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# SOME RESULTS IN NON-LINEAR VIBRATION OF STRETCHED STRINGS

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FOR more than two thousand years, the vibrations of stretched strings fascinated mathematicians, physicists and musicians alike. Pythagoras, Euler, D'Alembert, Helmholtz, Raman and many others made significant contributions to our knowledge of their behaviour. Conceptually simple and elegant, the vibration of a stretched string served as a mathematical model for our understanding of intuitively less tractable notions like electromagnetic waves and atomic wave functions when they were first conceived. The study of vibrations of strings of musical instruments not only marked the beginning of Raman's scientific work, but remained close to his heart throughout his long scientific career. It embodies in a sense his ethos of scientific work, *viz.*, simplicity of experimentation which marked all his work. It is in this spirit that we summarize here some recent results obtained on the non-linear behaviour of strings at the Indian Institute of Science,[1-5] of which he was the head for several years.

## RESONANCE RESPONSE OF STRETCHED STRINGS

If a stretched string is driven by a harmonic force $F \cos \omega t$ in the $xy$ plane as shown in Fig. 1, the classical theory tells us that the string would vibrate in the $xy$ plane and its frequency response would be as given by the curve AR..ST of Fig. 2. As a resonance frequency of the string is approached, the amplitude would grow to infinite values but for some amount of damping that is always present.

Actually, experiment reveals something entirely different. It is extremely simple to obtain the frequency response of a string by applying a constant magnetic field in a horizontal direction at right angles to the string and passing an alternating current from an oscillator of variable frequency. A force $F \cos \omega t$ then acts in the vertical or $y$-direction and the amplitude of the string can be measured with a microscope. If we start a little below a resonance frequency $\omega_n$ and slowly increase the frequency of the driving force, we find that the amplitude follows the curve AR in the beginning but before approaching the resonance it takes a path like APG....After reaching some point G on this curve, the amplitude suddenly falls down to the point E on the curve ST obtained from the linear theory and then continues on this curve. If the frequency $\omega$ of the driving force is decreased now, the amplitude does not retrace the path TEGPA, but follows the path TED'B'PA. At the point D' the amplitude suddenly jumps to the point B' after which the original curve B'PA is retraced.

In addition to the jumps in amplitude exhibited by the frequency response curve, the motion of the string ceases to remain in the plane of the driving force (the $xy$ plane) when a certain critical amplitude is reached. At some point P along the curve the string acquires a component of vibration in the perpendicular plane $xz$. This $z$-component of vibration increases continuously along with the $y$-component. The motion of the string which is planar until this stage becomes gradually a whirling, non-planar motion until the jump occurs at G, after which it becomes planar again. If the frequency of the driving force is decreased, the non-planar motion reappears after the upward jump takes place at D'. Unless the string is heavily damped, the non-planar motion and the jumps near the resonance frequency are always observed, however small the driving force may be. These are typical non-linear features which the classical theory cannot explain even qualitatively.

## NON-LINEAR EQUATIONS OF MOTION

Beginning with Carrier, the non-linear equations of motion of a string have been written down to different degrees of generality by several investigators. They differ essentially from the usual linear equations in considering a deformation of an element of the string into a space curve (instead of a plane curve as is normally done) and taking into account the local change in tension arising from an increase in the length of the element in accordance with Hooke's law. The momentum equations in the $x$, $y$ and $z$ directions lead to

$$u_{tt} = c_1^2 u_{xx} + c_1^2 \frac{\partial}{\partial x} \{\tfrac{1}{2} (y_x^2 + z_x^2)\}, \quad (1)$$

$$y_{tt} = c_0^2 y_{xx} + c_1^2 \frac{\partial}{\partial x} \{[u_x + \tfrac{1}{2} (y_x^2 + z_x^2)] y_x\}, \quad (2)$$

$$z_{tt} = c_0^2 z_{xx} + c_1^2 \frac{\partial}{\partial x} \{[u_x + \tfrac{1}{2} (y_x^2 + z_x^2)] z_x\}, \quad (3)$$

where $u$ is the longitudinal displacement, $y$ and $z$ are transverse displacements, $c_0 = (T/m)^{\frac{1}{2}}$ is the linear transverse velocity and $c_1 = (EA/m)^{\frac{1}{2}}$ is the linear longitudinal velocity. The first terms on the right-hand side of (1), (2) and (3) represent the usual linear equations of longitudinal and transverse motions. The second terms represent the coupling that arises between the longitudinal and transverse motion as well as between the two components of the transverse motions.

### SOLUTIONS OF FORCED MOTION

The last term on the right-hand side of Eqn. (1) represents an effective longitudinal force. Since, in most cases of practical interest, the fundamental longitudinal resonance frequency is small compared to the fundamental transverse resonance frequency, the frequency of the effective longitudinal force is normally well below the fundamental longitudinal resonance frequency. The term $u_{tt}$ in Eqn. (1) can, therefore, be ignored. This approximation facilitates separating the equations for longitudinal and transverse modes. However, the equations for the mutually perpendicular transverse vibrations cannot be separated.

The harmonic response to a simple harmonic driving force of frequency $\omega$ acting in the $xy$ plane can now be considered. For simplicity, we assume that the fundamental linear resonance frequency is unity and we seek solutions for the case $\omega \simeq 1$. Assuming the solutions

$$y = A_y \cos \omega t,$$
$$z = A_z \cos (\omega t + \beta), \quad (4)$$

substituting into the equations of motion, and applying the principle of harmonic balance, it is found that two distinct solutions are possible. These are

$$\eta A_y^3 - (\omega^2 - 1) A_y = F,$$
$$A_z = 0, \quad (5)$$

corresponding to a planar motion in the plane of the driving force and

$$\eta A_y^3 - \tfrac{3}{4} (\omega^2 - 1) A_y = \tfrac{3}{8} F,$$
$$\eta A_z^2 = \omega^2 - 1 - \frac{1}{3} \eta A_y^2, \quad \beta = \frac{\pi}{2}, \quad (6)$$

corresponding to elliptical motion. The parameter $\eta$ appearing in Eqns. (5) and (6) is a parameter of non-linearity.

The $y$-response curves ($|Ay|$ *vs.* $\omega$) for the two types of motion are shown in Fig. 3. The curve APB....CDE is the response curve for planar motion and A'PB'...C'D'E' corresponds to non-planar motion. From the second of Eqns. (6), it is evident that non-planar motion can exist only if $\eta A_y^2 < 3 \ (\omega^2 - 1)$. The boundary curve corresponding to

$$\eta A_y^2 = 3 (\omega^2 - 1) \quad (7)$$

is shown as QPF in Fig. 3. Non-planar motion cannot exist to the left of this curve. Hence, for a given amplitude of the driving force, as the driving frequency $\omega$ is increased, the response must remain planar until the curve QPF is reached. To the right of QPF, the response may be either planar or non-planar. However, an analysis of stability shows that planar motion is unstable in the region bounded by the curves QPF and QDH in Fig. 3, while non-planar motion is stable only in the region bounded by the curves QPF and QG. The equations of the curves QG and QDH are respectively

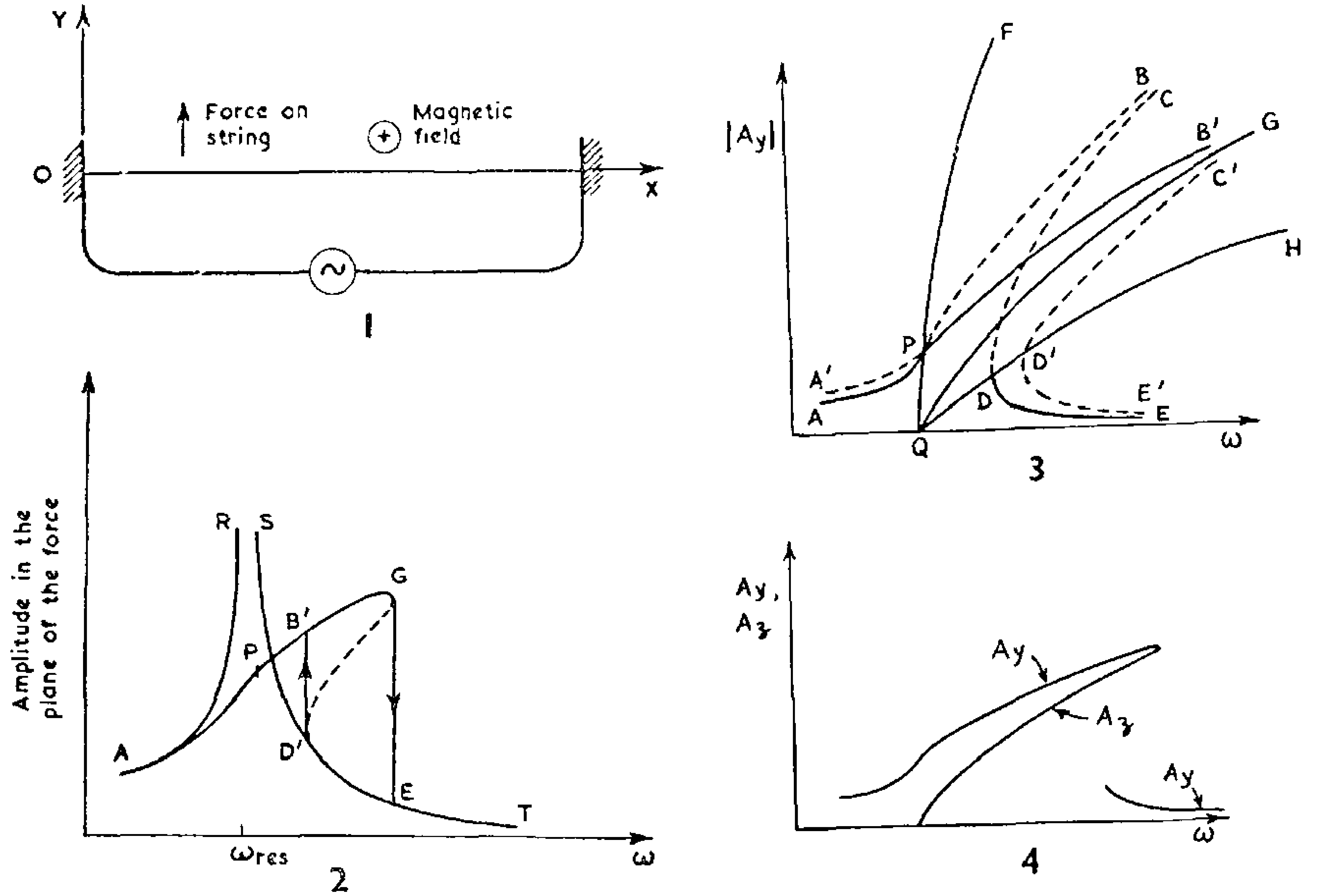
$$\eta A_y^2 = \tfrac{3}{4} (\omega^2 - 1), \quad (8)$$

and

$$\eta A_y^2 = \tfrac{1}{3} (\omega^2 - 1). \quad (9)$$

Hence, as the frequency is increased from a small value, the response curve actually traced is APB'. Due to the damping which is always present, the branches PB' and D'C' do not extend indefinitely but meet at some point close to QG. A downward jump in amplitude occurs at this point of intersection. This jump is accompanied by a transition to planar motion since non-planar motion is unstable below the curve QG. A decrease in frequency after the downward jump causes the response to increase along the curve ED until the point D is reached. At this point, the planar motion becomes unstable and an upward jump in amplitude occurs accompanied by a transition to non-planar motion.

The behaviour of the amplitude $A_z$ with frequency is shown in Fig. 4. It is seen that A increases with $\omega$ more rapidly than $A_y$ and just before the downward jump $A_y$ and $A_z$ are almost equal.




FIGS. 1-4

A theoretical analysis taking damping into account shows that the resonance frequency of the string is not a constant of the system as predicted by the linear theory, but increases with the driving force. Also, it can be shown that below a critical driving force, non-planar vibrations cannot exist. This critical driving force is very nearly proportional to the cube of the damping factor.

### FREE OSCILLATIONS

Analysis of the problem of free non-linear vibrations of the string also yields many interesting results. While the analysis for the most general initial conditions is very complicated all the salient features of non-linear behaviour can be brought out by studying the case of sinusoidal initial conditions. An approximate solution of this case can be obtained by assuming the amplitude and phase to be slowly varying functions of time. Such an analysis (ignoring damping) shows that $A_y^2$ and $A_z^2$ are sinusoidally varying functions of time.

$$A_y^2 = a_1^2 + a_2^2 \sin(pt + \phi),$$
$$A_z^2 = a_1^2 - a_2^2 \sin(pt + \phi), \qquad (10)$$

where $a_1^2$, $a_2^2$, $p$ and $\phi$ are functions of the initial values. Further, $p$ is proportional to the non-linearity parameter which means that $p$ is very small compared to the natural frequency of the string if the non-linearity is small. Equations (10) can be interpreted as representing an oscillation of part of the energy of vibration between the $y$ and $z$ components while the total energy remains constant. Also, the average energy of both components is equal and independent of the initial distribution of energy. It follows that the string cannot execute planar free vibrations since the quantity $a_2$ [appearing in Eqn. (10)] can never be made zero for any set of initial values. In fact, a stability analysis shows that the only stable periodic mode of vibration is the circular mode in which each point on the string describes a circle in the YZ plane. In general, the locus of a point on the string is an ellipse with slowly rotating axes, The results mentioned above, some of which are quite striking, bring into sharp focus the changes introduced by non-linearity.

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## SOME REFLECTIONS ON PROFESSOR RAMAN

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I HAD the privilege of being associated with Prof. Raman for over a decade from 1949-60. This was a period in which his great dream of establishing an independent research institute took the final shape. He devoted all his energy and time to the fulfilment of this dream. The building was just about ready in 1949 and the immediate task was to arrange the priceless crystals and mineral specimens that he had acquired. Soon began the flow of scientific apparatus and research students into the laboratory and the Raman Research Institute became an active centre for research. Professor Raman never stopped acquiring crystal specimens, nor building up the library. On both these counts he was lavish. Until death he was passionately involved in scientific research and towards the end he made inroads into fields of activity, conventionally not regarded as physics. But to him boundaries in science were artificial. Throughout his life, colour and order in Nature dominated his thought.

Professor Raman's love for crystals is universally known. There are very few scientific men in India who have not at one time or other seen his exquisite collection. The crystallographic and mineralogical museum at the Raman Research Institute is probably one of its kind in Asia and had a dual purpose. The specimens were not only for display but also meant for research. Professor Raman had an insatiable curiosity to understand crystal optical phenomena and was very much attracted by gems and gemstones. During my stay I had the privilege of assisting him in many of the optical and X-ray studies on gemstones. He had a fascination for opals, labradorite and moonstones. He explained the origin of the optical effects exhibited by iridescent feldspars, opals and agates and his researches on diamond are widely known. These studies fill volumes and have been issued as Memoirs of the Raman Research Institute.

He often worked with very simple equipment. For instance, when he needed a powerful beam of light he always turned to the sun. It took a year-and-a-half after I joined, to bring electricity to the Institute. This did not deter Professor Raman from doing research. A pair of polaroids, a heliostat to reflect the sunlight, a pocket spectroscope, an optical cell and a darkened room, were all that he needed to start a study of even the most complex optical phenomena.

For us the most enjoyable and rewarding moments were those when we were closeted with him, while he was making some observations and thinking aloud. During those moments, the most abstruse optical principles became crystal clear to us, an education that cannot come studying voluminous treatises on the subject. As his students and associates, we not only benefited by his guidance and suggestions in relation to a particular problem, but more importantly, we learnt the methodology of research. He advocated that if a particular problem was interesting, one should straight plunge into it irrespective of what other investigators have said or found. An unprejudiced mind can unearth something new, since each individual looks at a problem from a different angle. Time and again this has proved to be true within my own experience.

A year ago I heard a talk by a senior colleague of mine in Bell Laboratories and the title was, "How to do Nobel Prize Winning Work", indeed a most unusual title for a talk. It was a very interesting lecture and the prescriptions were not that simple. The lecturer surveyed the work and personal characteristics of many Nobel Prize winners and brought