RELATIVISTIC EQUATIONS FOR PARTICLES OF ARBITRARY SPIN

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THE problem has often been formulated as to how far the equations for a relativistic particle of any assigned spin can be put into the form

$$(p_k \alpha^k + \chi)\,\psi = 0 \qquad (1)$$

where $p_k = ih\dfrac{\partial}{\partial x^k}$, χ is some arbitrary constant and the α^k's are four matrices which satisfy a different set of commutation relations in each case. It is well known that the equation (1) is invariant for all transformations of the Lorentz group if the α's satisfy the commutation relations

$$[\alpha^k, I^{rs}] \equiv \alpha^k I^{rs} - I^{rs}\alpha^k = g^{kr}\alpha^s - g^{ks}\alpha^r \qquad (2)$$

where the metric tensor g^{kr} is defined by $kg^{00} = -g^{11} = -g^{22} = -g^{33} = 1, g^{kl} = 0, l \neq$ and the $I^{rs} = -I^{sr}$ are the six infinitesmal transformations of a particular representation of the Lorentz group satisfying the commutation rules

$$[I^{kl}, I^{rs}] = -g^{kr}I^{ls} + g^{ks}I^{lr} + g^{lr}I^{ks} - g^{ls}I^{kr} \qquad (3)$$

The further equation

$$[\alpha^k, \alpha^l] = I^{kl} \qquad (4)$$

can be shown to be consistent with (2) and (3) but it cannot in general be deduced from them. It should be noted that a possible numerical constant on the right of (4) can always be removed by absorption into the α's and results merely in a change of the value of χ in (1) which is without any significance.

I have investigated all possible equations of the form (1). It can be shown that these include a set equivalent to the one given by Dirac[1] and the alternative equivalent formulations in the force free case given by Fierz[2] for particles of any assigned spin. The necessary subsidiary conditions are not included. It also includes a set which is a generalisation to higher spins of the type of the scalar wave-equation. There are other more complicated sets. But I have proved that except for the case of spins 0, ½ and 1 equation (4) is not necessarily satisfied.

It can, however, be postulated that equation (4) shall hold for all spins. All the irreducible representations of the set of ten operators I^{kl} and α^k satisfying (2) to (4), *i.e.*, all possible irreducible wave equations of the form (1) can then be found by the following artifice. We introduce a new index 4 and define

$$I^{k4} \equiv -I^{4k} \equiv \alpha^k$$
$$g^{44} = -1,\ g^{k4} = 0 \qquad k \neq 4 \qquad (5)$$

The equations (2) and (4) are then included in the set (3) if we let the indices in the latter run from 0 to 4 instead of from 0 to 3. But the resulting ten matrices I^{kl} then satisfy the commutation rules for the infinitesimal transformations of the Lorentz group in five dimensions, and all irreducible representations of these are known. The problem is, therefore, completely solved. It can also be deduced immediately that the four α^k's and the six original $^kI^l$'s have the same eigenvalues (possibly multiplied by i to allow for the time-like character of the first co-ordinate). For example, by (2) and (3) the three quantities I^{kl}, α^k and α^l for $k,l = 1,2,3$ satisfy the three equations

$$[\alpha^k, I^{kl}] = -\alpha^l,\ [I^{kl}, \alpha^l] = -\alpha^k,$$
$$[\alpha^l, \alpha^k] = -I^{kl} \qquad (6)$$

which are just the commutation rules for the three components of angular momentum, and it follows from this that in any representation, irreducible or otherwise, I^{kl}, α^k and α^l have the same eigenvalues and satisfy the same characteristic equation. It can be proved further that for any irreducible representation the eigenvalues are always $s, s-1, \ldots\ldots -s+1, -s$ where s is any integer or half odd integer. One can define a particle of spin s as one for which the maximum eigenvalue of the I^{kl} is s. In that case more restricted commutation rules which the α^k's have to satisfy for a given value of s can be deduced from equations (2) to (4), as has been done by Madhava Rao[3] for $s = 3/2$ and $s = 2$.

The imposition of the condition (4) has very far-reaching consequences. It drastically cuts down the number of possible equations. The allowed set includes the Dirac equation and the scalar and vector Kemmer[4] equations, but it *excludes* the equations given by Dirac for particles of higher spin. The allowed equations for higher spins are such that each component of the wave-function in the force-free case does not satisfy the usual second order wave equation but a factorisable equation of higher order. To see this we note that since the α^k are matrices of a finite number of rows and columns, say n, the operator $P \equiv p_k\alpha^k$ must satisfy a characteristic equation of order $\leqslant n^2$ whose coefficients can only contain products of the four quantities p_k multiplied by pure numbers. It can also be seen quite easily that this characteristic equation must be invariant for all transformations of the Lorentz group and hence must contain the p_k only in powers of the combination $p^2 \equiv p_k p^k$. To find the numerical coefficients we consider the special case when $P = p_0\alpha^0$, the other three components of p_k being zero. Since the eigenvalues of α^0 for spin s are $+s, \pm(s-1), \ldots$ it follows that the characteristic equation of P must be

$$\{P^2 - s^2p^2\}\{P^2 - (s-1)^2p^2\} \ldots\ldots = 0 \qquad (7)$$

the last factor being either P or $P^2 - p^2/4$ depending on whether s is an integer or half odd integer. Our derivation shows that this is the lowest order characteristic equation that P can satisfy, for otherwise α^0 would also satisfy one of lower order. Letting this equation act on ψ and replacing every P in it by $-\chi$ through a repeated use of (1) we see that each component of ψ must satisfy the equation

$$\{\chi^2 - s^2p^2\}\{\chi^2 - (s-1)^2p^2\}\cdots$$
$$\{\chi^2 - p^2\}\chi\psi = 0 \qquad (8a)$$

if s is an integer, or

$$\{\chi^2 - s^2p^2\}\{\chi^2 - (s-1)^2p^2\}\cdots \{\chi^2 - {}^9/_4p^2\}\{\chi^2 - {}^1/_4p^2\}\psi = 0 \qquad (8b)$$

if s is half-odd integer. These equations show that a particle of spin s must necessarily appear with $2s$ and $2s+1$ values of the mass respectively, namely, $\pm\chi/s$, $\pm\chi/(s-1)$, ... Thus a particle of spin $^3/_2$ in this theory would necessarily be capable of appearing with two different values of the rest mass, the higher value being three times the lower. These higher values of the rest mass cannot be eliminated by an artifice any more than the states of negative mass (energy) in Dirac's theory of the electron, and we are, therefore, compelled to regard them as different states of the same particle. The above theory has the advantage over the theories of Dirac, Fierz and Pauli[5] that the equation (1) can be deduced naturally from a Lagrange function even in the presence of an electromagnetic field. There are no awkward subsidiary conditions.

1. Dirac, *Proc. Roy. Soc.*, A, **1936**, **155**, 447–59. 2. Fierz, *Helv. Phys. Acta*, **1939**, **12**, 3–37. 3. Madhava Rao, *Proc. Ind. Acad. Sci.*, A, **1942**, **15**, 139–47. 4. Kemmer, *Proc. Roy. Soc.*, A, **1939**, **173**, 91–116. 5. Fierz and Pauli, *Ibid.*, **1939**, **173**, 211–32.

SCIENTIFIC RESEARCH AND INDUSTRY IN U.S.A.

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THERE is an old Chinese saying that it is wiser to go abroad and learn than stay at home and teach. Accordingly we find that Chinese scholars for nearly a millennium, between the second century B.C. and eighth century A.D., came in large numbers to the famous universities of India, Taxila and Nalanda, staying there as long as they wished and seeing and learning whatever they wished to see and learn. The hazards of journey in those days were indescribable but where the spirit was daring, the flesh was never too week. The wheels of human progress move continuously and to-day have brought up to top the people of a country which only 300 years ago was a vast pathless jungle. We, from these ancient countries of India and China, cannot too often go to America and see for ourselves how the people there have shaped their lives and institutions which have enabled them to come to the forefront of progress. The Harvard University celebrated the ter-centenary of its foundation only few years ago. The Dean took us round what he called *the yard* of the University, round which the magnificent university buildings have been built. I asked him, "Why do you call it a yard and not a campus?" He said, it was the yard built by the Pilgrim Fathers with high walls all round, where they milked their cows and rested for night so that they might not be disturbed by ravenous wolves or red Indian poachers. The milk from the cows was fed straight to the children on the spot, and thus filled the need of a nursery school in the yard. That was the origin of the Harvard University. The few thousands of aboriginees there did not know that their problem of food and living could be solved in any way except by continuous wars of extermination between the tribes for small fields of maize or fishing grounds. Yet that country maintains to-day 150 millions of human beings with food, in such an excess that, some years ago, maize was burnt and milk thrown into streams to keep up the price level, with a standard of living so high that every family has a motor car of its own and perhaps two radio sets, one for the youngsters and the other for the elders, with one telephone for every eight persons, and with prophylactic and sanitary measures so perfect that the average expectation of life is sixty years. These have been achieved by the genius of the people in harnessing scientific knowledge for the development of the country. In Philadelphia, one cannot help admiring the statue of Benjamin Franklin on a long column which can be seen from miles; the central theme that he preached was that the most certain way of human betterment was improvement in natural knowledge. His countrymen have profited by this advice. They are making continuous efforts to learn the secrets of healthy living, to gain increased mastery over the force of nature, to make new materials having better qualities, to increase the productivity of the soil, and to improve the quality of crops and livestock by scientific breeding and management. All the Research Laboratories have one motto—"The impossible is only what we have not learned to do" and "what is impossible to-day will be commonplace tomorrow."

I believe it will be of more human interest if I were to touch lightly upon the activities of some of the institutions that we visited so that one may draw one's own conclusions regarding American enterprise in research. We visited, among other institutions in Washington—Bureau of Standards where the genial Dr. Lyman Briggs directs the activities of a magnificent group of workers. The average annual expenditure is about 3 million dollars. As a chemist, I was specially interested in the work of Rossini, who has practically revolutionised the technique of separation of hydrocarbons by fractional distillation. His long glass columns, often 60 feet high, give cuts whose boiling-points are constant within one-hundredth of a degree. Physico-chemical properties of these hydrocarbons were being studied with the greatest accuracy—their heats of combustion, free energies, specific heats, etc. This fundamental work is being undertaken because of a conviction in America that synthetic organic chemistry of the future will be based on the hydrocarbons of petroleum and natural gas. The Union Carbide and Carbon Corporation which has now absorbed the American Solvents Corporation, The American Cyanamide Company and the Bakelite Corporation, is now a giant chemical combine which has done pioneering work in this field, *e.g.*, in the chlorination of hydrocarbons, in the preparation of vinyl and acrylic resins, in the