Detailed measurements of related optical and electrical characteristics are under way and will be published in due course.

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RELATIVISTIC INVARIANCE AND DISCRETE SYMMETRIES*

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I HOPE to outline to you in this lecture briefly and rather imperfectly the method whereby geometric transformations like rotations and space time translations are implemented as invariance transformations in quantum mechanics. There are good reasons for such a study. The existence of such symmetries implies that of all possible theories, a certain subset sharing specific features is singled out as acceptable. This is an enormous simplification and in fact leads to definite predictions. Actually the significance of geometric symmetries is deeper. Thus, the Hamiltonian which governs the time evolution and hence the dynamics of the theory, corresponds on exponentiation to time translation. Thus, geometry in fact determines dynamics. It may be noted in this context that the conservation of angular or linear momentum is a consequence of geometrical invariance principles which decide the commutation relations of such operators with the Hamiltonian.

This is a conference on unsolved problems. The field which is surveyed now contains perhaps *the* unsolved problems in relativistic quantum mechanics. Let us briefly review a few of these.

1. Previous remarks suggest that the "correct" realization of geometric symmetries will lead in a direct way to the "correct" theory. This task has certainly not been carried out.

2. The analysis of continuous geometric transformations leads to certain relatively definite rules on how they are to be implemented. However, for discrete symmetries like parity P or time reversal T, the situation is more diffuse. In particular, that P is unitary and T anti-unitary is not a consequence of general principles alone, but requires the extra assumption that there are no negative energy states. There are further ambiguities regarding T and PT. It is known that T^2 can be $+1$ or -1. However, in all the theories we deal with in practice, the choice $T^2 = (-1)^{2j}$ for a particle of spin j is made. Whether this hypothesis is binding on us is not clear to me.

3. Experimentally, we know that P and T are not exact invariance properties, that is to say, that they cannot be implemented in the usual theories with the proper commutation relations with the Hamiltonian. Whether the preceding remarks have a bearing on this matter is not clear. If in fact what is observed is the impossibility of implementation of these geometric transformations, the nature of geometry itself may be different from the present concepts.

4. There are a whole group of symmetries like isospin and unitary spin transformations which do not seem to originate in an intimate way from geometry. There have been many attempts to bring these too into the geometric fold, but the successes have been limited. In fact, there are negative theorems (like McGlinn's, Sudarshan and co-workers', O'Raifeartaigh's, etc.) which indicate serious difficulties for such a program within the present framework.

5. Finally we may ask whether relativistic quantum theory should really be formulated so as to be generally covariant. The success in this task has been limited. It is often claimed that gravitation is too weak to be relevant in particle physics. This claim is ambiguous. Let us consider an example from another context to illustrate the ambiguity. At very low energies, we expect relativistic effects to be unimportant and nonrelativistic considerations to suffice. Consider now the CPT theorem which essentially says that CPT is always implementable as a symmetry transformation in a certain class of quantum field theories. Such a theorem cannot be proved using nonrelativistic dynamics although its effects persist at low energies. Conceivably similar effects could occur with general relativity.

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Let us now review how geometrical transformations are implemented in quantum mechanics. The material is almost entirely due to Wigner and to Bargmann.

We will call the group of coordinate transformations which we impose as an invariance group of the theory the relativity group $\mathcal{R}$. Let us now consider what is meant by invariance of a quantum theory under $\mathcal{R}$.

There are fundamentally two different ways we can implement geometric transformations on a system. We can transform the coordinate system leaving the object untouched or we can transform the object leaving the coordinate system untouched. The former view of the transformations is called passive and the latter, active. The two views of the transformations are equivalent only if there is an appropriate uniformity of space and time. For instance, consider a system in an external electrical field along the z-axis. Before rotations, let there be a state ψ with angular momentum component equal to 1 along z and a state ϕ with angular momentum component 1 along x. Then $|(\phi, \psi)|^2$, the probability for transition from ϕ to ψ, is unchanged if the coordinates are rotated, but not if the states are rotated. The point of view we shall adopt will always be the active one.

Consider the Hilbert space $\mathcal{H}$ of the quantum theory under study. Under a coordinate transformation r, each state ψ goes over into a state $U(r)\psi$ where the map $U(r)$ is not necessarily linear or anti-linear. Consider the set $\mathcal{R}$ of all coordinate transformations r with the property that for each $r \in \mathcal{R}$, we can find a map $U(r)$ such that $|(\phi, \psi)| = \{[U(r)\phi, \; U(r)\psi]\}$ for all ϕ, ψ in $\mathcal{H}$. This set is called the *relativity group* of the theory.

$U(r)$ is not unique since a state ψ and a state $e^{ia}\psi$ describe the same physical situation. Let us define a "phase operator" Ω depending on a real function on α by

$$\Omega_{\alpha} \; \psi = e^{i\alpha(\psi)} \; \psi.$$

Ω_{α} is not a linear operator unless $\alpha(\psi)$ is independent of ψ.

The non-uniqueness of $U(r)$ can be expressed by stating that both $U(r)$ and $\Omega\, U(r)$ implement the transformation r. We can use this ambiguity to standardize the choice of $U(r)$ in a particularly convenient way.

Theorem 1: There exists a phase operator Ω_{α} [where α may depend on r] such that $\Omega_{\alpha} U(r)$ is either unitary or anti-unitary.

Thus we shall assume hereafter that $U(r)$ is either unitary or anti-unitary.

We have yet to settle which transformations are unitary and which are anti-unitary. To decide this question, we note first that both the operators $U(r_2)\,U(r_1)$ and $U(r_2 r_1)$ implement the same transformation $r_2 r_1$ on $\mathcal{H}$. Therefore, they may differ from each other only by a phase:

$$U(r_2)U(r_1) = \omega(r_2, r_1)U(r_2\, r_1), \; |\omega(r_2, r_1)| = 1.$$

Consider first those $r \in \mathcal{R}$ which can be continuously deformed to identity (pure rotations, translations, etc.). Then, there is a theorem which for our purposes states essentially that each such r is the square of another such s:

$$r = s^2.$$

E.g. Each pure rotation r is of the form $e^{i\theta \vec{n} \cdot \vec{J}}$ which has the "square root" $s = e^{i(\theta/2)\vec{n} \cdot \vec{J}}$ which too is a pure rotation.

From the above follows the important

Theorem 2: For each r which can be continuously deformed to identity, $U(r)$ is unitary.

For we have

$$U(r) = \omega(s, s) \; [U(s)]^2$$

where ω is a phase. Regardless of the unitarity or otherwise of $U(s)$, $[U(s)]^2$ is unitary so that $U(r)$ is always unitary.

The third result concerns the phase ω.

Theorem 3: If $\mathcal{R}$ consists of the component of the Poincaré group continuously connected to the identity, then there exists $e^{i\alpha(r)}$, α real, such that if

$$U'(r) = e^{i\alpha(r)} U(r),$$

then

$$U'(r_2)U'(r_1) = \eta(r_2, r_1)U'(r_2, r_1)$$

for all $r \in \mathcal{R}$ where $\eta(r_2, r_1)$ is either $+1$ or -1.

The set $\mathcal{R}$ in the above theorem consists of pure rotations, four-dimensional translations, pure Lorentz transformations and all possible compositions of these transformations. The -1 values η may take cannot be removed by a choice of phase. The existence of half-integer spins is linked to the possibility that such minus signs are allowed.

Let us now consider those transformations which are normally called discrete. One such is parity $\mathcal{P}$, another is time reversal $\mathcal{T}$. We assume that $\mathcal{P}$ and $\mathcal{T}$ are in the relativity group $\mathcal{R}$. To decide whether the corresponding operators on $\mathcal{H}$ are unitary or anti-unitary, we have to examine their relation to time translations $V(t) \in \mathcal{R}$. It is clear that

$$\mathcal{P} V(t) \; \mathcal{P}^{-1} = V(t),$$

$$\mathcal{T} V(t) \; \mathcal{T}^{-1} = V(-t).$$

Suppose that on $\mathcal{H}$,

$$V(t) \rightarrow e^{iHt},$$
$$\mathcal{P} \rightarrow P,$$
$$\mathcal{T} \rightarrow T$$

where H is the Hamiltonian. Then

$$P\, e^{iHt}\, P^{-1} = e^{iHt},$$
$$T\, e^{iHt}\, T^{-1} = e^{iHt}$$

where it may be shown that no extra phases need be inserted in the above equations. The basic hypothesis required to decide the nature of P and T is that the eigenvalues of H are all nonnegative and that they are not all zero. This is a hypothesis not related to geometrical principles and may be questioned.

Now suppose that P is anti-unitary. Then the linear terms in t give

$$P(iHt)\, P^{-1} = iHt$$

or

$$PH\, P^{-1} = -H.$$

Therefore if $H\psi = |\lambda|\psi$, then

$$HP^{-1}\psi = -|\lambda|\, P^{-1}\psi.$$

Hence, P must be unitary.

Similarly, T must be anti-unitary.

The composition laws obeyed by P and T among themselves and with the operators which occur in theorem 3 have been discussed by Lurcat and Michel and by Wigner at the Istanbul Summer School (1964).

Let me conclude by observing that there seem to be several points regarding the implementation of discrete symmetries which are not entirely clear. In addition to those which were mentioned at the beginning of the talk, we may note here that if there were tachyons present, the spectrum of the Hamiltonian will no longer be nonnegative. Then the possibility arises of implementing for example $\mathcal{P}$ by an anti-unitary operator. To the best of my knowledge there is no exhaustive analysis of such possibilities in the literature.

PROBLEMS IN ELECTROMAGNETIC MASS-DIFFERENCE CALCULATIONS*

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THE basic idea of the electromagnetic mass difference is best expressed in the first two sentences of the initial paper by Feynman and Speisman[1] in 1954: "Suppose all deviations from isotopic spin symmetry are due solely to electromagnetic effects. Then such things as the mass difference of charged and neutral π-mesons, and the neutron-proton mass difference would have to be just electrodynamic." The elegance of this hypothesis is illustrated by the very small deviations (*see* Table I) in the masses of particles[2] belonging to the same iso-multiplet. In the last eighteen years several attempts[3] have been made to calculate this mass difference but all these calculations are either incomplete or have other difficulties. In this talk we shall briefly discuss how far we have progressed, at least in principle, in calculating the electromagnetic mass differences of elementary particles.

TABLE I

Mass differences of neutral and charged particles with same isospin

Particle	Spin and parity	Isotopic spin (I)	Average mass MeV	$m_I^Q - m_I^{Q+1} = \Delta m$ MeV	$\frac{100\Delta m}{m_I^{Q=0}}$ %
π^-, π^0	0^-	1	137·274	4·6943 ± ·0037	3·41
K^0, K^+	0^-	1/2	495·815	3·95 ± ·13	0·79
n, p	$1/2^+$	1/2	938·906	1·29344 ± ·00007	0·14
Σ^0, Σ^+	$1/2^+$	1	1190·95	3·06 ± ·16	0·26
Σ^-, Σ^0	$1/2^+$	1	1194·91	4·86 ± ·06	0·41
Σ^-, Σ^+	$1/2^+$	1	1193·38	7·92 ± ·13	0·66
Ξ^-, Ξ^0	$1/2^+$	1/2	1318·0	6·6 ± ·7	0·50
K^{*0}, K^{*+}	1^-	1/2	891·7	6·1 ±1·5	0·68
ρ^0, ρ^+	1^-	1	765	2·4 ±2·1	0·31
Δ^0, Δ^{++}	$3/2^+$	3/2	1236	2·9 ± ·85	0·23
Δ^-, Δ^{++}	$3/2^+$	3/2	1236	7·0 ±6·8	0·64
Y^{*-}, Y^{*+}	$3/2^+$	1	1385	6·3 ±2·0	0·45

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