MASS QUANTISATION OF HALF-SPIN PARTICLES

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1. Introduction

STEPHENSON[1] has considered the consequences of defining an elementary particle as a region of space-time in which the metric signature is $+4$ while outside the elementary particle the signature is -2. He concludes that a mass spectrum for zero-spin particles can be obtained by using this definition and the Klein-Gordon equation. Lele and Lagu[2] have shown that a similar conclusion can be arrived at by defining the elementary particle as a region of space-time in which the metric signature is $+2$, the metric signature outside the elementary particle being -2. In the present paper an attempt has been made to derive a Fermion (spin ½ particle) mass spectrum by treating the Dirac equation in the same way.

2. Spherical Wave Solutions of the Dirac Equation

Dirac equation for a free particle is

$$E\psi = (c\alpha\cdot p + \beta m_0 c^2)\,\psi \tag{1}$$

where the terms have their usual meanings and E and p should be taken as operators. This can also be written as

$$\frac{1}{c}\frac{\partial\psi}{\partial t} + \sum_{k=1}^{3}\alpha_k\frac{\partial\psi}{\partial x_k} + i\,\frac{m_0 c}{\hbar}\,\beta\psi = 0, \quad i = \sqrt{-1} \tag{2}$$

The above equation is equivalent to four equations for the four components of ψ. To write the equations in spherical polar co-ordinates, we make the following transformations[3]:

$$\left.\begin{aligned} \psi_1 &= i\,(l+m)\,F(r)\,Z_{l-1}^{m} \\ \psi_2 &= -\,i\,(l-m-1)\,F(r)\,Z_{l-1}^{m+1} \\ \psi_3 &= G(r)\,Z_l^{m} \\ \psi_4 &= G(r)\,Z_l^{m+1} \end{aligned}\right\} \tag{3}$$

where

$$Z_l^m = \sqrt{\frac{4\pi}{(2l+1)}\,(l+m)!\,(l-m)!}\;Y_{lm} \tag{4}$$

and Y_{lm} are the well-known Spherical Harmonics. The above equations for a free particle, therefore, reduce to

$$-\frac{(m_0c^2-E)}{\hbar c}\,F + \frac{dG}{dr} + \frac{(l+1)}{r}\,G = 0 \tag{5}$$

and

$$\frac{(-E-m_0c^2)}{\hbar c}\,G + \frac{dF}{dr} - \frac{(l-1)}{r}\,F = 0 \tag{6}$$

where $l = 1, 2, 3, \ldots$ but not zero.

Because we are interested only in static solutions, the above equations reduce to

$$-\frac{m_0c}{\hbar}\,F + \frac{dG}{dr} + \frac{(l+1)}{r}\,G = 0 \tag{7}$$

and

$$-\frac{m_0c}{\hbar}\,G + \frac{dF}{dr} - \frac{(l-1)}{r}\,F = 0 \tag{8}$$

Let $f(r) = r\,F(r)$ and $g(r) = r\,G(r)$, the equations (7) and (8) become

$$-\mu f + \frac{dg}{dr} + \frac{l}{r}\,g = 0 \tag{9}$$

and

$$-\mu g + \frac{df}{dr} - \frac{l}{r}\,f = 0 \tag{10}$$

where

$$\mu = \frac{m_0c}{\hbar}.$$

Substituting the value of f from (9) in (10), and similarly substituting the value of g from (10) in (9), we obtain

$$\frac{d^2f}{dr^2} - \left(\mu^2 + \frac{l\,(l-1)}{r^2}\right) f = 0 \tag{11}$$

and

$$\frac{d^2g}{dr^2} - \left(\mu^2 + \frac{l\,(l+1)}{r^2}\right) g = 0 \tag{12}$$

These are well-known differential equations and the solutions are

$$f = r^{1/2}\,[A_1 I_{(l-1/2)}(\mu r) + B_1 K_{(l-1/2)}(\mu r)] \tag{13}$$

and

$$g = r^{1/2}\,[C_1 I_{(l+1/2)}(\mu r) + D_1 K_{(l+1/2)}(\mu r)] \tag{14}$$

where A_1, B_1, C_1, D_1 are arbitrary constants. I and K are Modified Bessel Functions of first and second kind of the order indicated by suffixes.

Hence the spherical wave static solutions of the Dirac equation for a free particle are

$$\left.\begin{aligned}\psi_1 &= i\,(l+m)\,r^{-1/2}\,[A_1 I_{(l-1/2)}(\mu r) \\ &\quad + B_1 K_{(l-1/2)}(\mu r)]\,Z^{m}_{l-1} \\ \psi_2 &= -i\,(l-m-1)\,r^{-1/2}\,[A_1 I_{(l-1/2)}(\mu r) \\ &\quad + B_1 K_{(l-1/2)}(\mu r)]\,Z^{m+1}_{l-1} \\ \psi_3 &= r^{-1/2}\,[C_1 I_{(l+1/2)}(\mu r) + D_1 K_{(l+1/2)}(\mu r)]\,Z_l^{m} \\ \psi_4 &= r^{-1/2}\,[C_1 I_{(l+1/2)}(\mu r) + D_1 K_{(l+1/2)}(\mu r)]\,Z_l^{m+1}\end{aligned}\right\} \quad (15)$$

3. Mass Quantisation

Following Stephenson we make the imaginary space co-ordinate transformation

$$x \to ix, \quad y \to iy, \quad z \to iz, \quad t \to t$$

that is

$$r \to ir, \quad \theta \to \theta, \quad \phi \to \phi \quad t \to t$$

in the Dirac equation to obtain a condition which leads to mass quantisation. The equations (7) and (8), after transformation, become

$$\left.\begin{aligned} -i\,\frac{m_0 c}{\hbar}\,F_1 + \frac{dG_1}{dr} + \frac{(l+1)}{r}\,G_1 &= 0 \\ \text{and} \\ -i\,\frac{m_0 c}{\hbar}\,G_1 + \frac{dF_1}{dr} - \frac{(l-1)}{r}\,F_1 &= 0 \end{aligned}\right\}$$

and their solutions are

$$\left.\begin{aligned}\xi_1 &= i\,(l+m)\,r^{-1/2}\,[A_2 J_{(l-1/2)}(\mu r) \\ &\quad + B_2 Y_{(l-1/2)}(\mu r)]\,Z^{m}_{l-1} \\ \xi_2 &= -i\,(l-m-1)\,r^{-1/2}\,[A_2 J_{(l-1/2)}(\mu r) \\ &\quad + B_2 Y_{(l-1/2)}(\mu r)]\,Z^{m+1}_{l-1} \\ \xi_3 &= r^{-1/2}[C_2 J_{(l+1/2)}(\mu r) + D_2 Y_{(l+1/2)}(\mu r)]\,Z_l^{m} \\ \xi_4 &= r^{-1/2}\,[C_2 J_{(l+1/2)}(\mu r) + D_2 Y_{(l+1/2)}(\mu r)]\,Z_l^{m+1}\end{aligned}\right\} \quad (16)$$

where A_2, B_2, C_2, D_2 are again arbitrary constants. J and Y are Bessel Functions of first and second kind of the order indicated by suffixes.

Thus the wave function of the elementary particle within a sphere of radius R is given by (16) while outside the sphere it is given by (15). To exclude singularities at infinity in ψ and at the origin in ξ, we put $A_1 = C_1 = 0$ and $B_2 = D_2 = 0$ respectively. Imposing the requirement of continuity at $r = R$ which is

$$(\psi_i)_{r=R} = (\xi_i)_{r=R}$$

and

$$\left(\frac{\partial \psi_i}{\partial r}\right)_{r=R} = \left(\frac{\partial \xi_i}{\partial r}\right)_{r=R}$$

where R is the radius of the sphere across which the metric signature changes and $i = 1, 2, 3, 4$, we find four conditions which are identical in pairs and hence we get only two conditions. No specific conditions for mass quantisation can be obtained unless the value of l is fixed. Taking a simple case, we put $l = 1$, which is the lowest possible value of l, and obtain the two conditions as

$$\tan \mu R = -1 \qquad \text{for} \quad i = 1, 2 \qquad (17)$$

and

$$\tan \mu R = \frac{\mu R}{\mu R + 2} \quad \text{for} \quad i = 3, 4 \qquad (18)$$

The solutions for equation (17) are

$$\mu_n R = \frac{3\pi}{4} + n\pi \qquad (19)$$

where n is 0, 1, 2, 3, ... and the solutions of equation (18) can be found by approximate methods. These reduce to

$$\mu_n R = \frac{\pi}{4} + n\pi \quad \text{for large } n. \qquad (20)$$

4. Conclusion

It has been shown for zero-spin particles[2] that both real and imaginary time co-ordinate transformations lead to the same condition for mass quantisation although the two transformed wave equations become invariant under orthogonal and Lorentz transformations respectively. It can be shown for half-spin particles considered here, that the conditions (19) and (20) can again be obtained by considering the transformation

$$x \to ix, \quad y \to iy, \quad z \to iz, \quad t \to it$$

also.

It is interesting to note that the solution (19) is identical with that obtained by Stephenson for zero-spin particles. Further the solutions (19) and (20) lead to mass quantisation independently, but are incompatible with each other. An interpretation of this interesting result and the invariance of the wave equations will be discussed later.

The authors are grateful to Dr. P. Sah of the Department of Mathematics for taking keen interest in the work.

1. Stephenson, G., *Nature*, 1964, **201**, 589.
2. Lele, S. V. and Lagu, A. V., *Curr. Sci.*, **1964**, **33**, 520.
3. Persico, E., *Fundamentals of Quantum Mechanics*, Prentice-Hall, Inc., New York, 1954, p. 430.