ON SOME METHODS OF CONSTRUCTION OF ASYMMETRICAL FACTORIAL DESIGNS

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1. Introduction

KISHEN AND SRIVASTAVA[1,2] have given general methods of construction of asymmetrical factorial designs and Kishen[3] has discussed the method of constructing optimum designs of the class $q \times 2^2$. Some further optimum designs of the class $q \times 2^2$ and various methods of constructing different types of asymmetrical factorial designs which have since been developed are briefly discussed in this article.

2. Further Optimum Designs of the Class $q \times 2^2$

It has been demonstrated by Rao[4] that for $q = 13$, a BIB design with $v = 13$, $b = 26$, $k = 6$, $r = 12$, $\lambda = 5$ exists. This design, as in the case of the BIB designs associated with the 5×2^2 and 9×2^2 designs, can be split up into two PBIB designs with parameters $v = 13$, $b = 13$, $k = 6$, $r = 6$. Consequently, writing X_0 in the pattern of one of the PBIB designs in 13 blocks and filling the remaining 7 places in each block with X_1, and then writing X_1 in the pattern of the other PBIB design in 13 blocks and filling the remaining places with X_0, we obtain an optimum balanced design for 13×2^2 in 26 blocks of 26 plots each. This method was indicated by M. N. Das also, in a paper presented at the Fourteenth Annual Meeting of the Indian Society of Agricultural Statistics. As in the case of the 5×2^2 and 9×2^2 designs in 10 blocks and 18 blocks respectively, this design is non-resolvable.

It has been shown by Rao that for $q = 23$, 27 and 31, BIB designs with $b = 23$, 27 and 31, and $k = 11$, 13 and 15 exist. Consequently, for the cases 23×2^2, 27×2^2 and 31×2^2, optimum designs in 46, 54 and 62 blocks of 46, 54 and 62 plots each respectively have been obtained. Rao has also given solutions for BIB designs with parameters $v = 12$, $b = 22$, $k = 6$, $r = 11$, $\lambda = 5$, and $v = 16$, $b = 30$, $k = 8$, $r = 15$, $\lambda = 7$; by use of which optimum designs in the cases 12×2^2 and 16×2^2 in 22 and 30 blocks of 24 and 32 plots each respectively have been constructed.

3. Use of Pseudo-Factors

Designs of the type $l \times s^2$, where $l = s^m$, m being any prime positive integer, in blocks of ls plots each can be constructed by taking the first factor as equivalent to all combinations of m pseudo-factors, each at s levels. By this procedure, optimum balanced designs can be constructed in a minimum of $(l - 1)$ replications, or $(l - 1)s$ blocks of ls plots each, each replication being comprised of s blocks. Balancing is achieved on $(l - 1)(s - 1)$ degrees of freedom for the interaction ABC. Thus, the 8×2^2 design in 7 replications, or 14 blocks of 16 plots each, is given by the 14 equations

$$\left.\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &= 0, 1 \\
x_1 + x_2 + x_4 + x_5 &= 0, 1 \\
x_1 + x_3 + x_4 + x_5 &= 0, 1 \\
x_2 + x_3 + x_4 + x_5 &= 0, 1 \\
x_1 + x_4 + x_5 &= 0, 1 \\
x_2 + x_4 + x_5 &= 0, 1 \\
x_3 + x_4 + x_5 &= 0, 1
\end{aligned}\right\} \quad (1)$$

where x_1, x_2 and x_3 correspond to the three pseudo-factors taken as equivalent to the first factor A, and x_4 and x_5 correspond to the second and third factors B and C. Taking X_0 and X_1 to denote, as before, the treatment combinations b_0c_0, b_1c_1 and b_0c_1, b_1c_0 respectively, the design would be as under:

Table I

8×2^2 Design in 16-plot blocks

A	I	II	III	IV	V	VI	VII	
a_0	X_0 X_1	X_0 X_1	X_0 X_1	X_0 X_1	X_0 X_1	X_0 X_1	X_0 X_1	
a_1	X_1 X_0	X_0 X_1	X_1 X_0	X_1 X_0	X_0 X_1	X_0 X_1	X_1 X_0	
a_2	X_1 X_0	X_1 X_0	X_0 X_1	X_1 X_0	X_0 X_1	X_1 X_0	X_0 X_1	
a_3	X_0 X_1	X_1 X_0	X_1 X_0	X_0 X_1	X_0 X_1	X_1 X_0	X_1 X_0	
a_4	X_1 X_0	X_1 X_0	X_1 X_0	X_0 X_1	X_1 X_0	X_0 X_1	X_0 X_1	(2)
a_5	X_0 X_1	X_1 X_0	X_0 X_1	X_1 X_0	X_1 X_0	X_0 X_1	X_1 X_0	
a_6	X_0 X_1	X_0 X_1	X_1 X_0	X_1 X_0	X_1 X_0	X_1 X_0	X_0 X_1	
a_7	X_1 X_0	X_0 X_1	X_0 X_1	X_0 X_1	X_1 X_0	X_1 X_0	X_1 X_0	

Similarly, the 9×3^2 design in 24 blocks of 27 plots each can be readily constructed, to which reference has been made subsequently.

4. Optimum Designs of the Class $q \times 3^2$

For balanced designs of the type $q \times 3^2$ in blocks of $3q$ plots each, in which the sets

"J_0, J_1, J_2," "J_1, J_2, J_0," and "J_2, J_0, J_1," where J_0, J_1, J_2 are the three sets of treatment combinations corresponding to BC (J), are repeated t_1, t_2 and t_3 times respectively in each replication, it can be shown that the loss of information on BC and ABC is given by

$$L(BC) = \frac{t_1^2+t_2^2+t_3^2-t_1t_2-t_1t_3-t_2t_3}{q^2} \quad (3)$$

and

$$L(ABC) = \frac{3(t_1t_2+t_1t_3+t_2t_3)}{q^2}, \quad (4)$$

where

$$t_1 + t_2 + t_3 = q,$$

the total loss of information thus being 2, which is the necessary condition for a balanced design in this case. Now q can be of the form $3t$, $3t-1$ or $3t-2$, t being any positive integer.

When $q = 3t$, the optimum design is given by $t_1 = t_2 = t_3$, L (BC) being 0. (5)

When $q = 3t - 1$, the optimum design is given by

$$t_1 = t_2 = t,\ t_3 = t - 1,$$

L (BC) being $1/q^2$. (6)

Finally, when $q = 3t - 2$, the optimum design is obtained when $t_1 = t$, and

$$t_2 = t_3 = t - 1,$$

L (BC) being $1/q^2$. (7)

A balanced design cannot be optimum unless each of the replications follows one of the three patterns discussed above.

The procedure of pseudo-factors discussed in Section 3 gives optimum designs for the class of $l \times 3^2$ designs in which each of the three sets "J_0, J_1, J_2," etc., occurs $l/3$ times in a replication and L (BC) $= 0$. It would thus follow that where resolvable BIB designs with parameters $v = l$, $k = l/3$, b, r, λ exist, these would yield optimum balanced designs of the class $l \times 3^2$. Such BIB designs are known to exist for $l = 6$, 9 and 12, which thus enables construction of optimum 6×3^2, 9×3^2, 12×3^2 designs in 18, 27 and 36 plot blocks respectively.

It would also be seen that where resolvable BIB designs with parameters $v = sp$, $k = p$, b, r, λ exist, these enable construction of optimum balanced designs of the class $q \times s^2$, where $q = sp$. In this manner, the designs 10×5^2, 12×4^2, 15×5^2, 21×7^2 and 28×7^2 in blocks of 50, 48, 75, 147 and 196 plots respectively are readily obtained.

5. Method of Cutting

The method of cutting given by Kishen and Srivastava[2] and also independently by M. N. Das (unpublished) is a useful device for obtaining balanced asymmetrical factorial designs. However, it appears that for designs of the type $q \times s^2$, where $q = sp$, p being any integer, adopting a single cut, that is, cutting out in all the blocks all the treatment combinations which contain the last level of the factor A, gives an optimum design. Thus, from 8×2^2 design given in Section 3, the optimum 7×2^2 design is obtained by omitting all the treatment combinations involving a_7, *i.e.*, the last row of the design. This, in fact, is the design obtained from the associated BIB design with $v = b = 7$, $k = r = 3$, $\lambda = 1$. However, when we have two cuts, that is, we omit the row containing a_6 also in this design, the resulting 6×2^2 design is no longer an optimum design, the loss of information on BC in this design being 1/21 when it should have been zero if it were optimum. With the adoption of three cuts, that is, omitting the row containing a_5 also, we get a balanced 5×2^2 design which again is not optimum, the loss of information on BC being 3/35 when this loss in the case of an optimum design is 1/25. All these designs involve 7 replications.

From the 9×3^2 design referred to in Section 3, we can by one cut obtain the 8×3^2 design, which is optimum, the loss of information on BC in this case being 1/64. However, with two cuts, we obtain a balanced 7×3^2 design which, however, is not optimum, the loss of information on BC being 1/28 instead of 1/49 if the design were optimum. With three cuts, we get a balanced design for 6×3^2 in which the loss of information on BC is 1/16, instead of zero in the case of an optimum design. With a further cut, we obtain a 5×3^2 design in which the loss of information on BC is 1/10 instead of 1/25 in the case of an optimum design. All these designs have eight replications.

It would thus appear that in the case of designs of the types $q \times 2^2$, where $q = 2u$, and $q \times 3^2$, where $q = 3u$, u being any positive integer, only a single cut results in an optimum design and that further cuts yield only balanced designs which increasingly deviate from the optimum as the number of cuts adopted increases. The method of cutting is thus only useful in providing balanced designs where the corresponding optimum designs do not exist. Two or more cuts, therefore, never lead to optimum designs for the construction of which other methods, some of which have been indicated in the earlier Sections, have to be adopted. It may, however, be remarked that the optimum 5×2^2 and 9×2^2 designs in 10 and 18 blocks of 10 and 18 plots each respectively, which are non-resolvable, can be obtained direct from the optimum 6×2^2 and 10×2^2 designs by using a single cut.

6. Use of PBIB Designs

In any asymmetrical factorial design, the use of the associated BIB design usually results in a large number of replications. It is, therefore, necessary to use the associated PBIB designs for obtaining partially balanced asymmetrical factorial designs which lead to a marked reduction in the number of replications required for a completely balanced design. Thus, for instance, the partially balanced 6×2^2 design obtained by doubling the 3×2^2 design, requires only three replications instead of five replications required for the completely balanced optimum design. It is noticeable that in this partially balanced design, BC is unconfounded. Similarly, partially balanced designs for $q \times 2^2$, where $q = 4p$ $(p \geqslant 2)$, in three replications are obtainable from the completely balanced 4×2^2 design in three replications; and the partially balanced $q \times 3^2$ design, where $q = 3p$ $(p \geqslant 2)$, in two replications are derivable from the completely balanced 3×3^2 design in two replications. This technique, which has been given by Kishen and Srivastava,[2] thus enables a large class of practically useful partially balanced asymmetrical factorial designs to be constructed.

7. Designs of the Type $q \times t \times s$, where $t = s^m$

In this case, the $t \times s$ treatment combinations are divided into s sets corresponding to $s - 1$ degrees of freedom for the interaction BC. Then the problem of construction of optimum balanced designs in blocks of qt plots reduces to that of construction of optimum balanced designs for $q \times s^2$ in blocks of qs plots, methods of construction of which have already been discussed. Thus, for the $3 \times 4 \times 2$ design in blocks of 12 plots each, a balanced design is available, as in the case of the 3×2^2 design, in three replications. If, however, balance is desired on each of the three degrees of freedom belonging to the interaction BC, 9 replications would obviously be required.

1. Kishen, K. and Srivastava, J. N., *Curr. Sci.*, 1959, **28**, 98.
2. — and —, *J. Ind. Soc. Agric. Stat.*, 1960, **11** (1 and 2), 73.
3. Kishen, K., *Curr. Sci.*, 1960, **29**, 465.
4. Rao, C. R., *Sankhya*, 1961, **23 A**, 117.

THE INDUSTRIAL SPECTROSCOPE

In the pavilion of the German Democratic Republic at the Second Indian Industries Fair being held from the 14th of November 1961 until the 1st of January 1962, VEB Carl Zeiss JENA is exhibiting amongst other products of their extensive production programme the Industrial Spectroscope which deserves special mention.

The Industrial Spectroscope has been designed in a way which permits its application in the various fields of industrial material testing, primarily for rapid spectrochemical analyses for all kinds of metals. The principal field of application for the Industrial Spectroscope is in the small and medium-sized industrial and research inspection laboratories, *e.g.*, in metal and semi-finished product stores, in annealing and tempering plants, in scrap yards and scrap reclaiming depots, foundry laboratories, metallurgical, metallographic and mining experimental stations and similar inspection points where the speedy recognition of the material quality is frequently of great economic importance. The following technical data will indicate its optical performance: high dispersion by two double-effective flint-glass prisms, D = 0·9 Å/mm. at 5,000 Å; the resolving power $\lambda/\Delta\lambda = 17{,}000$ at 5,000 Å means that 0·3 Å can still be discerned; large aperture ratio of 1 : 11; two different magnifications: $\times$ 8 for low-power viewing and $\times$ 22 for exacting observations; direct wavelength determination to $\pm$ 2 Å; the coarse motion ensures a rapid survey of the spectrum—precise setting with the aid of the fine motion; izochromatic photometric evaluation with a neutral wedge of high pitch, thus no loss in light-intensity by polarising filters; convenient reading of wavelengths and extinction on frosted screens; two attached spark stages within easy reach from the operator's place; rapid change of the counter-electrodes; self-contained rigid design.