



Elements of Dynamical Systems: Texts and Readings in Mathematics, Vol. 79.

Anima Nagar, Riddhi Shah and Shrihari Sridharan (eds). Hindustan Book Agency, New Delhi, 2020. 214 pages. Price: Rs 600.

Ergodic theory and dynamical systems is a fast-growing area which is not yet adequately represented in mathematics curricula and research in India. The book under review carves its own niche in the broad area of dynamical systems. Written and edited by Indian mathematicians and researchers, the book is a unique text with a novel approach on several counts. It integrates basics and elementary concepts with advanced issues dealt with contemporary research works. Most of the material is presented with a sound pedagogical approach and a lot of examples and detailed proofs highlighting salient tools and tricks in the subject. The book is bound to enthuse students about the subject and attract young researchers to the area of dynamical systems. Another major plus is the cost of the book which is quite reasonable.

The publication is also well-timed when dynamical systems and its applicability to diverse fields of mathematics has caught the attention of the mathematical community at large – the Abel's Prize of 2020 was given to the two torchbearers of probability and dynamics – Furstenberg and Margulis.

The contents and pedagogy of the book are motivated by the experience of the contributors at a three-week long intensive advanced instructional school on 'Ergodic Theory and Dynamical Systems' at the Indian Institute of Technology Delhi, during December 2017, organized by the National Centre for Mathematics,

with the support of the National Board for Higher Mathematics, Department of Atomic Energy, Government of India. While the book is accessible to second year Masters and early Ph.D. students, research issues and related open problems are introduced in a seamless manner. The eloquent presentation style of the book will help a beginner to read and grasp the basics of the area, though working on the problems will require systematic efforts. Each chapter in itself opens up doors to a research area. The entire book or parts of the book can serve as a main reference for a year-long or a semester-long course.

Chapter 1, authored by V. Kannan, deals with dynamics of maps on the real line and interval maps. The proofs and ideas presented are specialized for the case of the real line and bring out the richness of the real line. The chapter begins by introducing elementary concepts and discussing primary examples in the subject. Five versions of understanding the notion of attraction are given. An interesting piece of the chapter is the section on topological transitivity. As a unique feature of this chapter, this concept has been presented through five views with distinct flavour, which are basically the same for a compact metric space without isolated points. The topological transitivity of the tent map is proved in five different ways leading to five different concepts such as strongly transitive or topologically exact maps, length expanding maps, expanding Markov interval maps and topological conjugacy. The mutual exclusiveness of the three constituent properties – topological transitivity, denseness of periodic orbits and sensitive dependence on initial conditions, in the definition of Devaney's chaos – is elucidated using eight different examples showing that, in general, none of them implies the other and moreover no two of them imply the third. These examples are also used to convey that, in interval maps, the above-mentioned three properties are not independent. Specifically, transitivity implies the other two, thereby implying chaos. An entire section is devoted to chaos for interval maps. Further the use of directed graphs and the beauty of the intermediate value theorem which is one of the most basic results taught in a first course in analysis is highlighted in the proof of Sarkovskii's theorem, a major result in dynamics of interval maps. A proof of

the Baire Category theorem for a complete metric space is also given. Several examples, non-examples and exercises aid the understanding of topics and bring out the excitement of the subject. At the end of the chapter, notes for each section are provided highlighting the important aspects of the results and possible extensions and generalizations with relevant references. The author's mastery in the subject is reflected at every stage of this chapter.

Chapter 2 is authored by Anima Nagar and C. R. E. Raja and is devoted to topological dynamics, a popular branch of dynamical systems formally active since the mid-last century. An abstract concept of a G -space is motivated through the fine dining problem. Several examples including linear dynamics, projective dynamics, a shift map, algebraic dynamics are presented. Important properties of G -spaces are highlighted. The section on minimal systems includes definitions, examples and key results. A classical construction of a minimal set arising from the Thue-Morse sequence is given. The statement and a proof sketch of the Birkhoff recurrence theorem is also provided. A proof by Furstenberg and Weiss of the *van der Waerden's* theorem, a celebrated result from combinatorial theory, based on ideas from topological dynamics is discussed. A section is dedicated to the algebraic theory of enveloping semigroups which is a fundamental tool in topological dynamics. Another section on proximal and distal systems is included. The key concepts of topological transitivity for cascades or semi-cascades and mixing in topological dynamics are discussed in the following section. In this context, a proof of the Furstenberg intersection lemma is presented. The relationship between properties of transitivity and mixing such as locally eventually onto, mixing strongly product transitive, weak mixing, transitive, strongly transitive and minimal are discussed, in detail. There is a brief discussion on the relationship between mixing and proximality stating an equivalent condition of weak mixing using proximality for minimal systems. The chapter discusses all major results on the topic with proofs, and provides a good set of references for advanced analysis.

Authors of Chapter 3 are C. S. Aravinda and Vishesh S. Bhat. This chapter is on basic ergodic theory. It gives an account of the major advances in the

subject in the first four decades of the twentieth century that set the pitch for further developments. The chapter begins with elementary concepts and definitions from measure theory and discusses all important results without letting the unnecessary technicalities digress the reader from the main theme of the chapter. In the process, the foundation for the next section on Hausdorff measures unifying the concept of Lebesgue measure is laid. The next two sections are the highlights of this chapter. The notion of recurrence of a measure preserving transformation on a probability space is discussed with a proof of the Poincaré recurrence theorem. A proof of the L^p version of the Birkhoff ergodic theorem is presented. The final section focuses on the geodesic flow on a closed surface of constant negative curvature, which is an important example of an ergodic system sitting in the realms of hyperbolic geometry. The Hopf's proof for ergodicity of this system is sketched. The author of this chapter beautifully unifies elements from measure theory and hyperbolic geometry in dynamics.

Chapter 4 deals with symbolic dynamics which is the study of shift spaces consisting of infinite one-sided or two-sided sequences. The author of this chapter is Siddhartha Bhattacharya. The framework in the chapter enables us to model a large class of abstract dynamical systems using shift spaces. One of the simplest examples is the conjugacy between the period doubling map on a circle and the shift space consisting of one-sided binary sequences. Symbolic dynamical systems play an important role in the theory of cellular automata. This chapter begins by introducing basic concepts in this area with lots of examples. The notion of entropy in the context of shift spaces is introduced first, and is then extended to a more general class of dynamical systems. An entire section is devoted to computation of entropy of Bernoulli shifts and translations on tori. Symbolic dynamics of tilings of Z^d are presented in detail. The unresolved Periodic tiling conjecture by Lagarias and Wang is discussed and a stronger statement in one dimension is proved. A proof due to Szegedy for a special case of this conjecture is also included. In the final section, the example of a 3-dot shift system introduced by F. Ledrappier to study mixing properties of algebraic dynamical systems is presented. Finally,

it is shown that the 3-dot shift system exhibits strong rigidity property, specifically the topological centralizer of this system consists of algebraic maps.

Complex dynamics is one of the most fertile areas in mathematics which emerged from the works of Julia and Fatou over a century ago. It is the study of the iterates of holomorphic functions. Chapter 5 on complex dynamics, by Shrihari Sridharan and Kaushal Verma, is aimed at introducing basics of this subject in one and several variables. The authors begin by introducing preliminaries and motivate the reader through a simple example of a map conjugate to the map on the Riemann sphere which takes each point to its square. The formal study starts with a discussion on the three versions of Montel's theorem for families of holomorphic and meromorphic functions. The concepts of the Fatou set and its complement, the Julia set, for a rational function are introduced and are illustrated using several examples. Some interesting properties such as the non-emptiness of the Julia set for a rational map of degree at least two are proved. An important issue is the existence of a rational map with an empty Fatou set. An entire section is devoted to demonstrate the existence and discuss the example of a map with an empty Fatou set by Samuel Lattes (1918). A stronger version of an earlier result on rational maps of degree at least two is revisited and it is proved that the Julia set is in fact infinite and each point in it can be approximated by a sequence of periodic points of the rational map. Local behaviour of a holomorphic map near its fixed point is discussed in detail and all the possible normal forms are derived. Broliin's theorem, one of the major results in complex dynamics, which gives a construction of a unique measure of maximum entropy of a polynomial map, is discussed. The chapter ends with a discussion on what happens in higher dimensions wherein a generalization of Broliin's theorem is given.

Authored by Anish Ghosh, Chapter 6 is a survey of recent research works at the intersection of homogeneous dynamics and number theory. Homogeneous dynamics is the study of dynamical and ergodic properties of group actions on homogeneous spaces of Lie groups. This has emerged as a very fertile area of research since 1980s, after the seminal works of Gregory Margulis (providing a

proof of the Oppenheim conjecture) and Marina Ratner (on conjectures of Raghunathan and Dani) involving unipotent flows on homogeneous spaces. The application of these works to number theory has opened a new frontier of research by bringing the number theorists and dynamicists together. This chapter provides deep and very interesting insights on the rich connection between homogeneous dynamics and number theory. It is bound to enthuse advanced readers to explore the subject further. A variety of results presented in this chapter will serve as a useful resource, particularly for young researchers. The author's expertise in this area guides the contents of the chapter making it crisp and a natural starting point for more advanced resources, such as the text by Einsiedler and Ward¹.

The final chapter, authored by S. G. Dani, introduces the idea of large sets in Euclidean and in more general spaces. The discussion is motivated by describing several perspectives for largeness – cardinality, measure, topological properties, etc. An illustrative example of a large set arising from Schmidt games is provided. It has connections with Diophantine approximation of numbers.

To sum up, the book leaves no scope for boredom. Each topic is presented with lucidity, yet contents are treated with utmost care without the reader feeling bogged down by unnecessary details and unwanted digression. For those wanting to delve deeper in a particular topic, enough references are provided. Given the diversity of topics and their relevance from the point of view of research, this book has a potential to become a leading text/reference book in the subject. It is a must read for researchers and teachers in mathematics, especially for those interested in ergodic theory and dynamical systems.

1. Einsiedler, M. and Ward, T., *Ergodic Theory: With a View to Number Theory*, Springer, 2011.

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