in a sufficiently general way, and yet with a comprehensive enough set of unifying principles is a major unsolved problem at present. However, a comparison with experiment of several two-body reactions up to reasonably high energies (~ 10 GeV, lab) at a pedagogical level, indicates that the variation of the form factors with energy could well be a valuable pointer to a more complete theory of the future. In particular, neither the predictions of the H.O. model (which gives Gaussian shapes), nor the constraints of analyticity (which lead to some sort of exponential structures) appear to be compatible with the data which seem to require a gentler fall with energy—something more akin to an inverse power law. One wonders if this indicates some "non-compact" features in the structure of form factors. (The "inverse power law" also appears to be compatible, in principle, with scaling behaviour in the triple Regge region.)

I conclude with a speculation on the status of quarks in the hierarchy of constituents of matter, assuming for the argument that they are physical entities. At this stage of development quarks represent a sort of "third stage" in the sequence

atoms → nuclei → nucleons → quarks.

The fundamental question is then the following: "Does the process stop at the quark stage, or does it continue to infinity?" The school which believes in mathematical quarks considers this question as meaningless. A subset of this school, which believes in elementary particle democracy, has a position diametrically opposite to the quark philosophy. However, if the quarks turn out to be merely a third stage in an infinite hierarchy of constituents, then it is amusing to speculate that the "quark" and "bootstrap" points of view may not be that different after all. In any case this is a big, *unsolved*, problem today.

THE GROUP-THEORETICAL ASPECTS OF QUANTUM MECHANICS *

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The unitary unimodular groups SU(n) occupy a position of paramount importance in the present-day physics. These groups have been used extensively to classify multi-electron states of atoms and states of nuclei. To have an idea of the importance of these groups in the theory of complex spectra one need only turn over the pages of the books written by B. R. Judd[1] on the subject. For $n = 3$, 4, 6, 12, they have been used as possible symmetries of elementary particles[2]. The experimental evidence in support of the SU(3) symmetry is now overwhelming, and some of the results obtained by using this symmetry are likely to be of permanent value. This is not quite true of the higher groups, though the SU(3) transformations form a subgroup of the larger symmetry. Nevertheless, persistent efforts are being made to gain a deeper insight into the nature of the elementary particles by using the higher groups, and it has become imperative to study the structural properties of these groups in greater detail. The mathematical problems that arise in this connection are more or less familiar and can be divided into the following broad categories:

1. Construction of state vectors.
2. Determination of the matrices of the generators.
3. Determination of the representation matrices for finite transformations.
4. Classification and construction of tensor operators.
5. Structure of the Clebsch-Gordan (CG) series.
6. Evaluation of the CG coefficients.

None of these problems appears to have been solved for general values of n. For solving problem 1 one may take[3,4], following Gel'fand, $n-1$ independent sets of variables a_j^i ($i = 1, 2, \ldots, n-1$; $j = 1, 2, \ldots, n$), each set transforming according to the basic n-dimensional representation of SU(n), and put a_j^1 in the first-row, a_j^2 in the second row, and so on, in a Young frame of $n-1$ rows. The standard Young diagrams corresponding to such a frame represent the basis states of an irreducible representation (IR) of SU(n) characterized by the row lengths m_{in}. However, in physical applications one is interested in IR's which are explicitly reduced with respect to a particular chain of subgroups, say, the canonical chain SU($n-1$) ⊃ SU($n-2$) ... ⊃ SU(1). To obtain such representations one must form appropriate linear combinations of the standard states. This is, in general, a difficult task and has been carried out only for values of n up to 4. The

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expression obtained for the general state of SU(4)[5,6] is sufficiently complicated to discourage any further attempt in this direction.

Next, let us come to problem 5, namely, the determination of the structure of the CG series[7,8]. I may be wrong, but there appears to be no serious difficulty in solving this problem and deriving a general formula that will tell us which IR's occur in the decomposition of the direct product of two IR's of SU(n) and how many times a particular IR is repeated.

A solution of problem 6 is of vital importance to theoretical physicists. In the study of reactions with two incoming and two outgoing particles belonging to the IR's of one of these groups and in relating the amplitudes in crossed channels[9] the CGC of the group play an important role. Unfortunately, no systematic investigation of the CGC has been carried out for values of n beyond 3. This is primarily due to the fact that the complexity of the computations increases extremely rapidly with n. For SU(3) general formulae for CGC have been given but not in a form which can be of any use to the physicist. They involve at least ten summations over quantities which are themselves summations. It is evident that the situation will be much worse for higher values of n. However, although the actual computation of the CGC for general values of n is a formidable task, there exists a straightforward and perfectly general method for doing this. This method is based on the theorem given in standard text-books that the integral of the product of three matrix elements of finite transformations belonging to three IR's of a group is equal to the product of two CGC. This method has been used by us to calculate the CGC of SU(3) in a recent paper[10]. At the end of the paper we have made the following remark, "For calculating the CGC of a group we require three things, (*i*) a set of parameters for labelling the elements of the group, (*ii*) the invariant volume element in the parameter space, and (*iii*) an analytic expression for the matrix elements of finite transformations. When these three things are known, the calculation of the coefficients becomes an exercise in elementary algebra." This method not only enables us, in principle, to calculate the CGC of any group, but also provides a simple and convenient scheme for orthogonalizing[11] them. Thus, problem 6 is seen to be intimately connected with problem 3 of finding the representation matrices. The latter problem has again been solved only for SU(3)[12]. In the general case it has been possible to introduce a set of parameters[13] in such a way that a special SU(n) matrix exp. $(-i\mu M)$ [where, M has unity in the $(n, 1)$, $(1, n)$ positions and zeros in all other positions] stands in the middle with two SU($n-1$) matrices on the two sides. The problem is, thus, reduced to the determination of the representation matrices of SU($n-1$) and of the 'boot operator' exp. $(-i\mu M)$.

To illustrate the points mentioned above we discuss SU(4) in some detail. This group is sufficiently more general than SU(2) and SU(3) to reveal some of the peculiarities of SU(n). An IR of SU(4) is characterized by three non-negative integers λ, μ, ν corresponding to a Young tableau of row lengths $\lambda+\mu+\nu$, $\mu+\nu$, and ν. The states within such a representation carry six labels and may be denoted by the symbol $|p\ q\ j\ m\ Y\ Z\rangle$, where, m Y Z are the eigenvalues of the diagonal generators, (p, q), j are the representation labels of the SU(3) and SU(2) subgroups, and $\mu \leqslant p+q \leqslant \lambda+\mu+\nu$. The group can be parametrized by writing its general element in the form

$$\tau = e^{-i\alpha_1 Z} e^{-i\alpha_2 Y} e^{-i\alpha_3 T_3} e^{-i\alpha_4 E_{21}} e^{-i\alpha_5 T_3} e^{-i\alpha_6 E_{13}} \times e^{-i\alpha_7 E_{34}} [e^{-i\alpha_8 T_3} e^{-i\alpha_9 E_{21}} e^{-i\alpha_{10} T_3} e^{i\alpha_{11} E_{13}} \times e^{i\alpha_{12} T_3} e^{i\alpha_{13} E_{21}} e^{i\alpha_{14} T_3} e^{i\alpha_{15} Y}]$$

where,

$$T_3 = \tfrac{1}{2}(A_1{}^1 - A_2{}^2),\ Y = \tfrac{1}{3}(A_1{}^1 + A_2{}^2 - 2A_3{}^3),$$
$$Z = \tfrac{1}{4}(A_1{}^1 + A_2{}^2 + A_3{}^3 - 3A_4{}^4),$$
$$E_{ij} = \frac{1}{2}(A_j{}^i + A_i{}^j),\ E_{ji} = \frac{i}{2}(A_j{}^i - A_i{}^j),$$

for

$$i < j;\ i, j = 1, 2, 3, 4$$

and

$$(A_i{}^j)_{mn} = \delta_{im}\delta_{jn} - \tfrac{1}{4}\delta_{ji}\delta_{mn}.$$

For the above parametrization the matrix element between two states of the same IR takes the form

$$\langle pqjmYZ| \overset{\lambda\mu\nu}{\tau} |p'q'j'm'Y'Z'\rangle = \sum_{\bar{j}\,\bar{m}\,\bar{Y}} e^{-i\alpha_1 Z} e^{-i\alpha_2 Y} e^{-i\alpha_3 m}$$
$$\times \langle jm| e^{-i\alpha_4 E_{21}} |\bar{j}\bar{m}\rangle e^{-i\alpha_5 \bar{m}}$$
$$\times \langle pqj\bar{m}Y| e^{-i\alpha_6 E_{13}} |pq\bar{j}\bar{\bar{m}}\bar{Y}\rangle$$
$$\times \langle pq\bar{j}\bar{\bar{m}}\bar{Y}Z| e^{-i\alpha_7 E_{34}} |p'q'\bar{\bar{j}}\bar{\bar{m}}\bar{\bar{Y}}Z'\rangle$$
$$\times \langle p'q'\bar{\bar{j}}\bar{\bar{m}}\bar{\bar{Y}}| \mathrm{SU}(3) |p'q'j'm'Y'\rangle.$$

All the quantities occurring in this expression are known with the exception of the matrix element of the 'boost' exp. $(-i\alpha_7 E_{34})$. To determine the latter one may express the generators as differential operators in the parameters of the group and then use the well-known theorem that the matrix elements are eigenfunctions of the Casimir operators. However, the procedure, even in the case of SU(3), leads to a system of simultaneous differential equations which are not easy to handle. An alternative procedure which gives the same result but keeps the calculations within reasonable length is to set up the first order recurrence relations satisfied by the boost matrix elements and build the general solution by iteration from a simple special case. These relations have been set up for SU(3) and SU(4). For SU(3) they are of the form

$$\left(\frac{j+m+1}{2j+1}\right)^{\frac{1}{2}}\Big[2ij\cot\nu - i(3\delta - p + q - m)\tan\nu - i\frac{\partial}{\partial\nu} - 4i\csc 2\nu\cdot m'\Big](jm\delta : m')^{*}$$
$$- 2i\left(\frac{(j-m)(j+m'+1)(j'-m')}{2j+1}\right)^{\frac{1}{2}} \times \csc\nu\cdot(jm+1\delta : m'+1)^{*}$$
$$= 2(2j+2)^{-\frac{1}{2}}\left[(j+\delta+q+2)(j+\delta+1)\times(p-j-\delta)\right]^{\frac{1}{2}}(j+\tfrac{1}{2}m+\tfrac{1}{2}\delta+\tfrac{1}{2} : m')^{*}$$

where,

$$Y = 2\delta - \tfrac{2}{3}(p-q),\quad (jm\delta : m') = \langle jm\delta|\, e^{-2i\nu E_{13}}\, |j'm'\delta'\rangle,$$

and ν is the variable in the boost operator exp. $(-2i\nu E_{13})$. The recurrence relations for SU(4) are longer and cannot be reproduced in this review.

The method described above works for any SU(n) though the complexity of the calculations increases enormously for higher values of n. Once the representation matrices are known the CGC can be calculated by the method of group integration mentioned earlier. The invariant volume required for the integration has been derived by the present author[13].

1. Judd, B. R., *Operator Techniques in Atomic Spectroscopy*, McGraw-Hill, New York, 1963.
2. Pais, A., *Rev. Mod. Phys.*, 1966, 38, 215.
3. Baird, G. E. and Biedenharn, L. C., *J. Math. Phys.*, 1963, 4, 1449.
4. Itzykson, C. and Nauenberg, M., *Rev. Mod. Phys.*, 1966, 38, 95.
5. Ciftan, M. and Biedenharn, L. C., *J. Math. Phys.*, 1969, 10, 221.
6. Ciftan, M., *Ibid.*, 1969, 10, 1635.
7. Mukunda, N. and Pandit, L. K., *Progr. Theor. Phys.*, 1965, 34, 46.
8. Gruber, B., *Nuovo Cimento*, 1967, 84 A, 23.
9. Sharp, R. T., *Phys. Rev.*, 1968, 168, 1753.
10. Majumdar, S. D. and Basu, B. K., *J. Math. Physi.*, 1973. (To be published).
11. —, *Progr. Theor. Phys.*, 1969, 42, 1475.
12. — and Basu, B. K., *Annals of Phys.* (N.Y.), 1970, 56, 464.
13. —, *J. Phys.*, A (General), 1972, 5.

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