, but also in intricate interactions among its economic, ecological and social dimensions. With a decade of experience in teaching SD to undergraduate students, Ossewaarde has brought together all the essential elements of SD in this book.

The book is divided into three parts more for convenience of grouping the issues; the fact remains that interrelationships and complex interactions among the three parts and their chapters cannot be treated distinctly from each other. Hence, this review is an overall analysis of the entire contents of the book.

The book starts with the UN Sustainable Development Goals (SDGs) set for the period 2015–2030. Ideally, it should be preceded by reference to the UN Millennium Development Goals (MDGs) for the period 2000–2015. It is important for students to understand that MDGs resulted in varied degrees of success across nations, and the overall impression is that failure to fulfil the target goals was mainly due to much greater emphasis on the economic dimension and much lesser on the ecological and social pillars of SD. Further, even before a reference to MDGs, the book (that is 'introductory in nature') could have ideally begun with a brief narrative of how Earth is at a cross-roads, brought about by anthropogenic activities leading to environmental degradation, biodiversity loss, depletion of finite natural resources, population growth beyond the 'carrying capacity' of the planet and technology-driven economic growth through production of largely inessential consumer goods, etc. Today, the threat of a 'tipping point' related to global warming and climate change is looming large.

The year 1968 is notably significant when M.S. Swaminathan, the architect of India's 'Green Revolution' referred to it as 'exploitative and unsustainable' in the long run. Then in 1972, publication of *Limits to Growth* (Meadows, D. *et al.*, Universe Books, New York, 1972, p. 211) for the Club of Rome, and the UN Conference on 'Human Environment' (Stockholm, Sweden, June 1972) moved the world leaders to realize that development by exploitation of the finite natural sources cannot go on indefinitely and that environmental degradation cannot be effectively tackled without also addressing poverty, especially the rural poverty due to lack of livelihoods. Consequently, the Gro Harlem Brundtland Report *Our Common Future* (Oxford University Press, Oxford, UK, 1987, p. 416) defined SD, and provided the base material for holding the UN Conference on Environment and Development (UNCED) in 1992 in Rio de Janeiro, Brazil. Among its outcomes, the 'Agenda 21' calling upon the Member nations to embark on SD provided major thrust for action.

The author's statement that the policies and actions of the World Bank, World Trade Organization and International Monetary Fund have not been conducive for promoting SD, especially in the developing world is appropriate. He has also pointed out that globalization encourages the spread of Western