Recent Researches in the Theory of Meromorphic Functions with Special Reference to the Picard-Borel Theorem.*

Part II.

(Concluded.)

SUPPOSE $f_1(z)$ and $f_2(z)$ be two rational functions which are such that the places at which they take two given values (for simplicity take these to be 0 and ∞) are identical. (The same order of multiplicity.) Then, it is easy to see that they are identical except for a constant. What is the corresponding theorem in the case of two meromorphic functions? The most important theorems in this connection are due to Nevanlinna. His first theorem shows that if for five different a_ν's the equations (1) $f_1(z) = a_\nu$ (2) $f_2(z) = a_\nu$, have identical solutions in z, then the functions are identical. (The functions are meromorphic.) This is proved as follows. Let us introduce the following functions:

$$m_r(f_1, f_2) = \frac{1}{2\pi} \int_0^{2\pi} \log [f_1, f_2]^{-1} \, d\theta.$$

$n(r, f_1 - f_2)$ = the number of zeros of $f_1 = f_2$ in $|z| \leqslant r$. $N(r, f_1 - f_2)$ being defined as before. By the method of proof of the first fundamental theorem it is obvious that

$T(r, f_1) + T(r, f_2) = m_r(f_1, f_2) + N(r, f_1 - f_2)$ $> N(r, f_1 - f_2)$. Let the order of $T(r, f_1)$ be greater than that of $T(r, f_2)$ for definiteness. Then $2\,T(r, f_1) > N(r, f_1 - f_2)$. Let us denote the functions corresponding to $f_1(z)$ by N, m, etc. Then a little consideration will show that $N(r, f_1 - f_2) > \Sigma N(r, a_\nu) - \bar{N}(r)$. $\therefore 2\,T(r, f_1) > \Sigma N(r, a_\nu) - \bar{N}(r)$. But from II, taking $q = 5$, $3\,T(r, f_1) \leqslant \Sigma N(r, a_\nu) - \bar{N}(r) + 0\,[\log r\, T(r)]$. Combining the two we obtain $\underline{\text{Lim}} \dfrac{\log T(r, f_1)}{\log r} < \infty$. $\therefore$ both f_1 and f_2 are rational functions and in that case they are obviously identical. [It is to be noted that we have not at all assumed that f_1 and f_2 take the values a_ν with the same orders of multiplicity. We have merely assumed that they take it at the same places.]

Next we take up the question of the identity of two functions if the distribution for four a_ν's are the same. The results obtained in this case are in a sense incomplete. Nevanlinna has proved that in case the functions take the four values at the same places with the same orders of multiplicity then the functions are identical except in a special case wherein the a_ν's are harmonic and two of them are lacunary values for both the functions; in that case, the two functions are connected by a homographic relation. But no corresponding result in case the restriction about the same order of multiplicity is removed is known. In order to prove these results we have to prove first of all some theorems in connection with meromorphic functions connected by a linear relation and having two lacunary values. This itself is an important chapter in the theory which was started by Borel and developed by Bloch and Nevanlinna; its application to the problems of unicity is due to Polya.

Before proceeding to the proofs of these results we mention a few results which are obvious from the definitions themselves.

(1) $T(r, f) = T\left(r, \frac{1}{f}\right) = T\left(r, \frac{af+b}{cf+d}\right) + 0\,(1)$.

(2) $T(r, f_1 + f_2) \leqslant 2\,T(r, f_2) + T(r, f_2) + 0\,(1)$.

(3) $T(r, f_1 f_2) \leqslant T(r, f_1) + T(r, f_2)$.

Now Picard's theorem can be stated in another form. Suppose an integral function does not take the values 0 and 1; *i.e.*, if $f = e^{g_1}$ $1 - f = e^{g_2}$ where g_1 and g_2 are integral functions, then $e^{g_1} + e^{g_2} = 1$. Picard's theorem asserts that such an equation cannot hold unless when g_1 and g_2 are suitable constants. Borel generalised this result in the following way. Suppose we have a relation of the form $\sum_1^n c_\nu \phi_\nu = 0$ where ϕ_ν's are integral functions which do not take the value zero say. Then, if they are linearly independent, their mutual ratios should be constant. [If they are not linearly independent it should be possible to break up the equation into a number of equations in each of which the functions that occur are linearly independent. The result will be true for each of the new equations.] In order to prove these results we have to deduce the second fundamental theorem in a form deduced by Nevanlinna originally by means of which he deduced II. The theorem is the following:

II'. $m\left(r, \frac{f'}{f}, \infty\right) = 0\,[\log r\, T(r)]$ [except for the exceptional intervals. To be always understood].

We give here a new proof by adopting the method of Ahlfors.

Now

$$\lambda(r) = \int_0^{2\pi} \frac{|f'|^2 \rho(f)}{[1 + |f|^2]^2} \, d\theta. \text{ Take}$$

$$\log \rho(f) = 2 \log [f, 0]^{-1} . [f, \infty]^{-1} - \beta \log [\log [f, 0]^{-1} . (f, \infty)^{-1}] + C$$

where C is such that the total density is unity, $\beta > 1$, so that the integral is convergent. Substituting the values for $[f, 0]$, etc., and simplifying we have

$$(1) \quad \lambda(r) = K \int_0^{2\pi} \left| \frac{f'}{f} \right|^2 \left[\log \frac{1 + |f|^2}{|f|} \right]^{-\beta} d\theta$$

and

* Abstract of lectures delivered by K. Venkatachaliengar to the Central College Mathematical Society, Bangalore.

$$(2)\ \frac{1}{2\pi}\int_0^{2\pi}\left[\log\frac{1+|f|^2}{|f|}\right]^{-\beta}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[\log\left(|f|+\frac{1}{|f|}\right)\right]^{-\beta}d\theta$$

$$\geqslant(\log 2)^{-\beta}.\quad \text{[K is a constant.]}$$

Utilising this we obtain

$$\frac{\lambda(r)+K(\log 2)^{-\beta}}{2K\pi}\geqslant\frac{1}{2\pi}\int_0^{2\pi}\left[1+\left|\frac{f'}{f}\right|^2\right]\times\left(\log\frac{1+|f|^2}{|f|}\right)^{-\beta}d\theta.$$

Taking logarithms and utilising the fact that the logarithm of the mean is $\geqslant$ the mean of the logarithms we obtain

$$\log\lambda(r)+0(1)\geqslant m\left(r,\frac{f'}{f},\infty\right)-\frac{\beta}{2\pi}\int_0^{2\pi}\log\cdots\cdots$$

$$[\log(f,0)^{-1}(f,\infty)^{-1}]\,d\theta.$$

$$\text{As }\beta>1>0,\ \geqslant m\left(r,\frac{f'}{f},\infty\right)-\beta\log\cdots\cdot$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log[f,0]^{-1}[f,\infty]^{-1}\,d\theta.$$

$$\therefore m\left(r,\frac{f'}{f},\infty\right)\leqslant\log\lambda(r)+0(1)+\beta\log[m(r,0)+m(r,\infty)].$$

As $\log\lambda(r)=0[\log r\,T(r)]$ except in the exceptional intervals we have II″. $[m(r,0)+m(r,\infty)<2T(r)]$. [For the deduction of II from II″ see Nevanlinna's excellent tract.]

We state and derive Borel's theorem in a slightly different way which is seen to be the same after a little reflection. Suppose $\phi_1,\phi_2,\cdots\phi_n$ be any n integral functions which are linearly independent ($\therefore$ their Wronskian is $\not\equiv 0$) and which are connected by the linear relation $\Sigma\phi_\nu=1$. Then, if they have a common lacunary value they are all constants. This is proved as follows. For simplicity let us assume that *$n=3$ and the common lacunary value is 0.* We have

$$-(1-\phi_1)+\phi_2+\phi_3=0.\quad \phi_1'+\phi_2'+\phi_3'=0,$$

and $\phi_1''+\phi_2''+\phi_3''=0.$

$$\therefore\frac{1}{\phi_1}=\begin{vmatrix}1 & 1 & 1\\ \frac{\phi_1'}{\phi_1} & \frac{\phi_2'}{\phi_1'} & \cdots\\ \frac{\phi_1''}{\phi_1} & \cdots & \frac{\phi_3''}{\phi_3}\end{vmatrix}\times\begin{vmatrix}\frac{\phi_2'}{\phi_2} & \frac{\phi_3'}{\phi_3}\\ \cdots & \frac{\phi_3''}{\phi_3}\end{vmatrix}^{-1}=\frac{D}{\Delta},\text{say}$$

$\therefore\phi_1=\dfrac{\Delta}{D}$ [let us use the symbol $T_n(r)$ for $T(r,\phi_n)$].

Let $T(r)$ be of the greatest order among all the characteristic functions that occur. Using the results on the characteristic functions of products and sums of functions, and writing

$$\frac{\phi_2''}{\phi_2}=\frac{\phi_2''}{\phi_2'}\cdot\frac{\phi_2'}{\phi_2},\ \text{etc.}\quad\text{we obtain by II″}$$

$$T_1(r)\leqslant N(r,\phi_1,\infty)+N(r,D,\infty)-N(r,D,0)+0[\log r.\,T(r)].$$

Now $D=W\phi_1\phi_2\phi_3$, where W is the Wronskian of the ϕ's. Writing the $N(r,D)$'s in terms of $N(r,\phi)$'s and $N(r,W)$ we obtain $T_1(r)=0[\log r.T(r)]$, utilising the hypothesis that the ϕ's do not take the values 0 or ∞. From this it is clear that the ϕ's are polynomials which contradicts the hypothesis that the ϕ's are not zero. Hence, the ϕ's are constants. [A slight generalisation is possible, *i.e.*, we can assume that ϕ's take 0 and ∞, also. They should only assume them relatively rarely. $[N(r,0)$ and $N(r,\infty)=0[\log r.\ T(r)].]$

Now we apply this result to the theorem on unicity mentioned earlier. f_1 and f_2 are two meromorphic functions which take four given values a_1, a_2, a_3 and a_4 at the same places with the same order of multiplicity. Then we prove either (1) They are identical or (2) two of these are lacunary values for both f_1 and f_2, the cross-ratio $(a_1\,a_2\,a_3\,a_4)=-1$, and f_1 and f_2 are connected by a homographic relation. Let us assume a_4 to be ∞.

$\left[\text{If it is not so, consider }\dfrac{1}{f_1-a_4}\text{ and }\dfrac{1}{f_2-a_4}\right].$ Then

$$\phi_r=\frac{f_1-a_r}{f_2-a_r},\ r=1,2,3$$

are integral functions with the common lacunary value 0. Eliminating f_1 and f_2 we obtain

$$\Sigma(a_2-a_3)\phi_1+\Sigma(a_2-a_3)\phi_2\phi_3=0.$$

Applying Borel's theorem to the six functions $\phi_1\ldots,\phi_2\phi_3\ldots$ all of which exclude the value 0 and dividing them into groups in all possible ways, we obtain one of the following types of alternatives

$$(1)\ \frac{f_1-a_1}{f_2-a_1}=K(\text{const.})\ (2)\ \frac{f_1-a_1}{f_2-a_1}=K.\ \frac{f_1-a_2}{f_2-a_1}$$

$$\text{or }(3)\ \frac{f_1-a_1}{f_2-a_1}=K.\ \frac{f_2-a_2}{f_1-a_2}.$$

If (1) is true then in case the functions are not identical both a_2 and a_3 are lacunary values and as there cannot be more than two lacunary values $a_2-a_1=k\ (a_3-a_1)$ and $a_3-a_1=k(a_2-a_1)$ $\therefore$ $k=1$ or the functions are identical. We can write (2) or (3) as $f_2=S(f_1)$ where S is the homographic transformation; say, $f_2=\dfrac{af_1+b}{cf_1+d}$. Then either ∞ is a lacunary value or else $c=0$. The latter case is disposed off as (1). $\therefore$ ∞ is a lacunary value. $\therefore$ $S(\infty)$ should also be a lacunary value for both. $S(\infty)$ should be one of the three a's or else there would be five values which has already been disposed off. And for the other two a's, $S(a)=a$ obviously. Hence, two of the values are fixed points of the homography and ∞ and the other a are corresponding points. Hence our theorem is completely proved. Nevanlinna has proved that if three functions take three values at the same places with the same orders of multiplicity then at least two of them should be identical. From our analysis it is also clear that there do exist functions having four identical distributions.

FUNCTIONS MEROMORPHIC IN THE UNIT CIRCLE.

Our preceding analysis confines itself to the distribution of values of a meromorphic function in the neighbourhood of an isolated singularity. A generalisation of that would be the problem of distribution of values of a meromorphic function in the neighbourhood of a line singularity; or in a *slightly more general way* we consider the problem of distribution of a function given to be meromorphic in a given region (not necessarily simply connected; and the function need not be one valued but should be capable of being continued indefinitely with the exception of only poles as singularities). In such a case we have to uniformise it; *i.e.*, we shall assume that we have transformed the region to the area of the unit-circle. Then our problem is divided into two. One is the nature of the polymorphic function which transforms the region and the other is that of the distribution of the values of a meromorphic function in the unit-circle. The former does not belong to the subject of the lecture. Therefore we take the latter problem and see how it differs from the earlier case.

The first fundamental *theorem* is easily seen to be true in this case also: but the fact that $T(r) \to \infty$ as $r \to \infty$ is not obviously true. For all functions which are analytic and bounded in the unit-circle $T(r)$ is certainly bounded. Nevanlinna has proved that if $T(r)$ is bounded then the function is the quotient of two bounded functions. [It is of course not necessarily bounded.] The following is a slightly simplified version of Nevanlinna's proof. $\therefore T(r)$ is bounded $N(r, 0)$ and $N(r, \infty)$ are both bounded. Evaluating the integral $N(r, 0) = \int_0^r \frac{n(r,0)}{r} dr$, and $N(r, \infty)$ we easily deduce the following; *i.e.*, if $r_1, r_2, r_3, \ldots\ldots$ be the absolute values of the roots of $f(z) = 0$, in the Unit-circle (multiplicity being taken into account). We obtain $N(1, 0) = \frac{1}{r_1 \; r_2 \; r_2 \cdots}$, $\therefore$ if $N(r, 0)$ is bounded $\Sigma(1 - r_n)$ is convergent. Let $a_1, a_2, \ldots, a_n, \ldots$ be the sequence of zeros of $f(z)$. $|a_n| = r_n$. Then, we show that the product $f_1(z) = \prod_1^\infty \frac{z-a}{1-z\bar{a}_n}$ converges uniformly in $|z| \leqslant r < 1$. This is easily proved by finding the maximum and minimum values of $\left|\frac{z-a}{1-z a}\right|$ in $1 < |a| < r \leqslant z$. We have

$$\frac{|a|+r}{1+r|a|} > \left|\frac{z-a}{1-\bar{a}z}\right| > \frac{|a|-r}{1-r|a|}.$$

From this, it is easy to show that

$$\Sigma\left[1 - \left|\frac{z-a\nu}{1-z\bar{a}\nu}\right|\right]$$

converges uniformly in the region considered. From this we deduce that the same is true of the product and $f_1(z)$ is a function which is bounded and analytic in the unit-circle. We similarly form $f_2(z)$ with the poles instead of the zeros of $f(z)$ in the unit-circle. Then it is easily seen $f(x) = e^{\psi(z)} \cdot f_1/f_2$, where $\psi(z)$ is analytic in $|z| \leqslant 1$. Now consider the circle $|z| = r$. Let A_r be the set of points on it for which $R(\psi)$ is non-negative and B_r its complement. Then we determine two functions which are analytic in $|z| < r$, say $\psi_1^{(r)}(z)$ and $\psi_2^{(r)}(z)$ which are such that $R\psi_1^{(r)} = R(\psi)$ on A_r and o on B_r and $R\psi_2^{(r)} = R(\psi)$ on B_r and o on A_r. Then in $|z| < r$. $\psi = \psi_1^{(r)} - \psi_2^{(r)}$. [We assume that a suitable imaginary constant is added.] For a sequence $r_n \to 1$ we determine ψ_1 and ψ_2 similarly. Then as $e^{-\psi_1^{(r)}}$ and $e^{-\psi_2^{(r)}}$ are bounded functions by Vitalis theorem [see *e.g.*, Bieberbach—*Lehrbuch der Funktionen Theorie*, Bd. I] there is a subsequence of (r_n) for which both $\psi_1^{(r)}$ and $\psi_2^{(r)}$ converge in $|z| < 1$ to two functions whose real parts are positive. Let these functions be ψ_1 and ψ_3 respectively. Then we can write $f(z)$ in the form $f(z) = \frac{f_1 \; e^{-\psi_2}}{f_2 \; e^{-\psi_3}}$. Both the numerator and the denominator are bounded.

We easily see that all other properties which are derived earlier to II are true for this case also; but the theorems on defective values, etc., are not true without some other restriction. We have examples for which $T(r) = 0 \cdots\cdots \cdots\left[\log \frac{1}{1-r}\right]$ which do exclude any number of values. A Fuchsian function with parabolic substitutions only, *viz.*, the function which transforms a circular polygon whose sides are arcs orthogonal to the unit-circle and which touch each other (on the unit-circle obviously) to the half-plane has the requisite property. [We omit the proof. See Nevanlinna, *loc. cit.*] The next point which needs amendment is the second fundamental theorem. The definition of the exceptional intervals needs amendment. Instead of the exceptional intervals being such that $\Sigma \int_{I_n} \frac{dr}{r}$ is finite (which is obviously meaningless in this case), we should have naturally $\Sigma \int_{I_n} \frac{dr}{1-r}$ is finite. $\left[\text{Note that } \int_0^1 \frac{dr}{1-r} \text{ is divergent.}\right]$ With this definition of the exceptional intervals the second fundamental *theorem assumes the following* form in this case:

$$\overline{\mathrm{II}}\; (q-2)\, T(r) \leqslant \sum_1^q N(r, a_\nu) - N_1(r) + 0$$

$$[\log T(r)] + (1+\epsilon) \log \frac{1}{1-r}. \quad (\epsilon, \text{ any constant } > 0)$$

except in the exceptional intervals. We see therefore that all our theorems, *viz.*, the theorems on defective values, unicity, multiple values, etc., remain the same *provided that* $\underline{\text{Lim}} \frac{T(r)}{\log (1-r)^{-1}} = \infty$. It is already mentioned that in case this is not true the theorems need not be valid. Many of our theorems can be amended suitably if $T(f) = 0 \left[\log \frac{1}{1-r}\right]$. We shall state and prove one such result. [See Ahlfors, *loc. cit.*] Suppose $\underline{\text{Lim}} \frac{T(r)}{\log \frac{1}{1-r}} = p$. Then we prove that the total defect is at most $2 + 1/p$. [See Nevanlinna's tract for examples of functions possessing the preceding property.] Dividing $\overline{\text{II}}$ by $T(r)$ we obtain

$$(q-2) \leqslant q - \Sigma\, \delta (a_\nu) + \frac{1+\epsilon}{p},$$ from which the result is at once apparent.

We close our lecture with one or two slight observations. There is no necessity for exceptional intervals in the case of meromorphic functions of finite order, *viz.*, in case $\frac{T(r)}{r^p} < \infty$ for a definite p. Now for all points which do not belong to the exceptional intervals

$\lambda(r) < [T(r)]^k . r^{k'}$. Suppose ρ is contained in an exceptional interval (a, b). Now $T'(r) = 0\,[r^{p+1}]$ obviously [as $T(r)$ is a convex increasing function of $\log r$]. And by a slight change we can adjust the exceptional intervals in such a way that $\Sigma \int_{I_n} r^{p+1}\, dr$ is finite. Now take an $r < \rho$, not belonging to any exceptional interval. Then

$$T(r) - T(\rho) = \int_\rho^r T^1(r)\, dr < k \int_\rho^r r^{p+1}\, dr.$$

Now we can so adjust r, in such a way that $T(r) - T(\rho) < M$ (independent of ρ). Now $\lambda(r) < [T(r)]^k . [r]^{k_1}$. But $\text{Lim} \frac{T(r)}{\log r} = \infty$. $\therefore r^{k'} < [T(r)]^{\alpha}$ for some α. $\therefore \lambda(r) < [T(r)]^{\beta}$ for some β.

$$\therefore [\lambda(\rho)]^{\frac{1}{\beta}} < [\lambda(r)]^{\frac{1}{\beta}} < [T(r) - T(\rho)] + T(\rho)$$
$$= T(\rho) + 0\,(1). \quad \therefore \log \lambda(\rho) = 0\,[\log T(\rho)].$$

We close the lectures with the remark that $T(r, f')$ is of the same order as $T(r, f)$ except in the exceptional intervals. This is too apparent if we put $f' = f . f'/f$ and apply II.

It is to be noted that we have unavoidably omitted many of the other branches of the subject. For a complete study the following books are recommended: (1) Valiron, *Lectures on the Theory of Integral Functions*, (2) Nevanlinna's tract, *loc. cit.* The latter is really a monumental book and also contains the bibliography till the year 1929.

Population Problem and Policy in India.

THE first Indian Population Conference was held at Lucknow on February 3 and 4, under the auspices of the Indian Institute of Population Research. A large number of delegates from the Universities, Provincial Governments and States attended. The Conference was convened by Dr. Radhakamal Mukerjee.

In his address of welcome to the delegates, Dr. R. P. Paranjpye, Vice-Chancellor, Lucknow University, emphasised the importance of the question of population in India in its quantitative, economic and biological aspects as underlying all sound progress. What the country wants, he observed, is a healthy vigorous population, every member of which should have a reasonable chance of living to a healthy old age and contribute to the general happiness of the people. For this, an adequate supply of nutritive food and other conditions of healthy life should be available to all, and the optimum population of a country should be determined by reference to these conditions.

In his inaugural address, the Hon'ble Mr. J. M. Clay, Finance Member, U.P. Government, traced how the pressure of population had been the motive power behind the innumerable migrations and incursions of the human race from prehistoric times. In Europe, we have Italy and Germany claiming the right to expand with their overflowing populations into Africa. In Asia, we find Japan following a similar policy towards China. In India itself, the rapid growth of population presents a problem serious enough to demand the earnest thought of her public men. At the last census of 1931, the population of the sub-continent was 352 millions; it has now increased to at least 370 millions; and unless some retarding factor impedes its natural progress, it will probably exceed 400 millions at the next enumeration in 1941. Indeed, it is not impossible that India may, before the 20th century is much more than half way through, have to support a population equal to that of China. These are staggering figures; they connote problems of the first magnitude for Government and for every thinking man.

Prof. Radhakamal Mukerjee, in the course of his address as convener, discussed at length the problem of India's population capacity. Prof. Mukerjee estimated that India's present food shortage was 18.4 billion calories and the present number of average men estimated without food in India, assuming that others obtained their normal daily ration, was 6.6 millions. India had 162 acres of waste lands which might grow food under an unremitting population pressure, but this could not increase the country's population capacity beyond 441 millions of persons.

Reviewing the growth of population in the country during the last 64 years, Prof. Mukerjee