Vol. 47, No. 1
Jan. 5, 1978

LETTERS TO THE EDITOR

STUDY OF THE VIBRATION CHARACTERISTICS OF A LOADED COMPOSITE STRING

1. *Introduction*

THE vibration of a composite string has been fully discussed by the present author[1]. The problem has been further extended to show the possibility of constructing new stringed instruments (plucked, struck and bowed) for musical sound[2]. In this paper, the same problem when small masses are attached to the joints has been considered and some interesting features of the vibrations of such study are pointed out. The author suggests that this problem will bring more important features for the study of stringed instruments.

2. *Solution of the Problem*

We consider a string of finite length $(0 \leqslant x \leqslant x_n)$ composed of n parts and small loads are applied at the $(n-1)$ junctions :

$$x_{k-1} \leqslant x \leqslant x_k,$$
$$k = 1, 2, \ldots, n, \quad x_0 = 0,$$

the densities of which per unit length are ρ_k respectively. Let τ be the uniform tension acting on the string. The equations of motion of such a string are

$$\frac{\partial^2 w_k}{\partial t^2} = c_k^2 \frac{\partial^2 w_k}{\partial x^2}, \tag{1}$$
$$x_{k-1} < x < x_k, \; t > 0$$
$$c_k^2 = \tau/\rho_k, \; k = 1, 2, \ldots, n.$$

For a string fixed at both the ends and loaded with particles of masses m_{k-1} at the junctions, the following conditions are to be fulfilled :

$$w_1(0, t) = 0, \; t > 0 \tag{2}$$
$$w_k(x_{k-1}, t) = w_{k-1}(x_{k-1}, t), \; t > 0$$
$$m_{k-1}\left(\frac{\partial^2 w_{k-1}}{\partial t^2}\right) = \tau\left\{\frac{\partial w_k}{\partial x} - \frac{\partial w_{k-1}}{\partial x}\right\},$$
$$x = x_{k-1}, \; k = 2, 3, \ldots, n, \quad t > 0 \tag{3}$$
$$w_n(x_n, t) = 0, \; t > 0. \tag{4}$$

Assuming the integrals $v_k = v_k(x, t)$ of equation (1) to be of the form

$$v_k = R_k(x)\, T(t), \tag{5}$$

we have

$$R_k(x) = A_k \sin(\beta_k x) + B_k \cos(\beta_k x), \tag{6}$$

and $T(t) = C \cos \omega t + D \sin \omega t, \; k = 1, 2, \ldots, n,$

$$\beta_k^2 = \omega^2/c_k^2. \tag{6a}$$

Using the conditions (2)-(4), we have[1]

$$B_1 = 0 \tag{7}$$
$$A_k \sin(\beta_k x_{k-1}) + B_k \cos(\beta_k x_{k-1})$$
$$= A_{k-1} \sin(\beta_{k-1} x_{k-1}) + B_{k-1} \cos(\beta_{k-1} x_{k-1}),$$
$$A_k \cos(\beta_k x_{k-1}) - B_k \sin(\beta_k x_{k-1})$$
$$= A_{k-1}\, q_{k-1} - B_{k-1}\, r_{k-1},$$
$$q_{k-1} \equiv \Lambda_k \cos(\beta_{k-1} x_{k-1}) - \frac{m_{k-1}\omega^2}{\tau\beta_k} \sin(\beta_{k-1} x_{k-1}),$$
$$r_{k-1} \equiv \Lambda_k \sin(\beta_{k-1} x_{k-1}) + \frac{m_{k-1}\omega^2}{\tau\beta_k} \cos(\beta_{k-1} x_{k-1})$$
$$\Lambda_k = \beta_{k-1}/\beta_k, \; \beta_k = \omega/c_k, \tag{8}$$

and

$$A_n \sin(\beta_n x_n) + B_n \cos(\beta_n x_n) = 0. \tag{9}$$

From equations (8) we get

$$A_k = (a_{k-1} A_{k-1} + \alpha_{k-1} B_{k-1})$$
$$B_k = (b_{k-1} A_{k-1} + \gamma_{k-1} B_{k-1})$$

where

$$a_{k-1} = a_{k-1}(\omega) = \cos(\beta_k x_{k-1})\, q_{k-1} + \sin(\beta_k x_{k-1}) \times \sin(\beta_{k-1} x_{k-1}),$$
$$\alpha_{k-1} = \alpha_{k-1}(\omega) = \sin(\beta_k x_{k-1}) \cos(\beta_{k-1} x_{k-1}) - \cos(\beta_k x_{k-1})\, r_{k-1},$$
$$b_{k-1} = b_{k-1}(\omega) = \cos(\beta_k x_{k-1}) \sin(\beta_{k-1} x_{k-1}) - \sin(\beta_k x_{k-1})\, q_{k-1},$$
$$\gamma_{k-1} = \gamma_{k-1}(\omega) = \cos(\beta_{k-1} x_{k-1}) \cos(\beta_k x_{k-1}) + r_{k-1} \sin(\beta_k x_{k-1}). \tag{10}$$

We write (10) in the matrix form[1] as follows:

$$K_k = M_{k-1}(\omega) K_{k-1}, \; k = 2, 3, \ldots, n,$$
$$K_k = \left\| \begin{matrix} A_k \\ B_k \end{matrix} \right\|$$
$$M_k(\omega) = \left\| \begin{matrix} a_k(\omega) & \alpha_k(\omega) \\ b_k(\omega) & \gamma_k(\omega) \end{matrix} \right\| \tag{11}$$
$$k = 1, 2, \ldots, (n-1).$$

After using (7), we have

$$K_k = P_{k-1}(\omega) \left\| \begin{matrix} A_1 \\ 0 \end{matrix} \right\|$$
$$k = 2, 3, \ldots, n,$$
$$P_k(\omega) = M_k(\omega)\, M_{k-1}(\omega) \ldots M_2(\omega)\, M_1(\omega)$$
$$k = 1, 2, \ldots, (n-1). \tag{12}$$

Let

$$x_1, x_2, \ldots, x_k = x^k, \quad k = 1, 2, \ldots, (n-1).$$

$$\mathrm{P}_k(\omega) = \| p_k^{[l', \sigma]}(\omega) \|, \qquad l', \sigma = 1, 2, \quad (13)$$

So, after using (12), we have

$$\begin{aligned} \mathrm{A}_k &= p_{k-1}^{[11]}(\omega)\mathrm{A}_1 \\ \mathrm{B}_k &= p_{k-1}^{[21]}(\omega)\mathrm{A}_1, \\ & k = 2, 3, \ldots, n. \end{aligned} \quad (14)$$

Therefore, from relation (9), we can now write the frequency equation of the vibrating loaded composite string as

$$p_{n-1}^{[11]}(\omega)\sin\beta_n x_n + p_{n-1}^{[21]}(\omega)\cos\beta_n x_n = 0. \quad (15)$$

By (14), we have[1]

$$\mathrm{A}_{km} = p_{k-1}^{[11]}(\omega_m)\mathrm{A}_{1m}$$

$$\mathrm{B}_{km} = p_{k-1}^{[21]}(\omega_m)\mathrm{A}_{1m}, \; k = 2, 3, \cdots, n,$$

which with A_{1m} and (6 *a*) gives the system of quantities

$$\beta_{km} = \omega_m / c_k, \; k = 1, 2, \ldots, n. \quad (16)$$

From (6), we now get

$$v_{km}(x, t) = \mathrm{U}_{km}(x)\mathrm{V}_m(t), \quad k = 1, 2, \ldots, n, \quad (17)$$

where

$$\mathrm{U}_{1m} = \sin(\beta_{1m} x)$$

$$\mathrm{U}_{km}(x) = p_{k-1}^{[11]}(\omega_m)\sin(\beta_{km} x) + p_{k-1}^{[21]}(\omega_m) \times \cos(\beta_{km} x), \quad k = 2, 3, \ldots, n,$$

$$\mathrm{V}_m(t) = \mathrm{A}_{1m}[\mathrm{C}\cos\omega_m t + \mathrm{D}\sin\omega_m t[. \quad (18)$$

For a string composed of two parts, we have

$$q_{k-1} \equiv \mathrm{A}_2\cos\beta_1 x_1 - \frac{m_1\omega^2}{\tau\beta_2}\sin\beta_1 x_1,$$

$$r_{k-1} \equiv \mathrm{A}_2\sin\beta_1 x_1 - \frac{m_1\omega^2}{\tau\beta_2}\cos\beta_1 x_1,$$

$$p_1^{[1,1]} = a_1(\omega) = q_1\cos\beta_2 x_1 + \sin\beta_2 x_1 \sin\beta_1 x_1$$

$$p_1^{[1,2]} = a_1(\omega) = \sin\beta_2 x_1 \cos\beta_1 x_1 - r_1\cos\beta_2 x_1$$

$$p_1^{[2,1]} = b_1(\omega) = \cos\beta_2 x_1 \sin\beta_1 x_1 - q_1\sin\beta_2 x_1$$

$$p_1^{[2,2]} = \gamma_1(\omega) = \cos(\beta_1 x_1)\cos(\beta_2 x_1) + r_1\sin\beta_2 x_1$$

$$\mathrm{P}_1(\omega) = \mathrm{M}_1(\omega) = \left\| \begin{matrix} a_1(\omega) & a_1(\omega) \\ b_1(\omega) & \gamma_1(\omega) \end{matrix} \right\|. \quad (19)$$

In this case, the frequency equation can be written after certain simplifications as

$$m_0\beta_2 l_2 = \left(\frac{\beta_1}{\beta_2}\right)\cot\beta_1 x_1 + \cot\beta_2(x_2 - x_1), \quad (20)$$

where $m_0 = m_1/\rho_2 l_2$,

(here m is in gm, τ is in dynes and ρ is in gm/cm). Writing $x_1 = l_1$, $x_2 = l_1 + l_2$ and $l_1/l_2 = a$, we have

$$m_0\beta_2 l_2 = \left(\frac{\beta_1}{\beta_2}\right)\cot\beta_1 l_2 a + \cot\beta_2 l_2. \quad (20\,a)$$

For $a = 1$, $\beta_1 = \beta_2 = \beta$, we get

$$m_0\beta l/2 = \cot\beta l. \quad (21)$$

This is the frequency equation for the vibration of a homogeneous string of length $2l$ when loaded at the middle and fixed at both the ends.

The first six roots of equation (21) are given in Table I[3].

From Table I, we find that when the mass load is zero, the allowed values of frequency are in the ratio 1:3:5:7:9. The same is true when $m_0 = 0 \cdot 1$. Though the mass load here lowers the pitch, but still rich in harmonics. For higher loading, of course, the pitch is lowered and also not rich in harmonics. Thus, if a small mass load is attached to the middle of the string, the harmonics are not destroyed.

TABLE I

$m_0/2$	βl					
	1	2	3	4	5	6
0·00	1·571	4·712	7·854	10·996	14·137	17·279
0·05	1·496	4·491	7·495	10·512	13·542	16·586
0·10	1·429	4·306	7·228	10·200	13·214	16·259
0·15	1·368	4·155	7·041	10·012	13·039	16·101
0·20	1·314	4·034	6·910	9·893	12·935	16·011
0·25	1·265	3·935	6·814	9·812	12·868	15·954
0·30	1·220	3·855	6·742	9·754	12·821	15·914

The above discussion suggests that we can now analyse the vibration characteristics of plucked, struck and bowed strings of composite nature, when small masses are attached to the joints[2].

In Fig. 1, the change in frequencies for various mass loads is shown.

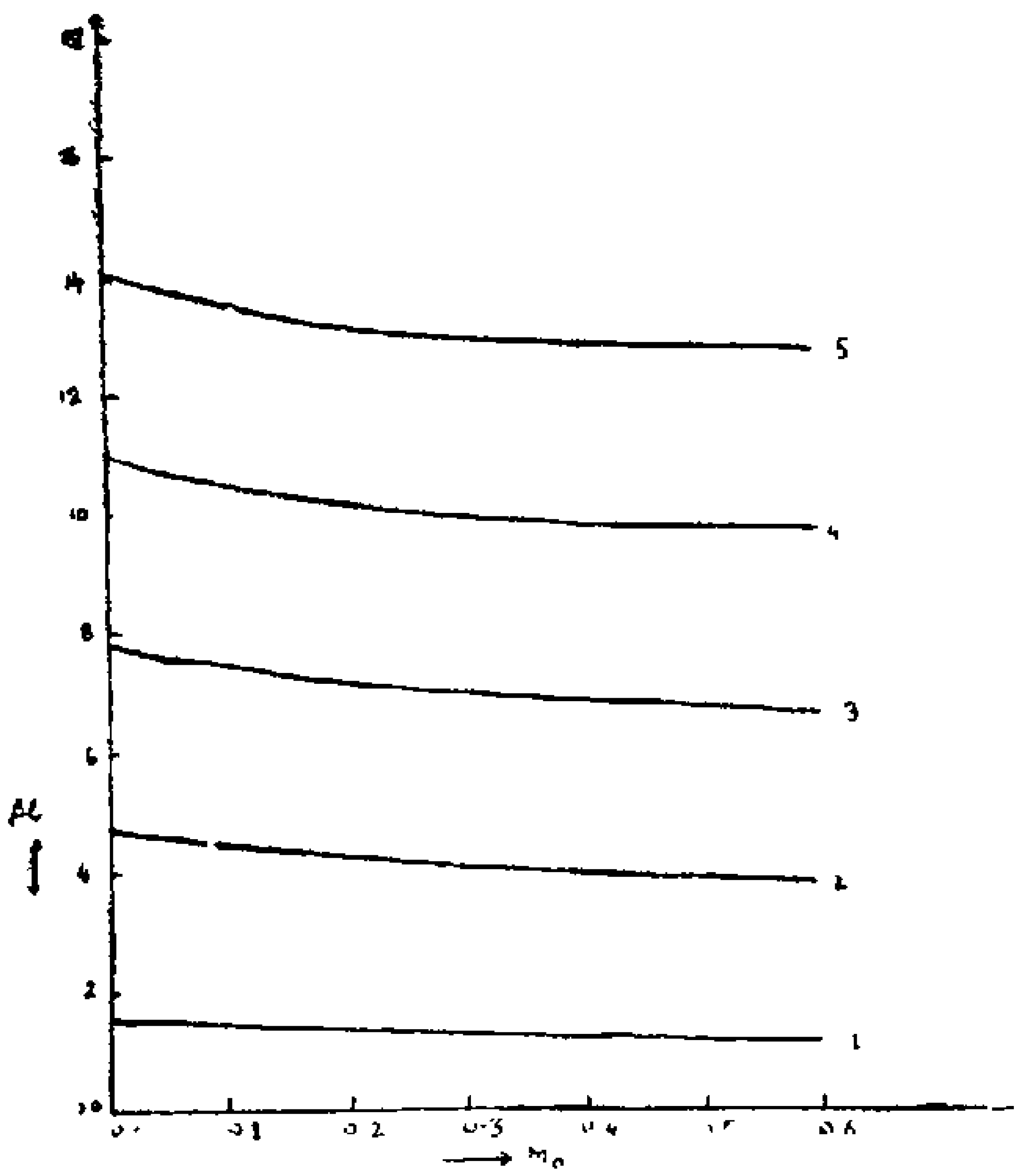


FIG. 1. Plot of m_0 *vs.* βl.

In the absence of the mass load, equation (20) can be written as

$$\frac{\beta_1}{\beta_2} \cot \beta_1 x_1 + \cot \beta_2 (x_2 - x_1) = 0,$$

i.e.,

$$\frac{\beta_1}{\beta_2} \sin \beta_2 (x_2 - x_1) \cos \beta_1 x_1 + \cos \beta_2 (x_2 - x_1)$$
$$\sin \beta_1 x_1 = 0,$$

where $x_1 = l_1$ and $x_2 = l_1 + l_2$.

This tallies with De's previous investigation[1].

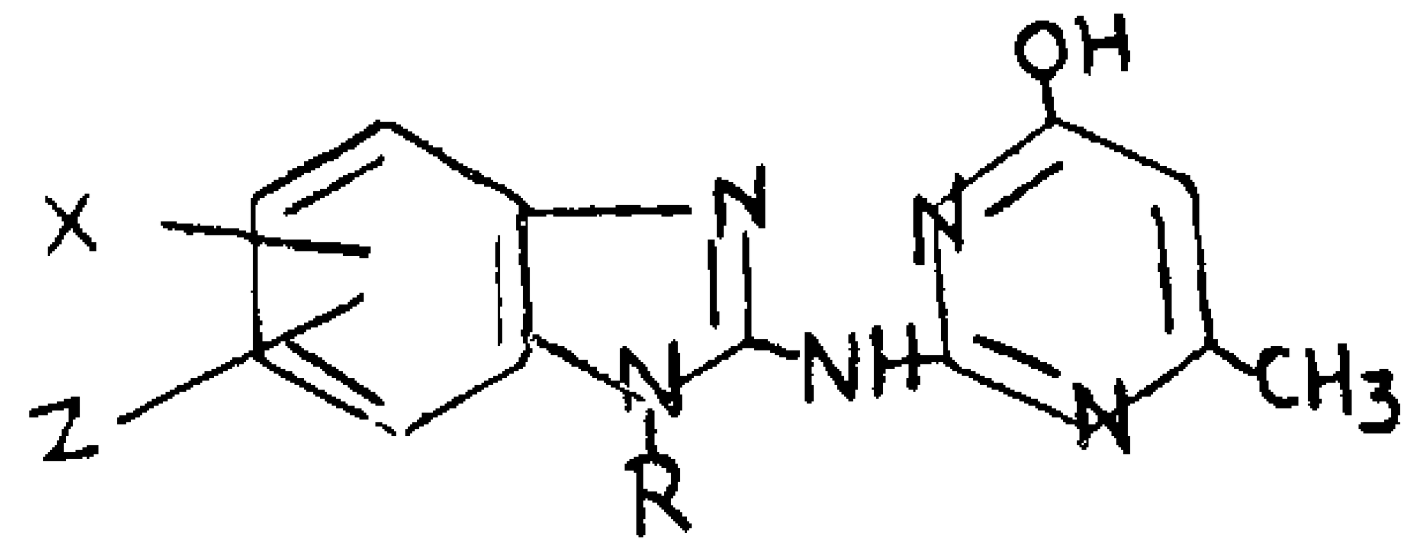
The author is grateful to Prof. P. K. Ghosh, Head of the Department of Applied Mathematics, Professor S. N. Ghosh of Applied Physics, Calcutta University and Dr. M. Ghosh, Director of the Institute of Theoretical Physics, for their keen interest and constant encouragement.

Old Engineering Office (Qrs.), SASADHAR DE.
Santiniketan, Birbhum,
West Bengal, India, *September* 3, 1977.

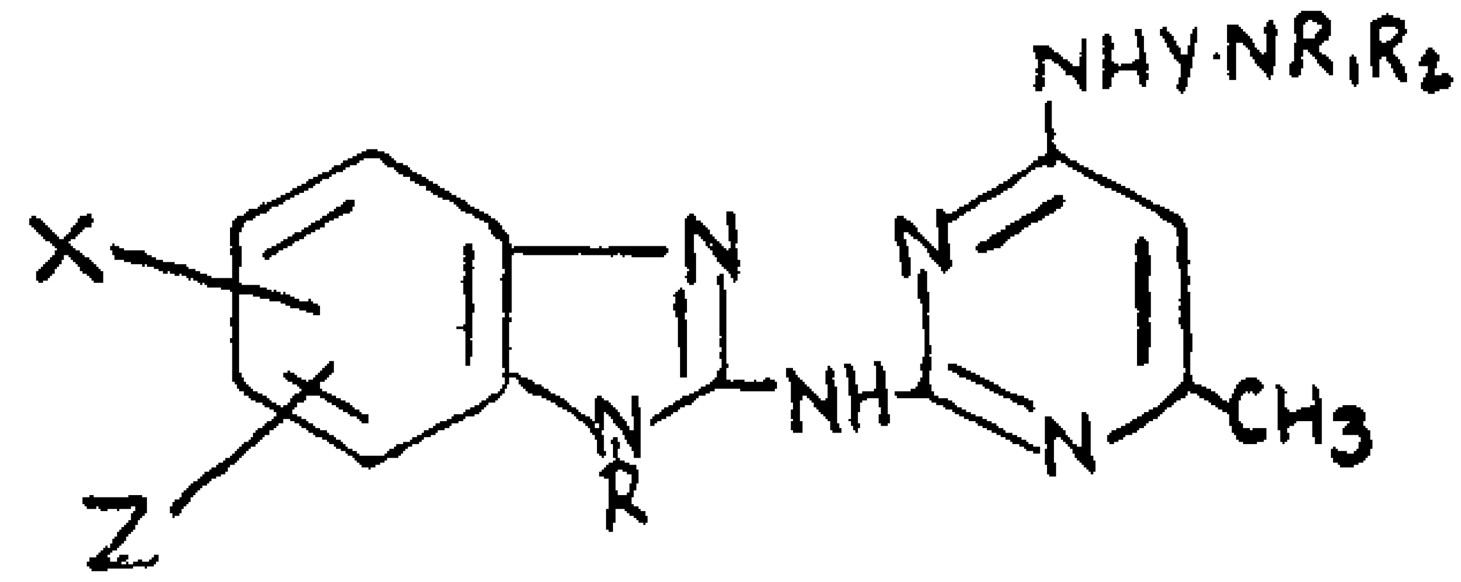
1. De, S., "Vibration of a composite string," *Ind. J. Phys.*, 1972, 46, 149.
2. —, "Problem of the vibrations of stringed instruments," *Bull. De L'Acad. Pol. Des Sciences* (Submitted), 1976.
3. Abramowitz, M. and Stegun, I. A. (Editors). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, Inc., New York, 1965, p. 225.

A CONVENIENT METHOD FOR THE SYNTHESIS OF 2-[(2-BENZIMIDAZOLYL) AMINO]-6-METHYL-4-PYRIMIDINOLS

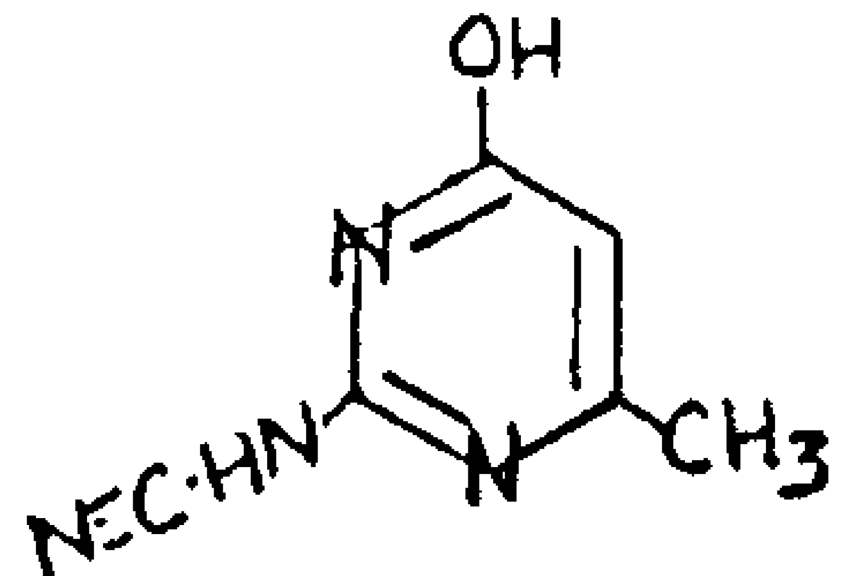
2-[(2-Benzimidazolyl) amino]-6-methyl-4-pyrimidinols (I) are valuable precursors for the preparation of various potent antimalarial drugs such as 2-[(4) {(dialkylamino) alkyl] amino}-6-methyl-2-pyrimidinyl-amino]-benzimidazoles[1,2] (II)



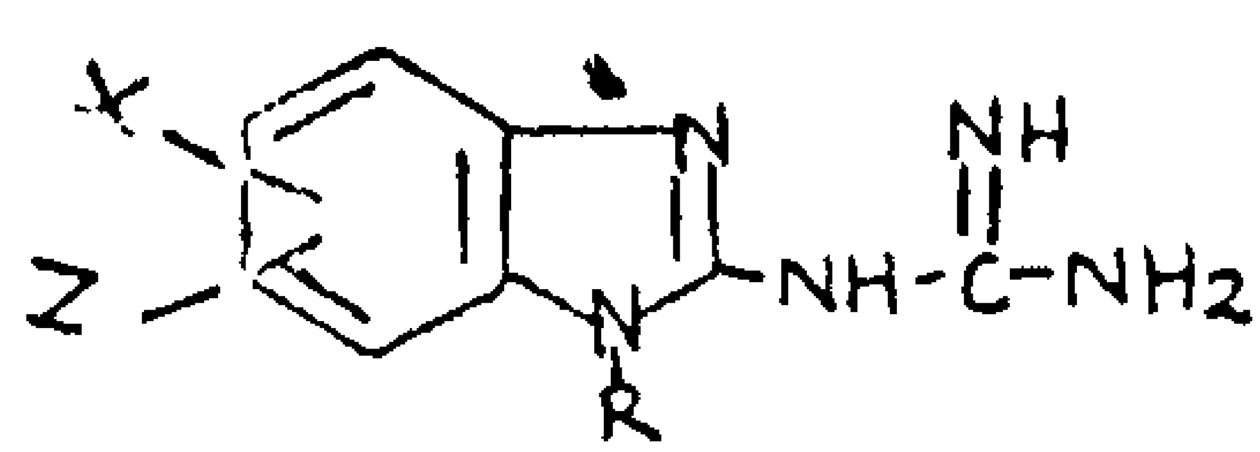
I



II



III



IV

Recently Werbel and coworkers[1] prepared I in 11–51% yields by the condensation of suitably substituted *o*-phenylenediamines with 2 (cyanoamino-4-hydroxy-6-methylpyrimidine (III) which in turn is obtained (in less than 50% yield) by the condensation of dicyandiamide with ethyl acetoacetate in the presence of sodium ethoxide.