THE GENERALIZED DESCRIBING FUNCTION AND ITS APPLICATIONS

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THE technique of the describing function as a tool of design has been emphasized and demonstrated by several authors.[1-3] Johnson[4] has discussed at length the necessary assumptions and the accuracy attainable by this method. Klotter[5] has extended this concept to include the case of dynamic non-linearities. West[6] and his colleagues have developed the idea of dual input describing functions. A transformation method capable of dealing with all the above categories has been developed by the author.[7-9]

However, the describing function is valid only in the steady state. Therefore it is used as a design technique to ascertain desirable performance characteristics in the steady state *only*. *It is incapable of* giving any information on the transient performance, as it cannot discriminate between different sets of initial conditions. It is well known that initial conditions play a vital role in non-linear systems.

The object of this paper is to extend the concept of the describing function for purposes of analysis and synthesis even in the transient state. This extended concept may be termed the generalized describing function or the g.d.f.

With this view, a damped sinusoid like ae^{-pt} sin wt, $(p > 0)$, has been considered as the input to the non-linear component. The important feature of this concept is that the conventional describing function can be obtained as a special case of the generalized describing function as p tends to zero. It is also interesting to notice that the g.d.f. is in the time domain, and the corresponding Laplace Transform has to be used for design in the s-plane. Thus it is the presence of s in the Transformed g.d.f. that extends the scope of this function to the transient state also.

As the non-linear component can now be represented by an s-function, the effect of the initial conditions can be accounted for, and the analysis of the non-linear control system can now be carried out on the basis of the root loci with its inherent advantages. The following procedure clarifies the sequence of operations necessary in arriving at the g.d.f. in the s-domain.

It has been shown in reference 7 that the characteristic of a static component can be represented by an algebraic equation. Therefore with reference to Fig. 1, let

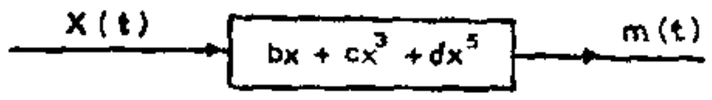

FIG. I. CHARACTERISTIC OF THE NON-LINEAR COMPONENT.

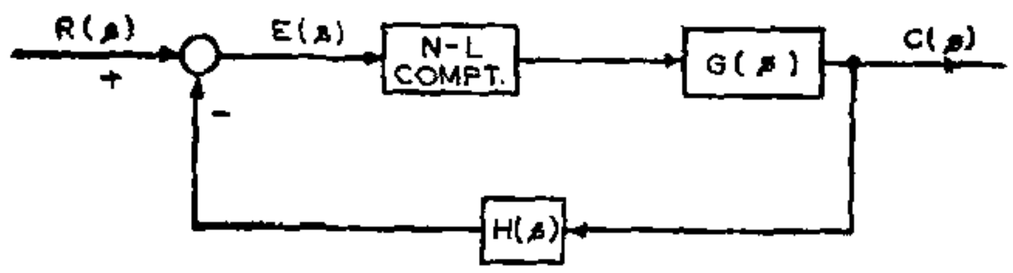

FIG. II. TYPICAL NON-LINEAR CONTROL SYSTEM.

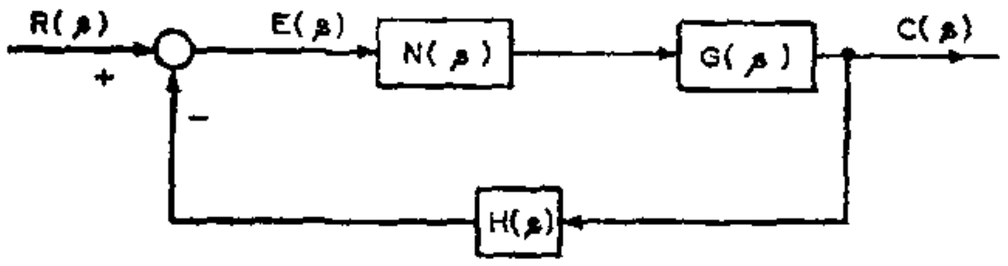

FIG. III. THE NON-LINEAR COMPONENT OF FIG II IS REPLACED BY ITS g.d.f. IN THE s-DOMAIN

FIGS. 1–3

$$f(x) = bx + cx^3 + dx^5, \quad -1 \leqslant x \leqslant +1, \quad (1)$$

be the characteristic of the component.

Let

$$x = ae^{-pt} \sin wt, \quad p > 0, \quad (2)$$

be the input to the non-linear component.

This expression for x is substituted into the non-linear characteristic, giving rise to the response of the component as follows.

$$m(t) = bae^{-pt} \sin wt + ca^3e^{-3pt} \times \left[\frac{3}{4} \sin wt - \frac{1}{4} \sin 3wt\right] + da^5e^{-5pt}\left[\frac{10}{16} \sin wt - \frac{5}{16} \sin 3wt + \frac{1}{16} \sin 5wt\right]. \quad (3)$$

The generalized describing function in the time domain may be defined as the ratio of terms containing sin wt in the output, to the

exponentially time varying amplitude of the sinusoidal input to the non-linear component. This expression may be denoted by $n(t)$. From equations (1), (2) and (3), $n(t)$ may be written as follows :

$$n\,(t) = b + \frac{3}{4}\,ca^2e^{-2pt} + \frac{10}{16}\,da^4e^{-4pt}. \qquad (4)$$

The justification for the above procedure is exactly similar to that of the conventional describing function.

At this stage it may be observed that the g.d.f. is in the time domain, and for p tending to zero, the conventional describing function is obtained. Hence the justification in the choice of the name 'g.d.f.' to denote this concept. However, it must be emphasized that the conventional describing function, being a function of amplitude and frequency only, may directly be used for a study of the frequency response of the system containing the non-linear component. Since the g.d.f. derived as above is an explicit function of time, the Laplace Transform of the g.d.f. is necessary for the s-plane study of the system. Thus the Laplace transformed g.d.f. may be denoted by

$$\mathrm{N}\,(s) = \mathcal{L}\,[n\,(t)]. \qquad (5)$$

From equations (4) and (5)

$$\mathrm{N}\,(s) = \frac{b}{s} + \frac{3}{4}\,\frac{ca^2}{s+2p} + \frac{10}{16}\,\frac{da^4}{s+4p}\,. \qquad (6)$$

Even when $n(t)$ has a phase shift with reference to the input $x(t)$, $\mathrm{N}(s)$ may be expressed as the ratio of two polynomials in s.

It must also be emphasized that the effect of initial conditions comes into picture only after obtaining the expression for $\mathrm{N}(s)$, and while working in the s-plane. This restriction is necessary to preserve the uniqueness of the g.d.f. in the time domain. The effect of such a restriction on dynamic non-linear components requires careful examination.

Figure 2 shows a typical closed loop non-linear control system. Having derived the expression for $\mathrm{N}(s)$ as per equation (6) Fig. 2 may be re-drawn as in Fig. 3, which is in the required form for study in the s-plane.

From equation (6), it may be noticed that in general $\mathrm{N}(s)$ is a ratio of two polynomials in s, and depends on a, p and possibly w. Therefore the forward transfer function may now be written as the product $\mathrm{N}(s)\ \mathrm{G}(s)$; and the general equation for the root locus[10] may be written as

$$\underline{|\,GH'} = 180° - \underline{|\,N.} \qquad (7)$$

Figure 3, along with equations (6) and (7), brings out the important role of the g.d.f. in the root locus study of closed loop non-linear control systems.

As transient performance is of interest only when steady state stability has been ascertained, the conventional describing function may first be used for the frequency response analysis. This analysis gives an indication of the range of interest for a and w. At this stage, to reduce the number of the arbitrary parameters, a, p and w, it may be logical to fix up the value of p as a positive number p_0, comparable to the inverse of the largest time constant of the system. Now equation (7) may be written as

$$\underline{|\,G\,(s)\,H\,(s)} = 180° - \underline{|\,N\,(p_0,\,a,\,w,\,s)}. \qquad (8)$$

Equation (8) indicates that a modification is necessary in drawing the root loci based on the g.d.f. This may be briefly stated as follows :

The conventional root loci are plots of the variations of the closed loop poles with changes in open loop gain. Thus the open loop gain is used as a parameter to be adjusted suitably to yield relative stability.

While applying the principles of root loci, in the context of equation (8) it is best to specify the open loop gain at a particular value, and obtain the loci with a, the scalar amplitude of the sinusoidal input to the non-linear component, as the parameter. The significant values of w can be obtained from the conventional frequency response analysis. These values of w give rise to a family of root loci, on each member of which a is treated as a parameter. This is possible since p has already been fixed. Therefore the relative stability of the system may now be analysed in terms of a as parameter, and compensation by means of scalar gain factor is also possible.

Further details of constructing the root loci, possible alternative techniques of transient analysis along with computer solution of specific problems will be published elsewhere.

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ISOTOPE SHIFTS AND INTERNAL CONVERSION OF γ-RAYS*

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INVESTIGATION of isotope shifts in energy of atomic spectral lines provides a tool for studying nuclear charge distribution. Isotope shifts due to finite mass of the nucleus are important in light elements but become negligible for atoms of heavy elements. For elements in medium and heavy mass region, $A > 80$, isotope shifts due to non-zero volume of the nucleus, called volume effect, become important. It is the study of this volume effect in isotope shifts which can be used for investigating properties like nuclear charge distribution, deformation of nuclear shape and compressibility of nuclear matter. In these areas, therefore, measurements on volume effect in isotope shifts will provide valuable information, supplementary to the results obtained from nuclear spectroscopic studies of nuclear energy levels. Detailed interpretation of volume effect in isotope shifts in the mass region, where nuclear shape deformation is pronounced, will also involve consideration of specific nuclear models, which is of great interest to nuclear physicists.

For a nucleus treated as a point charge, the potential at a distance r is $-Ze^2/r$. For a finite nucleus with a certain charge distribution of radius r_0, the potential at $r < r_0$ will be different from this value. The energy of an electron in an orbit which penetrates the nuclear volume, *i.e.*, s-states and to a slight degree $p_{\frac{1}{2}}$-state, will be different from that calculated for a point charge nucleus. The addition of neutrons to a given nucleus alters its radius and charge distribution and the effect of penetration on the position of energy level of the s-electron is different in different isotopes. This gives rise to volume effect in isotope shifts. On the basis of such considerations, the change in energy ΔE, of an s-state electron is obtained as follows :

$$\Delta E = \int P(r) \left[V(r) + \frac{Ze^2}{r} \right] d\tau$$

where $d\tau$ is volume element and $P(r)$, which is s-electron density in the neighbourhood of a point charge in Dirac theory is

$$P(r) = \frac{2(2\rho + 1)}{[\Gamma(2\rho + 1]^2} \cdot \psi^2(0) \left(\frac{2Zr}{a_H} \right)^{2\rho - 2}$$

where $\rho = (1 - Z^2\alpha^2)^{\frac{1}{2}}$, α being the fine-structure constant.

a_H = first Bohr radius and $\psi(0)$ is the non-relativistic Schrodinger wave-function at $r = 0$.

$$\therefore \quad \Delta E = F(Z)\, \psi^2(0) \cdot R_1^{2\rho}$$

where

$$F(Z) = \frac{12\pi Z e^2 (\rho + 1)}{[\Gamma(2\rho+1)]^2 \cdot \rho(2\rho+1)(2\rho+3)} \left(\frac{2Z}{a_H} \right)^{2\rho-2}$$

$$R_1 \equiv \left[\left(1 + \frac{2\rho}{3} \right) \langle r^{2\rho} \rangle \right]^{1/2\rho}$$

$$\langle r^{2\rho} \rangle = \int f(r)\, r^{2\rho} \cdot \frac{d\tau}{Z}$$

and $f(r)$ is normalized so that $f(r)\, d\tau = Z$. The perturbation energy shift between two isotopes differing by δR_1 in equivalent radius, *i.e.*, the isotope shift due to volume effect is then given as follows :

$$\delta \Delta E = 2\rho F(Z)\, \psi^2(0)\, R_1^{2\rho} \frac{\delta R_1}{R_1}.$$

If nuclei are spherical and the nuclear radius varies as $A^{1/3}$, one gets certain results about isotope shifts, which are, however, different from what is actually observed. For even isotopes of a given element, for instance, the addition of successive two neutrons should give equal shifts. Actually this is not the case. There are fluctuations in the value of the ratio of observed to theoretically predicted (on spherical model) shifts, $\delta \Delta E_{obs.}/\delta \Delta E_{th.}$ in isotopes

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* Based on a Review talk given at Nuclear Physics Symposium, Madras, in February 1962.