Research Notes.

Zeros of Legendre-Polynomials.

SZEGO (*Trans. Am. Math. Soc.*, **39**, 1-17) has derived very interesting results concerning the zeros of Legendre-polynomials, Bessel's functions and certain trigonometrical polynomials in an entirely elementary way. The method consists in applying in a modified form the classical theorem of Sturm concerning the zeros of functions defined by means of a linear second order equation. [For an analogous method, see *e.g.* Courant-Hilbert, *Methoden der Mathematischen Physik.*, Ed. II.] The modified form runs as follows: Let $y(x)$ and $Y(x)$ satisfy the equations $y'' + f(x)y = 0$, $Y'' + F(x)Y = 0$. Further let $y(x) > 0$ in $a < x < b$, $y(b) = 0$ and $\lim_{x \to a+0} \{y'Y - yY'\}$ exists and $\geqslant 0$. Then either Y is identically zero or else it will be negative in some subintervals of $\{a, b\}$. As an immediate consequence of this is the following theorem, *viz.*, if $\phi(x)$ be continuous and decreasing then the sequence of zeros of $y(x)$, any solution of $y'' + \phi(x)y = 0$ is convex. (Szego makes the conditions on $\phi(x)$ less restrictive.) The method of application of these results consists in comparing the equations that are satisfied by $P_n(x)$ or $J_n(x)$ with equations, the nature of the roots of the solutions of which are known. For instance, the second order equation satisfied by $y = \sqrt{\sin\theta} \times P_n(\cos\theta)$ is $y'' + \{(n+\tfrac{1}{2})^2 + (2\sin\theta)^{-2}\}y = 0$; this is compared with the equation $y'' + (n+\tfrac{1}{2})^2 y = 0$. By applying these two theorems Szego proves the results of Markoff-Bruns-Stieltjis in a slightly sharpened form. He has proved that if θ_ν is the νth zero of $P_n(\cos\theta)$ in the interval $(0, \pi/2)$ then

$$\frac{(\nu - \frac{1}{4})\pi}{n + \frac{1}{2}} < \theta_\nu < \frac{\nu}{n+1}\pi.$$

Analogous results are proved in the case of Bessel's functions of all orders and associated Legendre functions. By comparing the equations satisfied by $J_0(x)$ and $P_n(\cos\theta)$, he has also deduced interesting relations connecting the zeros of these two functions. In the second part of the paper he has obtained results concerning the uniformity of distribution of zeros of polynomials of the type

$$\sum_{(t)} \lambda_{m-t} \begin{matrix}\cos\\ \sin\end{matrix} \theta t \quad \text{and} \quad \sum_{(t)} \lambda_{m-t} \begin{matrix}\cos\\ \sin\end{matrix} (t+\tfrac{1}{2})\theta$$

where λ_r is a monotonic decreasing sequence.

For the proofs the only result that is used is the known elementary result that $\sum_{m=0}^{\nu} \sin \frac{(2m+1)\theta}{2}$ is $\geqslant 0$ for all ν. Sharper inequalities are derived for the case when the λ's form a convex sequence. He has also obtained simple proofs, with generalisations of some known theorems of Polya and of some results concerning the regularity of distribution of zeros of polynomials of the type

$$\sum_{0}^{[n/2]} a_k a_{n-k} \cos(n-2k)\theta \quad (a > 0).$$

He has obtained some more precise results in case the a_n's are capable of being defined as the moments {interval (0, 1)} of a positive function.

K. V. I.

Trigonometric Series and Power Series with (Multiply) Monotonic (*Mehrfach-monotone*) Sequence of Coefficients.

FEJER (*Trans. Am. Math. Soc.*, **39**, 18-59) has obtained a series of theorems concerning polynomials and power series of the type mentioned in the title in a very elementary way. He has deduced a great number of results concerning the rest series [*e.g.*, $\Sigma\, c_\nu \cos(\nu + r)\theta$] and in particular has obtained the result of Szego concerning the distribution of zeros of $P_n(\cos\theta)$. There are some results concerning the positivity of partial sums of various orders of trigonometric polynomials whose coefficients are multiply-monotonic. Incidentally he has given a very simple proof of Heine's sine-series derivation for $P_n(\cos\theta) = \Sigma\, c_\nu \sin(n + 2\nu + 1)\theta$, and has obtained the interesting result that the sequence c_ν is 3-ply monotonic. He has made this too evident by obtaining the formula

$$\frac{\pi}{4} \cdot c_\nu = \Sigma \frac{a_k a_{n-k}}{2k + 2\nu + 1}$$

where

$$a_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k}.$$

(It can also be obtained by expressing Heine's formula for c_ν in terms of partial fractions.) It is not possible to reproduce here all the important results contained in the paper. The following illustrate the types of important results obtained:—

1. If a_ν is a monotonic sequence then the cesaro-means of the first order of the sum of the series $\Sigma\, a\nu \sin(2\nu+1)\theta$ are all non-negative.

2. If $f(\theta)=\Sigma\, a_\nu \sin(\nu+1)\theta$, and $\lim_{n\to\infty} a_n=0$. Then $f(\theta)$ cannot be throughout negative in any interval $(0, \alpha)$.

3. If a_ν is a positive non-increasing sequence, and $\lim a_n = 0$, then if $\theta_1, \theta_2, \cdots \theta_n$ are the zeros in $(0, \pi)$ of

$$f(\theta) = \sum_0^\infty a_\nu \sin(n + 2\nu + 1)\theta \text{ are such that}$$

$$\frac{(k-1)\pi}{n} < \theta_k < \frac{k\pi}{n}.$$

If, moreover, a_ν is 2-ply monotonic (then the zeros are symmetrically distributed about $\pi/2$) then

$$\frac{(m-1)\pi}{n+1} < \theta_m < \frac{m\pi}{n+1};$$

if the sequence is 3-ply monotone

$$\frac{(m-\frac{1}{2})\pi}{n} < \theta_m < \frac{m\pi}{n+1}.$$

There are two results concerning the *schlicht* nature of power-series whose coefficients are monotonic. These are,

If $f(z)=\sum_1^\infty c_\nu\, z^\nu$, and c_ν is 4-ply monotonic, then $f(z)$ is *schlicht* in the unit-circle; and if $f(z)$ is an odd-function then $f(z)$ is *schlicht* in the unit-circle even if the coefficient-sequence is only 3-ply monotonic.

These results are proved by investigating the *bild-curves* of $|z| = r < 1$, and showing that they have no multiple points.

K. V. I.

The Neutrino Theory of Light.

FOLLOWING a suggestion of De Broglie (*Comptes Rendus*, 1934, 199, 813) but altering his ideas in an essential way, P. Jordan has recently developed a neutrino theory of light. (*Zs. f. Phys.*, 1935, **93**, 464; 1936, **98**, 709 and 759.) The neutrino is conceived as an elementary particle with a spin $\frac{1}{2}\frac{h}{2\pi}$ and a rest-mass zero; its existence was originally suggested by Pauli in order to account for the continuous β-ray spectra of radioactive bodies. (The idea has been employed by Fermi to develop a theory of β-decay.) De Broglie assumed that a quantum $h\nu$ of light was made up of two neutrinos of energy $h\nu/2$ and he attempted to deduce Maxwell's equations for the quantum from the Dirac equation of the neutrino. Jordan, on the other hand, does not consider the quantum as a real entity at all, but thinks that only neutrinos produce the effects usually attributed to a quantum when a pair of them enter simultaneously into the reaction. Thus the absorption of a quantum $h\nu$ is described in the neutrino-theory as either a simultaneous absorption of any two neutrinos of energy $h\nu_1$ and $h\nu_2$ such that $\nu_1 + \nu_2 = \nu$ or the absorption of one neutrino $h\nu'$ and the emission of another of less energy $h\nu''$ such that $\nu' - \nu'' = \nu$ (Raman effect of the neutrinos as Jordan calls this latter process). From this point of view Jordan has been able to prove that while the neutrinos having spins of $\frac{1}{2}$ obey Fermi-Dirac Statistics, the apparent entity—the quantum—must follow Bose-Statistics. In the neutrino processes considered above, not only should the conditions $h\nu_1 + h\nu_2 = h\nu = h\nu' - h\nu''$ be satisfied but also the neutrinos $h\nu_1$ and $h\nu_2$ must have parallel spin-axes as also the neutrinos $h\nu'$ and $h\nu''$. Kronig has been able to show how to set up a Lorentz invariant connection between the field, E, H obeying Maxwell's equations and the spinor field $\psi_1, \psi_2, \psi_3, \psi_4$ of the neutrinos which obeys a Dirac wave-equation. Now Jordan has tried to develop the physical significance of his theory by assuming two kinds of neutrinos—positive and negative—and assumes that in every light process the difference between the numbers of positive and negative neutrinos remains constant and shows that the mathematical results of Kronig receive a simple interpretation in terms of this hypothesis. In this way a close analogy has been set up between electron-positron pairs and γ-rays on the one hand and a positive-negative neutrino pair and the quantum on the other hand. By considering a unidimensional cavity to which Thermodynamical considerations are applied Jordan then shows that when and only when there are equal numbers of positive and negative neutrinos do we get a pure radiation field.

T. S. S.

Direct Measurement of the Absolute Amount of Adsorption in Liquid Surfaces.

MCBAIN has been associated with the development of several new experimental methods useful in adsorption studies. To mention a few, we have the quartz spring technique for following the rate of adsorption, the precision beam-type quartz microbalance, the ultra-filtration method for the determination of bound water and the microtome method for measuring adsorption at the plane liquid-vapour interface. In a recent communication (*J. Am. Chem. Soc.*, 1936, **58**, 378), has been described two new methods for measurement of adsorption which involve the use of the Hilger Raleigh Interferometer. The two methods, referred to as the 'compressed surface' and the 'submerged surface' method, are both claimed to be applicable to any type of solution whatsoever. The method has given good results with an aqueous solution of β-phenylpropionic acid.

K. S. G. D.

Determination of Melting Point of Organic Substances.

A COMMUNICATION on the above subject by F. Francis and F. J. E. Collins in the *Journal of the Chemical Society* (1936, p. 137) should be of great interest to all chemists as it discusses in detail the classical determination of melting points by the capillary tube method. The authors have designed an apparatus by means of which different observers can obtain results which do not differ by more than 0·03°. The essential modification, apart from such improvements as electrical heating and mechanical stirring, is in the method of observation for which a telescope incorporating a periscope device is used so that in the same eyepiece both capillary and thermometer scale can be viewed together. There is also described another apparatus by means of which the setting points can be determined correct to ± 0·01° with but 2 gr. of material. It is found that invariably the melting point determined by the capillary tube method is higher than the setting point, but the latter is nearer to the temperature at which the molten material allowed to cool very slowly commences to solidify in the capillary tube. This temperature thus affords a better value for the melting point of a substance than the criterion used at present.

M. A. G. RAU.

Purification of Gallium by Fractional Crystallisation of the Metal.

JAMES I. HOFFMAN AND BOURDON F. SCRIBNER [*Journal of Research of the National Bureau of Standards*, U. S. Department of Commerce, September 1935, **15**, (3), 205] give the results obtained in a systematic investigation of the suitability of a process of fractional crystallisation for obtaining gallium in a state of high purity starting from the metal containing small amounts of various impurities. It was found that most of the impurities (some 20 other metals) tended to concentrate in the crystalline portion, and only a few impurities like silver and lead remained behind in the molten residue. Copper and thallium were found to be distributed about equally between the crystals and the residue. It was also found that the removal of iron, platinum, iridium and lead was impracticable except when the elements were present in very small amounts.

The detection and estimation of the impurities were carried out by examination of the sensitive lines of the elements, employing the stigmatic concave-grating spectrograph described previously by Meggers and his co-workers.

K. R. K.

A Study of Sagger Clays and Sagger Bodies.

RAYMOND A. HEINDL [*Journal of Research of the National Bureau of Standards*, U. S. Department of Commerce, September 1935, **15**, (3), 255] presents in detail the results obtained in a laboratory investigation of various specimens of simple and blended clays, this being undertaken with a view to determine their suitability for the production of saggers of desired quality.

Each of the various grogs employed in the mixtures had been heated to about 1200°C. before being crushed and graded. The test specimens consisted of bars of 1 sq. in cross-section and of different lengths up to 12", and also of oval saggers 4" × 4" × 6" × ½". The tests carried out included (*a*) water of plasticity, (*b*) the volume and linear shrinkage during drying, (*c*) porosity, (*d*) modulus of rupture, (*e*) pyrometric cone equivalent (softening point), (*f*) linear thermal expansion from room temperature to 1000° C., (*g*) relative resistance to thermal shock, (*h*) Young's modulus of elasticity and (*i*) plastic deformation. The addition of talc or magnesia in small quantities to sagger mixes was found to be beneficial but larger additions were injurious owing to their detrimental effect on the refractoriness of clay.

The large amount of valuable data obtained in this investigation are presented graphically as well as in tabular statements.

K. R. K.

(*Research Notes continued on p. 683*)