LETTERS TO THE EDITOR

STATICAL MODELS IN BIMETRIC GRAVITATION THEORY

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THE most general statical space-time is given by

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{1}$$

where

$$\frac{\partial}{\partial x^4} g_{\alpha\beta} = 0 \quad \text{and} \quad g_{i4} = 0.$$

The Greek suffixes take the values (1, 2, 3, 4) and the Latin suffixes have the range (1, 2, 3). The line-element (1) can be written as (see Synge[1])

$$ds^2 = g_{ij}\, dx^i\, dx^j - g_{44}\,(dx^4)^2. \tag{2}$$

The part $g_{ij}\, dx^i dx^j$ gives the geometry of a three-dimensional space and we shall denote the quantity belonging to space by putting a bar over that quantity.

The bimetric relativity (BR) theory of Rosen[2] assigns two metrics: corresponding to curved space-time and flat space-time at every point of a manifold. The flat space-time corresponding to (2) is

$$d\sigma^2 = \gamma_{ij}\, dx^i\, dx^j - (dx^4)^2. \tag{3}$$

For the line-element (3),

$$\Gamma^4_{ij} = 0 = \Gamma^i_{4j},$$

where Γ's are the γ-Christoffel symbols of second kind. The vacuum field equations of BR are

$$N^\beta_\alpha = \tfrac{1}{2}\, \gamma^{\mu\nu}\, (g^{\epsilon\beta} g_{\epsilon\alpha|\mu})\,|_\nu = 0, \tag{4}$$

where a bar ($|$) stands for γ-covariant derivative. Noting

$$g_{ij} = \bar{g}_{ij},\ g^{ij} = \bar{g}^{ij},\ \gamma_{ij} = \bar{\gamma}_{ij},\ \gamma^{ij} = \bar{\gamma}^{ij},$$

$$g_{i4} = 0,\ g^{i4} = 0,\ \gamma_{i4} = 0,\ \gamma^{i4} = 0$$

we have

$$2N^j_i = \bar{\gamma}^{lm}\,(\bar{g}^{jk} g_{ik|l})|_m = 2\bar{N}^j_i \tag{5}$$

and

$$2N^4_4 = g^{44}\, D\, g_{44} + E\, g_{44} \tag{6}$$

where

$$D g_{44} = \bar{\gamma}^{lm} g_{44|lm},\ E\, g_{44} = \bar{\gamma}^{lm} g_{44|l}\, g^{44}|_m.$$

Equations (5) and (6) yield

$$2N = 2N^\alpha_\alpha = 2\bar{N} + g^{44} D\, g_{44} + E\, g_{44}. \tag{7}$$

It is easy to see from (6) and (7) that the vacuum field equations ($N = 0$, $N^4_4 = 0$, etc.) give $N = 0$ implying that space has a vanishing invariant $\bar{N}$. This result has an analogue in general relativity[1]: space has a vanishing curvature invariant $\bar{R}$. In case of conformastat metric the above vacuum field equations give interesting results. A conformastat line-element is obtained from (2) by letting $g_{ij} = \delta_{ij}\, e^{2a}$. Also writing $g_{44} = e^{2b}$, the resulting conformastat metric is

$$ds^2 = e^{2a}\, dx^i\, dx^i - e^{2b}\, dt^2, \tag{8}$$

the functions a and b being independent of t.

The flat metric with reference to (8) is $\gamma_{ij} = \delta_{ij}$, $\gamma_{44} = -1$. It can then be seen that the γ-covariant derivatives ($|$) become usual partial derivatives which we denote by commas. We find that

$$N^j_i = \delta^j_i\, a_{,mm} \quad N^4_4 = b_{,mm},$$

$$N = 3a_{,mm} + b_{,mm}.$$

The vacuum field equations then imply

$$a_{,mm} = 0, \quad b_{,mm} = 0. \tag{9}$$

The unknown quantities a and b turn out to be harmonic functions with respect to the flat space. Therefore the choice of harmonic functions completely determines a conformastat model. It is interesting that a set of conformastat universes can be generated just from harmonic functions. On the contrary in general relativity the situation is not satisfactory. The contrast between the two theories: BR and general relativity is remarkable as far as the above case is considered. The solution (2·9) of Rosen[2] is one of the examples of the conformastat metrics. The other examples can easily be constructed from harmonic functions. For similar results one may refer to the equations (13) and (14) of Rosen and Rosen[3].

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1. Synge, J. L., *Relativity: The General Theory*, North-Holland Pub. Co., Amsterdam, 1971, p. 338.
2. Rosen, N., "Bimetric theory of gravitation," *Topics in Theoretical and Experimental Gravitation Physics*, edited by V. De Sabbata and J. Weber, Plenum Press, New York, 1975, p. 273. All other references of Rosen N. cited in the above article.
3. Rosen, J. and Rosen, N., "Incompressible matter in the bimetric gravitation theory," *Astrophys. J.*, 1977, **212**, 605.