ANALYSIS OF SAINT VENANT TORSION FOR REGULAR POLYGONAL CROSS-SECTIONS

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1. Introduction

SETH[1] analysed the Saint Venant torsion of a prismatic bar whose cross-section is an n-sided regular polygon by mapping the polygon on to a unit circle. The procedure was elaborate and the computation, if numerical results are desired, would be substantial. A simple exact solution is now presented for the same problem.

To take advantage of the cyclic character of the regular polygon, the solution is initially taken as a series in polar co-ordinates. However, to facilitate satisfaction of the boundary conditions on the straight edges, it is converted into polynomials in Cartesian co-ordinates. Thereafter a very simple procedure is applied to obtain systematic convergence and accurate results with only a few terms of the series. It is found that the true values of the torsional stiffness and maximum stress are rapidly approached from below.

The solution for the equilateral triangle ($n = 3$) is easily determined in the well-known closed form. Numerical examples are given for the square and the hexagon for which values from alternative analyses are available for comparison.

Series solutions in polar co-ordinates were used before to analyse torsion and bending involving straight boundaries.[2–4] However, in these contributions, the constants of the series were determined mostly by collocation and sometimes by least squares—procedures whose limitations, particularly in the matter of systematic convergence and computational labour, are too well known to need further discussion. The analytical procedure now used eliminates the disadvantages of the above two methods. It is simple and ensures systematic and rapid convergence with relatively little computation.

Our method may be described as one of generating simple harmonic polynomials to deal with straight edge boundary conditions. As such it finds wide application in dealing with rectilinear fields governed by Laplace, Poisson and biharmonic type equations. This procedure appears to have been effectively exploited for the first time in the study of diffusion of loads into swept panels.[5] It has since been applied to a wide range of problems as in references 6 and 7 and in other work to be published.

2. Solution

Referring to the regular polygon of n-sides in Fig. 1, considerations of symmetry would

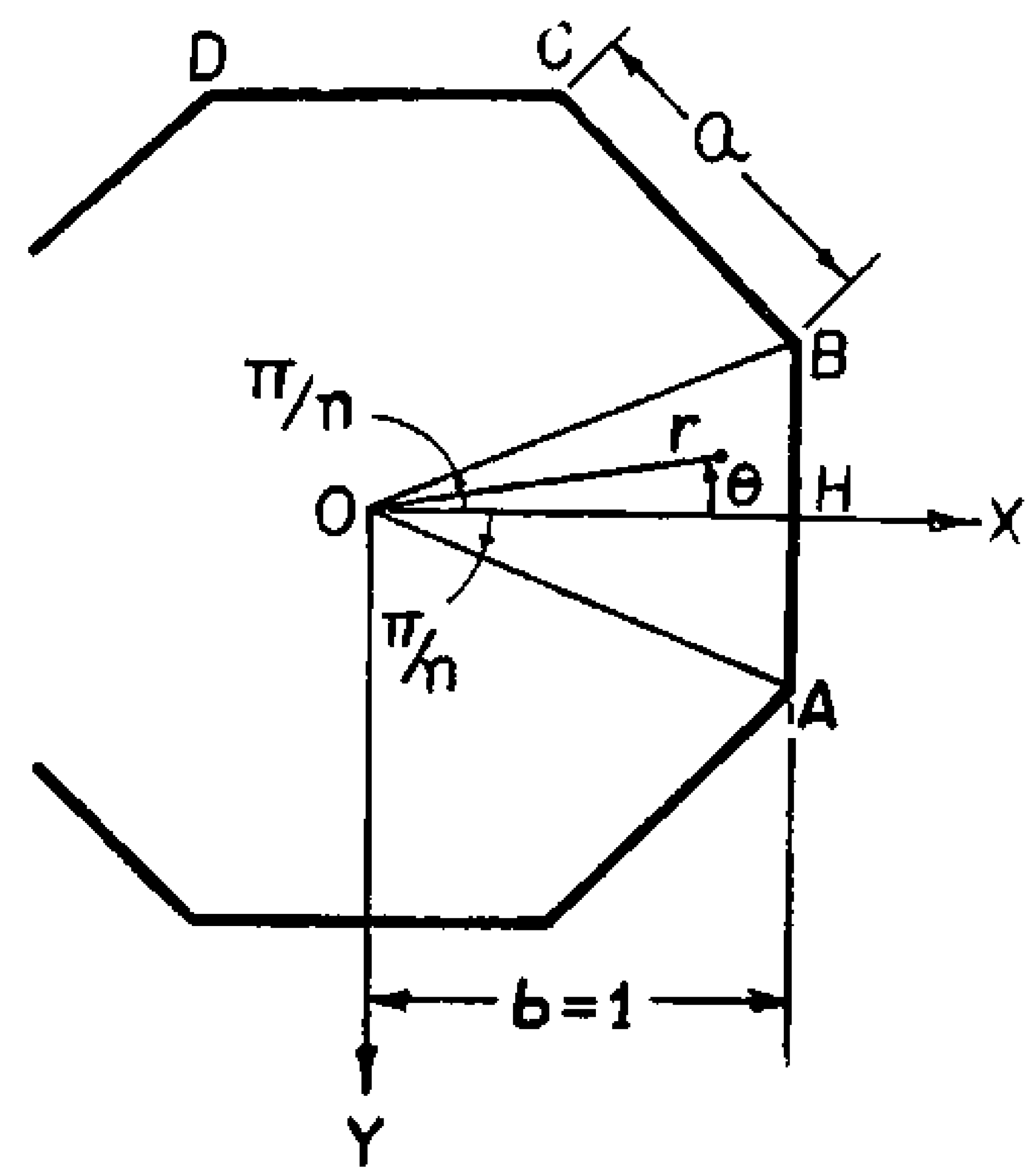

FIG. 1

indicate that the St. Venant torsion problem for such a cross-section is solved when a stress function ψ is determined in the triangular region AOB to satisfy the differential equation

$$\nabla^2\psi = -2 \tag{1}$$

and the boundary conditions,

$$\psi_\theta = 0 \text{ on OA, OB } (\theta = \mp \pi/n) \tag{2}$$

and

$$\psi = 0 \text{ on AB } (x = b). \tag{3}$$

Without any loss of generality, one may, for convenience, put $b = 1$ and $a = 2 \tan \pi/n$. A solution to identically satisfy Eqns. (1) and (2) can be written down as

$$\psi = \psi_0 - \tfrac{1}{2}r^2 + \sum_{m=1}^{\infty} (-1)^m A_m r^{mn} \cos mn\theta. \tag{4}$$

Application of the second boundary condition, Eqn. (3), will determine the constants A_m and ψ_0. $r^{mn} \cos mn\theta$ can be expanded into polynomials in x and y if either mn is a positive

TABLE I

Details of solution for square section

Sl. No.	Type of solution	$A_5 \times 10^5$	$A_4 \times 10^4$	$A_3 \times 10^3$	$A_2 \times 10^2$	$A_1 \times 10^1$
	Soln. with					
1	1 term	..	..	..	..	0·83333333
2	2 ,,	..	..	..	0·12755102	0·89285714
3	3 ,,	..	..	0·04984051	0·16447368	0·90460526
4	4 ,,	..	0·02907996	0·07920827	0·17821861	0·90837071
5	5 ,,	0·02056817	0·05529571	0·09514961	0·18432020	0·90992376
6	Exact values from Eqns. (157) and (160) in Ref. 8					

Sl. No.	Type of solution	ψ_0	Stiffness (J/a^4) Value	Error %	Maximum stress ($\sigma/G\theta' b$) Value	Error %
	Soln with					
1	1 term	0·58333333	0·1361111	3·177	1·3333333	1·281
2	2 ,,	0·58801020	0·1394700	0·788	1·3469388	0·273
3	3 ,,	0·58886563	0·1401375	0·313	1·3492824	0·100
4	4 ,,	0·58913118	0·1403577	0·156	1·3499948	0·047
5	5 ,,	0·58923900	0·1404514	0·090	1·3502813	0·026
6	Exact values from Eqns. (157) and (160) in Ref. 8		0·1405777		1·2506294	

TABLE II

Details of solution for regular hexagonal section

Sl. No.	Type of solution	$A_5 \times 10^5$	$A_4 \times 10^4$	$A_3 \times 10^3$	$A_2 \times 10^2$	$A_1 \times 10^1$
	Soln. with					
1	1 term	..	..	..	..	0·33333333
2	2 ,,	..	..	..	0·116[illegible]5012	0·38461538
3	3 ,,	..	..	0·08427443	0·17365633	0·40114613
4	4 ,,	..	0·07966215	0·15693358	0·20367263	0·40840785
5	5 ,,	0·08679712	0·17803708	0·20982224	0·22199279	0·41521086
6	Exact values quoted by Conway (Ref. 3)					

Sl. No.	Type of solution	ψ_0	Stiffness (J/a^4) Value	Error %	Maximum stress ($\sigma/G\theta' b$) Value	Error %
	Soln. with					
1	1 Term	0·53333333	1·00[illegible]9263	2·80	1·2000000	2·11
2	2 ,,	0·53729604	1·0243418	1·12	1·2167832	0·74
3	3 ,,	0·53846232	1·0295927	0·61	1·2213659	0·36
4	4 ,,	0·53895203	1·0318390	0·39	1·2232376	0·22
5	5 ,,	0·53929405	1·0333899	0·24	1·2246629	0·09
6	Exact values quoted by Conway (Ref. 3)		1·0359		1·22581	

integer or θ is within the range $(-\pi/4)$ to $(+\pi/4)$. For a regular polygon, mn being a positive integer, we make the expansion and rewrite the stress function and the Eqn. (3) as

$$\psi = \psi_0 - \tfrac{1}{2}(x^2 + y^2) + \sum_{m=1}^{\infty} (-1)^m A_m \sum_{p=0,2,4,\ldots} (-1)^{p/2} C_p^{mn} x^{mn-p} y^p. \tag{5}$$

and

$$[\psi]_{x=1} = 0 = \psi_0 - \tfrac{1}{2} - \tfrac{1}{2} y^2 + \sum_{p=0,2,4,\ldots} (-1)^{p/2} y^p \times \sum_{m} (-1)^m C_p^{mn} A_m \tag{6}$$

where C_p^{mn} is the coefficient of y^p in $(1+y)^{mn}$ and ψ_0 is the value of ψ at the origin.

Eqn. (6) yields the following set of linear simultaneous equations in ψ_0 and A_m whose solution completes the desired analysis:

$$\begin{aligned}
p = 0 \quad &: \quad \psi_0 + \sum_{m=1}^{\infty} (-1)^m A_m = \tfrac{1}{2} \\
p = 2 \quad &: \quad \sum_{m=1}^{\infty} (-1)^m C_2^{mn} A_m = -\tfrac{1}{2} \\
p = 4, 6, \ldots &: \quad \sum_{m \geqslant p/n} (-1)^m C_p^{mn} A_m = 0
\end{aligned} \tag{7}$$

In practice the series in the stress function is restricted to a finite number M and hence the order of Eqn. (7) is $(M+1)$.

The torsion constant (J/a^4) for the regular polygon is easily determined as

$$\frac{J}{a^4} = \frac{2n}{(2\tan\pi/n)^4} \int_{\Delta BOH} 2\,\psi dA$$

$$= \frac{h}{4(\tan\pi/n)^4} \int_{x=0}^{1} \int_{y=0}^{x\tan\pi/n} \psi dx\,dy$$

$$= \frac{n}{4(\tan\pi/n)^3} \Big[\tfrac{1}{2}\psi_0 - \tfrac{1}{8} - \tfrac{1}{24}(\tan\pi/n)^2$$

$$+ \sum_{m=1}^{\infty} (-1)^m (mn+2)^{-1} A_m \sum_{p=0,2,4,..} (-1)^{p/2}$$

$$\times\, C_p^{mn} (p+1)^{-1} (\tan\pi/n)^p \Big]$$

$$= \frac{n}{4(\tan\pi/n)^3} \Big[\tfrac{1}{2}\psi_0 - \tfrac{1}{8} - \tfrac{1}{24}(\tan\pi/n)^2$$

$$+ \sum_{m=1}^{\infty} A_m (mn+1)^{-1} (mn+2)^{-1}$$

$$\times\, (\sec\pi/n)^{mn} \Big] \qquad (8)$$

The maximum stress, occurring at $x = b$, $y = 0$, is

$$\frac{\sigma}{G\theta' b} = -1 + \sum_{m=1}^{\infty} (-1)^m mnA_m \qquad (9)$$

3. Equilateral Triangle

It is readily verified that

$$\psi_0 = \tfrac{2}{3},\ A_1 = \tfrac{1}{6},\ A_2 = A_3 = \ldots = 0 \qquad (10)$$

4. Square and Hexagon

Complete numerical analysis has been carried out for the square and the regular hexagon retaining successively 1, 2, 3, 4 and 5 terms of the series. The computations were done on a semi-automatic desk computer with $10 \times 10 \times 20$ capacity. The constants ψ_0 and A_m, the torsion constant (J/a^4) and the maximum stress coefficient $(\sigma/G\theta' b)$ (occurring at the middle points of the sides) are tabulated for the two cases in Tables I and II respectively. Exact values from alternative solutions are also included for comparison. Large numbers of significant figures have been retained to facilitate further comparisons with other methods.

5. Discussion

It is clear from the tables that the convergence of the solution is systematic and rapid. With only a few terms of the series one obtains very close lower bounds for the torsional stiffness as well as for the maximum stress in the section, the rate of convergence being faster for the maximum stress than for the torsional stiffness.

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