THE UNPUBLISHED MANUSCRIPTS OF SRINIVASA RAMANUJAN

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SRINIVASA RAMANUJAN, the greatest Indian mathematician of modern times, was born on December 22, 1887 at Erode in Tamil Nadu. After several unsuccessful attempts to pass the Intermediate examination of the Madras University, he joined, in February 1912, the Madras Port Trust as a grade 4 clerk on a monthly salary of rupees thirty. Even from the age of 18 or 19, Ramanujan began keeping his, now famous, Notebooks in which he recorded his discoveries. These Notebooks after several years of hibernation in the archives of the University of Madras were published and brought to the notice of the scientific public in 1957 by the Tata Institute of Fundamental Research, Bombay, with financial assistance from the Sir Dorabjee Tata Trust. Ramanujan's correspondence with G. H. Hardy of Cambridge, England, who was boundless in his appreciation of Ramanujan's mathematical work, resulted in his going to England in 1914 and staying there for five years during which he did work of "profound and invincible originality". In 1917 he fell seriously ill, partly because of the damp English climate and partly because of his reluctance—almost amounting to stubbornness—to take even vegetarian food which was not highly spiced. He was confined, for the greater part of his stay in England, to Nursing Homes and Sanatoria and returned to India early 1919, emaciated and in poor health. He passed away in Madras on April 26, 1920 hardly 33 years old.

During his stay at the Sanatoria in England where he was most of the time confined to bed and in comparative solitude, he had been working on many problems related to the partition function and the 'tau' function and on general aspects of, what is now known as, 'Hecke Theory', in which he was indeed a pioneer. He seems to have begun there a long memoir (A) entitled "Congruence properties of $p(n)$ and $\tau(n)$ defined by the equations

$$\sum_{0}^{\infty} p(n)\, x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$$

$$\sum_{1}^{\infty} \tau(n)\, x^n = x\{(1-x)(1-x^2) \times (1-x^3)\ldots\}^{24}"$$

He had also been working on various aspects of the continued fraction

$$1 + \frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\ldots,\quad |x| < 1$$

which was discovered independently by Rogers and Ramanujan. Its evaluation, however, for special values of x, is Ramanujan's own and has intimate connection with *complex multiplication*.

He had, in addition, been writing letters to Hardy fairly frequently. His letters were invariably full of mathematics and are very interesting. They deserve to be made public.

After his return to India in precarious health, he had been staying in Nursing Homes, especially in Madras. He seems to have done an enormous amount of work on q-series, continued fractions, etc. In fact he wrote a long letter to Hardy containing his results on mock-theta functions. He recorded his discoveries during this period in loose sheets of paper where, as was his wont, statement followed statement. A long manuscript, which is called by G. E. Andrews, the "Lost" notebook of Ramanujan has a strange history and seems to have been written during the last year of his life.

Thanks are due to the Master and Fellows of the Trinity College, Cambridge, England, for according us permission to print the three pages from Ramanujan's manuscript.

(3)

I am now considering another problem I told you, viz the expression of the various powers of $\Gamma(\tfrac{1}{4})$ and $\Gamma(\tfrac{1}{8})$ as products in primes similar to the powers of $\Gamma(\tfrac{1}{2})$ i.e the powers of π. For instance we know that

$$\pi^2\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots = 6$$

$$\pi^4\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\cdots = 90$$

$$\pi^6\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\cdots = 945$$

$$\pi^8\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\cdots = 9450$$

$$\pi^{10}\left(1-\frac{1}{2^{10}}\right)\left(1-\frac{1}{3^{10}}\right)\left(1-\frac{1}{5^{10}}\right)\cdots = 93555$$

and so on, the right hand side being a rational number. I find that

$$\left\{\frac{\Gamma(\tfrac{1}{4})}{2\pi^{\frac{3}{4}}}\right\}^4\left(1-\frac{1}{3^2}\right)\left\{\left(1-\frac{1}{(1+2i)^2}\right)\left(1-\frac{1}{(1-2i)^2}\right)\right\}$$
$$\times\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left\{\left(1-\frac{1}{(3+2i)^2}\right)\left(1-\frac{1}{(3-2i)^2}\right)\right\}$$
$$\times\left\{\left(1-\frac{1}{(1+4i)^2}\right)\left(1-\frac{1}{(1-4i)^2}\right)\right\}\left(1-\frac{1}{19^2}\right)\cdots\cdots$$
$$=4$$

Here we take the odd primes 3, 5, 7, 11, 13, 17, 19 ... and retain the primes of the form $4k-1$ viz 3, 7, 11, 19 ... and instead of the primes of the form $4k+1$ viz 5, 13, 17, ... we write the conjugate factors $1+2i$ and $1-2i$; $3+2i$ and $3-2i$; $1+4i$ and $1-4i$ and so on

(*)

A part of Ramanujan's letter to Hardy

Shortly after Ramanujan's death in Madras on April 26, 1920, all his manuscripts were acquired by the University of Madras on payment of rupees twenty per month for life to his widow. (It appears that this amount had been enhanced in the subsequent decades.) These manuscripts were later transmitted to G. H. Hardy. Since G. N. Watson of the University of Birmingham and B. M. Wilson intended to edit the Notebooks of Ramanujan, the manuscripts were entrusted with G. N. Watson. Hardy intended to edit the manuscript (A) mentioned above. He, however only published a short note on some results of Ramanujan's in the manuscript (A). A few other results from (A) were supplied with proofs by J. M. Rushforth in his thesis in 1951 written with Watson. All the manuscripts remained with Watson and were, perhaps, forgotten by him. On Watson's death in 1965, these manuscripts were deposited by his widow, at the instance of R. A. Rankin of Glasgow University, with the Trinity College, Cambridge, of which Ramanujan was a Fellow. They were strangely found by G. E. Andrews[1] whose preprint made me aware of their existence at Trinity. The Tata Institute of Fundamental Research acquired photo copies of all these manuscripts in 1978.

Watson had also copied some manuscripts of Ramanjuan (whose originals seem to be not available now) and these papers are now in the Oxford mathematical library. They were brought to light, a few years ago, by B. J. Birch[2].

To our knowledge, thus, there are the following Ramanujan manuscripts which are to be found in the libraries of the Trinity College, Cambridge and the Oxford mathematical library in England. Whether there are any other unpublished manuscripts of his, available elsewhere, we do not know.

MS (A) is the long memoir of 43 pages of foolscap-size on the congruence properties of $p(n)$ and $\tau(n)$. A detailed discussion of this was already given by us in ref. 6.

MS (B) are papers copied from loose sheets by Watson. They are 32 pages in length and are mentioned by Birch[2].

MS (C) Ramanujan's letters to Hardy written at various times while in Sanatoria and later from India.

MS (D). Miscellaneous notes available a Trinity College together with the edited, but unpublished, versions of some chapters of Ramanujan's Notebooks by B. M. Wilson.

MS. (E). The so-called "Lost" Notebook of Ramanjuan[1].

It would be difficult to discuss, in detail, the contents of these manuscripts. We shall, however, mention some of the interesting results in them which might give a flavour of their contents. As stated above, we shall not discuss the manuscript (A).

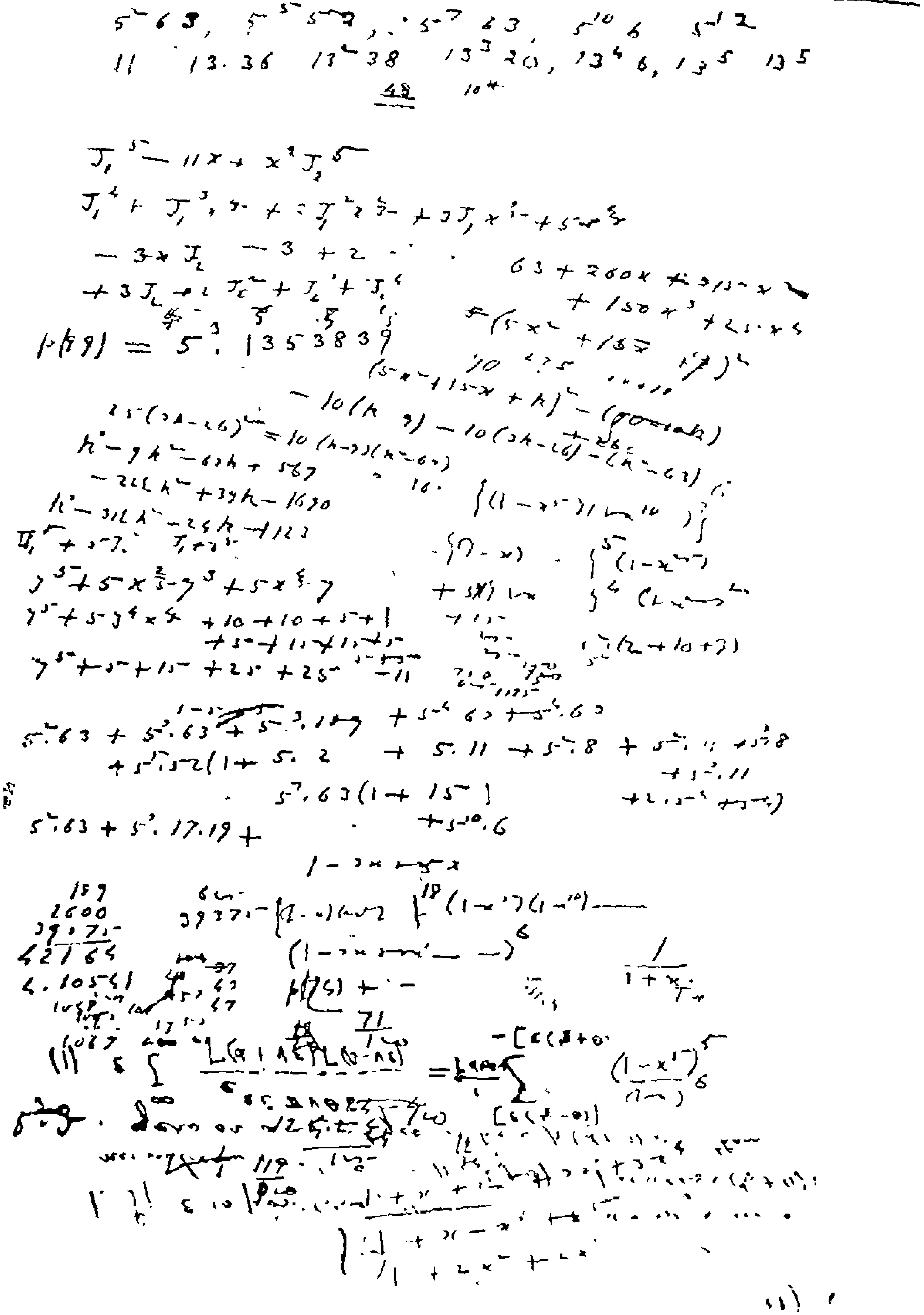
MS (B) was discussed by Birch. He had published the results in (B) relating to Hecke theory and the identities connected with the functions $G(x)$ and $H(x)$ given by

$$G(x) = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \frac{x^9}{(1-x)(1-x^2)(1-x^3)} + \ldots$$

$$H(x) = 1 + \frac{x^{1\cdot 2}}{1-x} + \frac{x^{2\cdot 3}}{(1-x)(1-x^2)} + \frac{x^{3\cdot 4}}{(1-x)(1-x^2)(1-x^3)} + \ldots$$

In his note in the *Proceedings of the London Mathematical Society*, Records for March 1919 Ramanujan has given two identities between $H(x)$ and $G(x)$ and says "Each of these formulae is the simplest of a large class" Brich in his note[2] has published 40 such formulae from (B). It would be very interesting to give a systematic, and perhaps a unified, but not *ad hoc* proof of all these identities. Watson give proofs of 9 of them[11].

The importance of manuscript (B) is however due to the large number of results on Dirichlet series with Euler product associated with cusp forms belonging to the modular group and its subgroups. One could



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A page from the 'Lost' Notebook of Ramanujan

justifiably say, that on the formal side, this is one of Ramanujan's most beautiful creations. In his only published paper[8], which, as Birch says, is one of the most beautiful published by the Cambridge Philosophical Society, Ramanujan gives Dirichlet series with Euler product. The interesting fact is that *the p-factor of this Euler product is a quadratic irreducible polynomial.* In (B) Ramanujan writes down many more examples of Euler products of this type related to subgroups of the modular group. There are many such examples in (D) and (E). It is really remarkable to see that Ramanujan with uncanny insight, chooses just those cusp forms whose associated Dirichlet series have Euler products. How he arrived at these cusp forms is a mystery. For instance, in (A) on page 28, he defines

$$\sum_{1}^{\infty} \Omega_2(n)\, x^n = x^2 \{(1-x)(1-x^2)\ldots\}^{48} \times \left(1 + 240 \sum_{1}^{\infty} \frac{n^3 x^n}{1-x^n}\right)$$

and

$$\sum_{1}^{\infty} \Omega_3(n)\, x^n = x^2 \{(1-x)(1-x^2)\ldots\}^{48} \times \left(1 - 504 \sum_{1}^{\infty} \frac{n^5 x^n}{1-x^n}\right)$$

and remarks: "The functions

$$\sum \frac{\Omega_2(n)}{n^s},\ \sum \frac{\Omega_3(n)}{n^s}$$

are obviously not capable of a single product...; but they are, as a matter of fact, the differences of two such products." It is indeed a great tragedy that the war intervened during Ramanujan's stay in England and prevented him from meeting or even corresponding with E. Hecke in Germany who, as is well known, independently investigated 20 years later in 1937, the relationship between modular forms and Dirichlet series with Euler products[4]. He had many things in common with Hecke, at least on the formal side, which might have got exploited, had the two mathematicians (who were both born in 1887) met or corresponded.

Dr. S. S. Rangachari[10] has recently been able to elucidate the statements made in (B), (D) and (E) and relate them to the work of Hecke and others. As Rangachari shows, Ramanujan had been able to write down, just how one is unable to say, a basis of *eigenfunctions* (of Hecke operators) which alone possess *Dirichlet series with Euler products.* It is an astonishing piece of work by Ramanujan.

MS (C) contains a large number of letters related to *asymptotic formulae for fourier* coefficients of modular forms of positive dimension (negative weight). Ramanujan's letters deserve to be scrutinised further. There is one particular letter (as usual with no date) written in 1918 which is of interest. In it, Ramanujan evaluates series of the type

$$Z(s) = \left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{(1+2i)^s}\right) \times \left(1 - \frac{1}{(1-2i)^s}\right)\left(1 - \frac{1}{7^s}\right) \times \left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{(3+2i)^s}\right) \times \left(1 - \frac{1}{(3-2i)^s}\right)\ldots$$

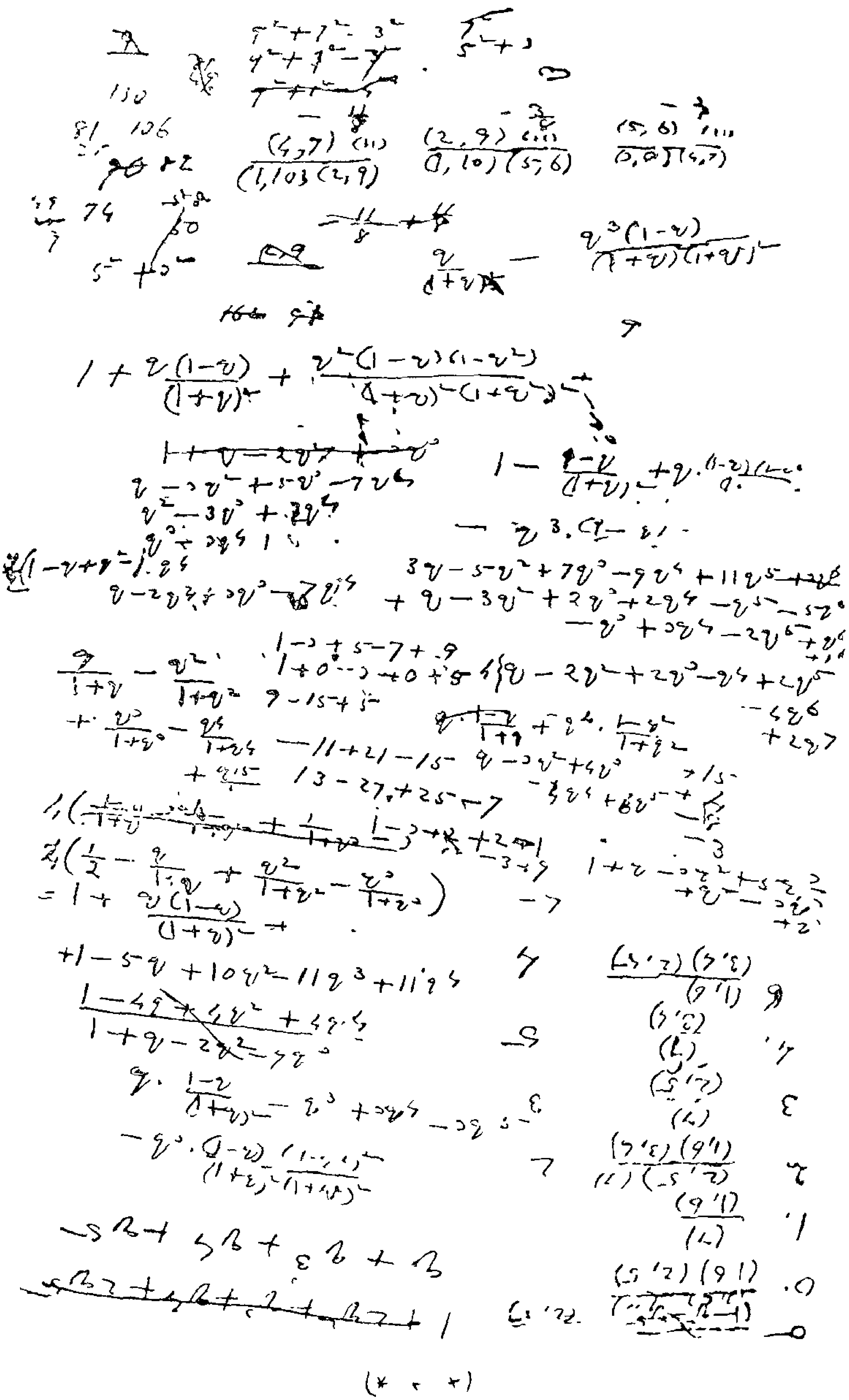
for $s = 2, 4, 6, 8, \ldots$ The numbers 3, $1+2i$, $1-2i$, 7, 11, $3+2i$, $3-2i, \ldots$ are all the 'odd' primes in the ring of Gaussian integers. Ramanujan says that

$$Z(s) \cdot \left(\frac{\Gamma(\frac{1}{4})}{2\pi^{1/4}}\right)^{2s} = \lambda_s$$

is a rational number for $s = 2, 4, 6, \ldots$ In fact $\lambda_2 = 4$, $\lambda_4 = 12, \ldots$ This theme also occurs in (E). He evaluates λ_s for $s = 2, 4, \ldots 60$ and observes that for $s = 4k$, they are related to Eienstein series

$$\sum_{a,b}{}' (a+ib)^{-4k}, \quad i = \sqrt{-1}.$$

a and b running through all integers not simultaneously zero. Ramanujan also knows



Another page from the 'Lost' Notebook

that this is related to the lemniscate integral

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}}$$

(see[9] for $k = 2$). In this case they are related to what are called Hurwitz numbers[5]. But Ramanujan's viewpoint is independent and goes much further. The series

$$(Z(s))^{-1} = \prod_{p} \left(1 - \frac{\chi(p)}{(Np)^s}\right)^{-1}$$

where $\chi(p) = \bar{p}^s$, the bar denoting complex conjugation and $Np = p \cdot \bar{p} > 0$ and p runs through the primes mentioned earlier, is a Hecke L-series with 'Grössencharaktere' related to the imaginary quadratic field $Q(\sqrt{-1})$. Curiously enough, Hecke was investigating such series precisely in 1918 in his researches on algebraic number theory[3,4]. Ramanujan's is a remarkable achievement and he writes to a letter Hardy, a part of which is reproduced on page 204.

A detailed discussion of these results is to be found in ref. 7.

MS (D) is a collection of unpublished parts of several papers which were published in the British periodicals during Ramanujan's stay in England. These papers had not been published in full presumably because of their length. For example, as Hardy mentions[8] (p. 339), part of the long paper 'Highly composite Numbers' which was suppressed is to be found in (D) though some sections ((62)–(68)) are missing. It might be interesting to study these anew in the light of later work of P. Erdös and others.

A most interesting part of MS (D) is the section due to B. M. Wilson who had 'edited' twelve chapters from Ramanujan's Notebooks.

The most important manuscript, without doubt, is MS (E). It is a mine of formulae written in the style of the Notebooks containing a very large number of results on mock-theta functions, continued fractions, q-series, complex multiplication, etc. G. E. Andrews wrote to us last year that he had proved a large number of formulae in this manuscript. We just quote one formula related to continued fractions; the formula is indicative of the fertility of his ideas and the beauty of his results:

$$\frac{2\sqrt{(x)}}{1 -}\ \frac{1}{e^{x} + e^{-x} -}\ \frac{1}{e^{2x} + e^{-2x} -} \cdots$$

$$= \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} e^{G(x)}$$

where

$$G(x) = a_2 x^2 + a_4 x^4 + \ldots$$

and

$$a_n = \frac{4\Gamma(n)}{\pi^{2n+1}} \left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\right) \times \left(\frac{1}{1^{n+1}} - \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} \cdots\right)$$

$$a_2 = \frac{1}{48},\ a_4 = \frac{1}{1152}, \ldots$$

It is to be noticed that Ramanujan knew not only the values of the zeta function at even positive integers but also of the L-series with real characters at odd positive integers. There is another example like the above in (E).

There are any number of variations related to Ramanujan's continued fraction mentioned earlier.

However, one of the most interesting features of the manuscript (E) is the insight, albeit slight, that it affords us into the way Ramanujan must have arrived at the beautiful results in (E). While Ramanujan had tremendous intuition about and insight into formulae, one does not *even now* know of the tremendous amount of computations and work that must have preceded before the many beautiful formulae crystallized. We reproduce (see pages 206 and 208) two facsimiles of "working". It is quite possible that the many papers in which he had done the "working" have either perished or been thrown away.

In conclusion, the fascinating life of Ramanujan and the beauty of the many results obtained by him in his several unpublished manuscripts encourages us to hope that many young and able mathematicians in India will study his work more carefully and relate Ramanujan's work to contemporary work in modular forms and arithmetic. Such work besides being greatly rewarding will be a signal service to the mathematical community.

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