EQUILIBRIUM CONFIGURATIONS OF OBLATE FLUID SPHEROIDS UNDER THE INFLUENCE OF MAGNETIC FIELD

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RECENTLY G. Gjellestad[1] discussed the equilibrium configurations of gravitating incompressible fluid spheroids (homogeneous, inviscid and infinitely conducting) subject to a uniform magnetic field H inside and a dipole field outside. However, in her paper the sign of the integral in the first part of the equation (65) is incorrect. We give, in this note, the results of a more general study of the problem of the equilibrium of *oblate fluid spheroids* in the presence of a magnetic field which is assumed to be described by

(*i*) a uniform field H inside the spheroid in the z-direction, and

(*ii*) an external field made up of a uniform field kH along the z-direction and that due to a dipole of moment $(1-k)\ \dfrac{\mathrm{H}\,(1+\mathrm{E}^2)^{\frac{1}{2}}}{\mathrm{Q}_1^{\,1}(i\mathrm{E})}$ (the axis of the dipole being parallel to the z-direction) in order to make the normal component of the field continuous on the surface of the spheroid. Here E gives the boundary of the spheroid and the function $\mathrm{Q}_1^{\,1}(i\mathrm{E})$ is defined as

$$\mathrm{Q}_1^{\,1}(i\mathrm{E}) = (1+\mathrm{E}^2)^{\frac{1}{2}}\,\frac{dQ_1(i\mathrm{E})}{d\mathrm{E}}$$

$\mathrm{Q}_1(i\mathrm{E})$ being the Legendre function of the second kind.

If we take $k=0$, we get a spheroid with a uniform magnetic field inside and a dipole field outside. This case is the same as that discussed by G. Gjellestad. But due to the inconsistency mentioned above, our subsequent results for the case $k=0$ are different. The case $k=1$ corresponds to a uniform field of the same value both inside and outside the spheroid. If we let $k\to\infty$, $\mathrm{H}\to 0$ but $k\mathrm{H}=\mathrm{H}_0$ remaining finite, it corresponds to an oblate spheroid under only an external field made up of a uniform field H in the z-direction superposed by a field due to a dipole of moment $\dfrac{\mathrm{H}_0\,(1+\mathrm{E}^2)^{\frac{1}{2}}}{\mathrm{Q}_1^{\,1}(i\mathrm{E})}$ in the antiparallel direction.

Under the influence of magnetic field of the characteristics mentioned above, we find that there exists a sequence of gravitating oblate fluid spheroids.

The spheroids are assumed to be infinitely conducting, incompressible non-rotating and situated in infinite empty space. Following G. Gjellestad we have used oblate spheroidal co-ordinates which are defined in terms of the triple infinity of orthogonal surfaces provided by the confocal spheroids

$$\frac{x^2+y^2}{1+\xi^2}+\frac{z^2}{\xi^2}=c^2 \qquad (0\leqslant\xi\leqslant\infty) \tag{1}$$

the confocal hyperboloids

$$\frac{x^2+y^2}{1-\mu^2}-\frac{z^2}{\mu^2}=c^2 \qquad (-1\leqslant\mu\leqslant+1) \tag{2}$$

and the planes $\phi=$ constant $(0\leqslant\phi\leqslant 2\pi)$ (3) through the Z-axis. Here c is a constant equal to half the distance between the foci.

We investigate the stability of an oblate spheroid of boundary given by

$$\xi=\mathrm{E} \tag{4}$$

by subjecting it to a general P_n deformation so that its boundary changes to one given by

$$\xi=\mathrm{E}+\epsilon\,\frac{1+\mathrm{E}^2}{\mathrm{E}^2+\mu^2}\,\mathrm{P}_n(\mu) \qquad (n>0) \tag{5}$$

where ϵ is a non-dimensional constant.

Because of the deformation (5), there shall be a change $\Delta\Omega$ in the gravitational potential energy of the spheroid and a change Δm in the total magnetic energy, which consists of two parts—the change, $\Delta m^{(i)}$ in the magnetic energy inside the spheroid and the change, $\Delta m^{(e)}$ in the external magnetic energy. We then employ the equilibrium condition

$$\Delta\Omega+\Delta m=0 \tag{6}$$

in order to define the equilibrium spheroids.

We find that the total change, Δm, in the magnetic energy vanishes for odd values of n, whereas it is of the order ϵ for all even (P_{2n}) deformations, and is given by

$$\Delta m_{2n} = -\frac{\mathrm{H}^2c^3(1+\mathrm{E}^2)}{\mathrm{Q}_1^{\,1}(i\mathrm{E})}\sum_{n=1}^{\infty}\left\{\frac{\left(1-\frac{k}{3}\right)}{\mathrm{P}_{2n}^{\,1}(i\mathrm{E})} - \frac{(1-k)\left(1+\frac{k}{3}\right)}{4\,[\mathrm{Q}_1^{\,1}(i\mathrm{E})]}\int_{-1}^{+1}\frac{\mathrm{P}_{2n}(\mu)}{\mathrm{E}^2+\mu^2}\,d\mu\right\}\epsilon_{2n} \tag{7}$$

where $\mathrm{P}_{2n}^{\,1}(i\mathrm{E})$ are defined as

$$\mathrm{P}_{2n}^{\,1}(i\mathrm{E}) = (1+\mathrm{E}^2)^{\frac{1}{2}}\,\frac{d\mathrm{P}_{2n}(i\mathrm{E})}{d\mathrm{E}} \tag{8}$$

The functions $\mathrm{P}_{2n}(i\mathrm{E})$ denote the Legendre functions of the first kind.

The expression for the change, $\Delta\Omega$ in the gravitational potential energy of a spheroid as derived by G. Gjellestad, is

$$\Delta\Omega = -\frac{3}{10}\,\frac{M^2G}{c}\,\epsilon_2\left[\frac{3E^2+1}{E}\cot^{-1}E-3\right] \quad (9)$$

where M denotes the mass of the spheroid. The change in the gravitational potential energy of the spheroid is of the first order in ϵ only for a P_2 deformation and of higher order for all higher order deformations.

For a P_2 deformation of the spheroid, for which both Δm and $\Delta\Omega$ are of the order ϵ, the condition

$$\Delta\Omega + \Delta m = 0$$

for equilibrium gives that a configuration is stable for P_2 deformation if

$$H = H_{eq.}\left[\frac{f(e)}{e^2F_2(e)}\right]^{\frac{1}{2}} \quad (10)$$

where, for convenience, we have put

$$H_{eq.} = \sqrt{\frac{3}{10}}\,\frac{M\sqrt{G}}{a^2}$$

a, being the major half-axis of the spheroid, and G the constant of gravitation.

Here the functions $f(e)$ and $F_2(e)$ are defined as

$$f(e) = \frac{3-2e^2}{e(1-e^2)^{\frac{1}{2}}}\cot^{-1}\left(\frac{1-e^2}{e^2}\right)^{\frac{1}{2}} - 3 \quad (11)$$

and

$$F_2(e) = \frac{e^2\left(1-\frac{k}{3}\right)}{3(1-e^2)^{\frac{1}{2}}Q_1'(iE)} - \frac{(1-k)\left(1+\frac{k}{3}\right)}{4[Q_1'(iE)]^2}\,f(e) \quad (12)$$

(e denotes the eccentricity of the spheroid).

The function $H/H_{eq.}$ is plotted against e for the case $k=0$ in Fig. 1, and for the other two cases in Fig. 2. We find that $H/H_{eq.}$ increases with increase in the eccentricity for the three types of magnetic field discussed.

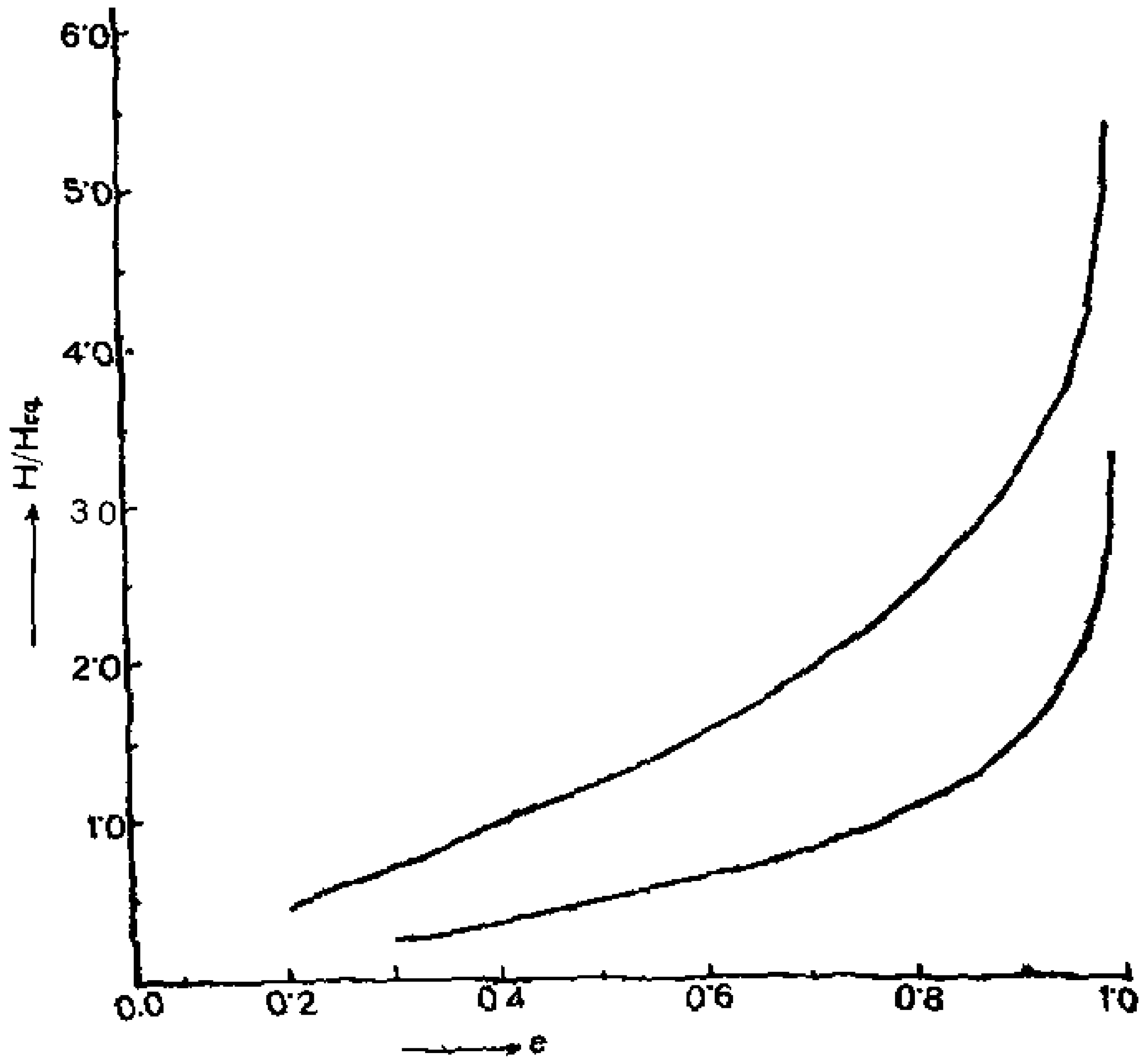

FIG. 2

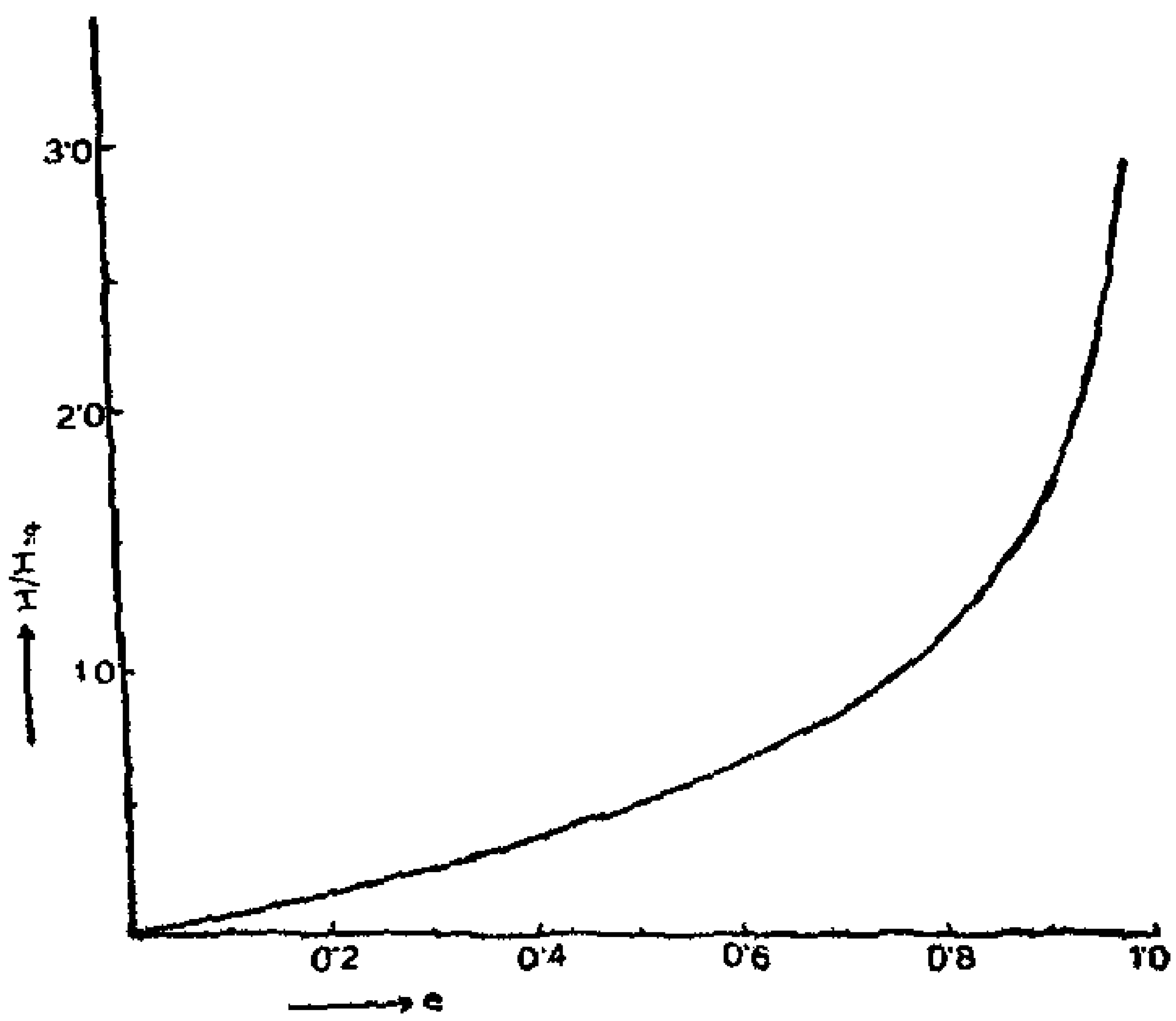

FIG. 1

However, $H/H_{eq.}$ required for stability of the spheroid is more for the case when $k\to\infty$, $H\to 0$ but kH remaining finite $(=H_0)$. Thus we find that there exists a unique configuration for a spheroid which is stable for a P_2 deformation for each of the three types of magnetic field under consideration.

The detailed paper shall be published elsewhere.

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1. Guro Gjellestad, *Astrophys. J.*, **1954, 119, 14.**

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