COMBINATION OF REGRESSION AND RATIO ESTIMATE

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ABSTRACT

In this note an estimator dual to Mohanty estimator has been proposed. The properties of the proposed estimator have been discussed under the large sample approximations.

INTRODUCTION

Let a simple random sample of n units be drawn from a population of N units. Denote the variable of interest by y, and two auxiliary varibales correlated with y, by x and z. Let $\bar{Y}, \bar{X}, \bar{Z}$ be the population means and $\bar{y}, \bar{x}, \bar{z}$ be the sample means. It is desired to estimate $\bar{Y}$ using supplementary information on x and z. The usual regression estimator $(\bar{y}_{1r})_{yx}$ and ratio estimator $\bar{y}_R$ can be defined as

$$(\bar{y}_{1r})_{yx} = \bar{y} + b(\bar{X} - \bar{x}), \tag{1}$$

and

$$\bar{y}_R = (\bar{y}/\bar{x})\bar{X}, \tag{2}$$

where b is the coefficient of regression of y on x in the sample. Srivenkataramana[1] proposed a dual-to-ratio estimator as

$$\bar{y}_{Rd} = (\bar{y}/\bar{X})\bar{x}^*, \tag{3}$$

where $\bar{x}^* = (N\bar{X} - n\bar{x})/(N-n)$. Mohanty[2] proposed a regression ratio estimate based on two auxiliary variables x and z as

$$\bar{y}_{RR} = [(\bar{y}_{1r})_{yx}/\bar{z}]\bar{Z}. \tag{4}$$

In this note we propose a dual to Mohanty[2] estimator and define it as

$$\bar{y}_H = [(\bar{y}_{1r})_{yx}/\bar{Z}]\bar{z}^*, \tag{5}$$

where $\bar{z}^* = (N\bar{Z} - n\bar{z})/(N-n)$. It is assumed that the population means $\bar{X}$ and $\bar{Z}$ are known.

BIAS AND MEAN SQUARE ERROR (MSE) OF $\bar{y}_H$:

Following Mohanty[2], the bias and MSE of $\bar{y}_H$ to terms of order $O(n^{-1})$ are obtained as

$$B(\bar{y}_H) = (\bar{Y}/N)(\rho_1\rho_3 - \rho_2)c_y c_z - \text{cov}(b, \bar{x}), \tag{6}$$

$$M(\bar{y}_H) = (1-f)(\bar{Y}^2/n)[c_y^2(1-\rho_1^2) - 2g(\rho_2 - \rho_1\rho_3)c_y c_z + g^2 c_z^2], \tag{7}$$

where $f = n/N$, $g = n/(N-n)$, $c_y = \sigma_y/\bar{Y}$, $c_x = \sigma_x/\bar{X}$, $c_z = \sigma_z/\bar{Z}$, $\rho_1 = \text{cov}(y,x)/\sigma_y\sigma_x$, $\rho_2 = \text{cov}(y,z)/\sigma_y\sigma_z$ and $\rho_3 = \text{cov}(x,z)/\sigma_x\sigma_z$.

Mohanty[2] obtained the MSE of $\bar{y}_{RR}$ to terms of order $O(n^{-1})$ as

$$M(\bar{y}_{RR}) = (1-f)(\bar{Y}^2/n)[(1-\rho_1^2)c_y^2 + c_z^2 - 2c_y c_z(\rho_2 - \rho_1\rho_3)]. \tag{8}$$

The MSE of the estimators $\bar{y}$, $(\bar{y}_{1r})_{yx}$ and $(\bar{y}_{1r})_{yz}$ (regression of y on z) to terms of order $O(n^{-1})$ are

$$M(\bar{y}_{1r})_{yx} = (1-f)(\bar{Y}^2/n)(1-\rho_1^2)c_y^2, \tag{9}$$

$$M(\bar{y}_{1r})_{yz} = (1-f)(\bar{Y}^2/n)(1-\rho_2^2)c_y^2, \tag{10}$$

$$V(\bar{y}) = M(\bar{y}) = (1-f)(\bar{Y}^2/n)c_y^2. \tag{11}$$

REMARKS

The consistent estimators of the MSEs (7)–(11) can be obtained by replacing $\bar{Y}$, c_y, c_x, c_z, ρ_1, ρ_2 and ρ_3 by sample statistics $\bar{y}$, $\hat{c}_y(= s_y/\bar{y})$, $\hat{c}_x(= (s_x/\bar{x}))$, $c_z(= s_z/\bar{z})$ r_1, r_2 and r_3, where $s_u^2 = (n-1)^{-1}\sum_{i=1}^{n}(u_i - \bar{u})^2$, $\bar{u} = \sum_{i=1}^{n} u_i/n$; $u = x, y, z$; $r_1 = \text{cov}(y,x)/s_y s_x$, $r_2 = \text{cov}(y,z)/s_y s_z$ and $r_3 = \text{cov}(x,z)/s_x s_z$.

Comparisons

The proposed estimator $\bar{y}_H$ is more efficient than the estimators $\bar{y}$ or $\bar{y}_{RR}$ according as

$$\{\rho_1^2 c_y^2 + 2g\, c_y c_z(\rho_2 - \rho_1\rho_3) - g^2 c_z^2\} > 0; \tag{12}$$

either $(\rho_2 - \rho_1\rho_3) < \{(1+g)c_z/(2c_y)\}$;

$$(1-g) > 0; \tag{13a}$$

or $(\rho_2 - \rho_1\rho_3) > \{(1+g)c_z/(2c_y)\}$;

$$(1-g) < 0. \tag{13b}$$

The condition (13a) can also be written as

$$\frac{\{2(\rho_2 - \rho_1\rho_3)c_y - c_z\}}{c_z} < g < 1. \tag{14}$$

Let $\rho_{i(1)} > 0$, $(i = 1,2,3)$; $c_{y(1)} > 0$ and $c_{z(1)} > 0$ be the estimated (or guessed) values of ρ_i $(i = 1,2,3)$; c_y and c_z

respectively, from the data at hand such that

$$0 < \rho_{i(1)} \leqslant \rho_i, (i = 1,2,3); \quad 0 < c_{y(1)} \leqslant c_y$$

and

$$0 < c_{z(1)} \leqslant c_z.$$

It may be shown that a set of sufficient conditions for $\bar{y}_H$ to be more efficient than $\bar{y}_{RR}$ would be given by

$$\frac{[2(\rho_{2(1)} - \rho_{1(1)}\rho_{3(1)})c_{y(1)} - c_{z(1)}\}}{c_{z(1)}} < g < 1. \quad (15)$$

From the expressions for $M(\bar{y}_{RR})$ in (8) and $M(\bar{y})$ in (11) it follows that $\bar{y}_{RR}$ is more efficient than $\bar{y}$ when

$$(\rho_2 - \rho_1\rho_3) > \{c_z/(2c_y)\}. \quad (16)$$

Since g is usually small, (13a) and (16) imply that for most part $\bar{y}_H$ is superior, in terms of MSE, to $\bar{y}$ just when $\bar{y}_{RR}$ is inferior to $\bar{y}$. In this sense $\bar{y}_H$ and $\bar{y}_{RR}$ are complementary.

DOUBLE SAMPLING

Let $\bar{x}'$ and $\bar{z}'$ be the means of x and z for the first phase large sample of size n' ($> n$). n is the size of the second phase sample and $\bar{y}$, $\bar{x}$ and $\bar{z}$ are associated sample means for y, x and z. We estimate $\bar{Y}$ first by $(\bar{y}_{1r})_{yx} = \bar{y} + b(\bar{x}' - \bar{x})$ and this estimate is used to get the estimator

$$\bar{y}_{ds} = [(\bar{y}_{1r})_{yx}/\bar{z}']\bar{z}^{*\prime}; \quad (17)$$

where $\bar{z}^{*\prime} = (n'\bar{z}' - n\bar{z})/(n' - n)$.

The bias and mean square error of (17) are given by the following theorems:

Theorem 1—The bias of the two phase sampling estimator (17) to terms of order O (n^{-1}), is given by

$$B(\bar{y}_{ds}) = \left(\frac{\beta g'}{\bar{Z}}\right)\{(\text{cov}(\bar{x},\bar{z}) - \text{cov}(\bar{x},\bar{z}')) - (\text{cov}(\bar{x}',\bar{z}) - \text{cov}(\bar{x}',\bar{z}'))\} - \{g'(\text{cov}(\bar{y},\bar{z}) - \text{cov}(\bar{y},\bar{z}')) + (\text{cov}(b,\bar{x}) - \text{cov}(b,\bar{x}'))\}, \quad (18)$$

where $\beta = \text{cov}(x,y)/\sigma_x^2$, $g = n/(n' - n)$.

Theorem 2—In double sampling if the second phase sample is a subsample of the first phase sample, MSE of the estimator (17) to terms of order O (n^{-1}), is given by

$$M(\bar{y}_{ds}) = \{(n' - n)/n'n\}\bar{y}^2[c_y^2(1 - \rho_1^2) + g'^2c_z^2 + 2g'(\rho_1\rho_3 - \rho_2)c_yc_z] + \{(N - n')/Nn'\}\bar{Y}^2c_y^2. \quad (19)$$

Theorem 3—In double sampling if the first and second phase samples are independent, MSE of (17), to terms of order O (n^{-1}), is given by

$$M(\bar{y}_{ds}) = (\bar{Y}^2/n)[g'^2c_z^2 + (1 - \rho_1^2)c_y^2 + 2g'(\rho_1\rho_3 - \rho_2)c_yc_z] + (\bar{Y}^2/n')[g'^2c_z^2 + \rho_1^2c_y^2 + 2g'\rho_1\rho_3c_yc_z]. \quad (20)$$

STRATIFIED SAMPLING

Let N_h denote the number of units in the h^{th} stratum and n_h the size of the sample to be selected therefrom, so that

$$\sum_{h=1}^{L} N_h = N \text{ and } \sum_{h=1}^{L} n_h = n. \quad (21)$$

Let $\bar{y}_h$, $\bar{x}_h$ and $\bar{z}_h$ ($h = i \ldots L$) be the sample means of the variables y, x, z for the h^{th} stratum. $\bar{y}_{st} = \sum_{h=1}^{L} W_h\bar{y}_h$, $\bar{x}_{st} = \sum_{h=1}^{L} W_h\bar{x}_h$ and $\bar{z}_{st} = \sum_{h=1}^{L} W_h\bar{z}_h$, where $W_h = N_h/N$, are stratified sample means of y, x and z respectively. Then the following two types of estimators of $\bar{Y}$ can be formed.

(i) Separate estimator

$$\hat{\bar{Y}}_s = \sum_{h=1}^{L} W_h(\bar{y}_{1rh}/\bar{Z}_h)\bar{z}_h^*, \quad (22)$$

where $\bar{y}_{1rh} = \bar{y}_h + b_h(\bar{X}_h - \bar{x}_h)$ and $\bar{z}_h^* = (N_h\bar{Z}_h - n_h\bar{z}_h)/(N_h - n_h)$.

(ii) Combined estimator

$$\hat{\bar{Y}}_c = [(\bar{y}_{1r})_{st}/\bar{Z}]\bar{z}_{st}^*; \quad (23)$$

where $(\bar{y}_{1r})_{st} = \bar{y}_{st} + b(\bar{X} - \bar{x}_{st})$ and $z_{st}^* = (N\bar{Z} - n\bar{z}_{st})/(N - n)$.

The bias and MSE of (22) and (23) are given by the following theorems:

Theorem 4—Bias and MSE of the separate estimator to the first order, are

$$B(\hat{\bar{Y}}_s) = \sum_{h=1}^{L}\{(\bar{Y}_h/N)(\rho_{1h}\rho_{3h} - \rho_{2h})c_{z_h}c_{y_h} - \text{cov}(b_h,\bar{x}_h)\}; \quad (24)$$

and

$$M(\hat{\bar{Y}}_s) = \sum_{h=1}^{L}\{(N_h - n_h)/N_h\}W_h^2 \times (\bar{Y}_h^2/n_h)[(1 - \rho_{1h}^2) \times c_{y_h}^2 - 2g_h(\rho_{2h} - \rho_{1h}\rho_{3h}) \times c_{y_h}c_{z_h} + g_h^2c_{z_h}^2]; \quad (25)$$

where $g_h = n_h/(N_h - n_h)$, $\rho_{1h} = \text{cov}(y_h, x_h)/\sigma_{y_h}\sigma_{x_h}$,

$$\rho_{2h} = \text{cov}(y_h, z_h)/\sigma_{y_h}\sigma_{z_h},\ \rho_{3h} = \text{cov}(x_h, z_h)/\sigma_{x_h}\sigma_{z_h},$$

$$c_{y_h} = \sigma_{y_h}/\overline{Y}_h,\ c_{x_h} = \sigma_{x_h}/\overline{X}_h,\ c_{z_h} = \sigma_{z_h}/\overline{Z}_h.$$

Theorem 5—Bias and MSE of the combined estimator to the first order are

$$B(\hat{\overline{Y}}_c) = \sum_{h=1}^{L} g\left(\frac{N_h - n_h}{N_h n_h}\right) W_h^2 (\beta\rho_{3h}\sigma_{x_h} - \rho_{2h}\sigma_{yh})(\sigma_{z_h}/\overline{Z}) - \text{cov}(b, \overline{x}_{st}), \qquad (26)$$

and

$$M(\hat{\overline{Y}}_c) = \sum_{h=1}^{L}\left(\frac{N_h - n_h}{N_h n_h}\right) W_h^2 \Big[\sigma_{y_h}^2 + \beta^2\sigma_{x_h}^2 + g^2(\overline{Y}/\overline{Z})^2\sigma_{z_h}^2 - 2\beta\rho_{1h}\sigma_{x_h}\sigma_{y_h} + 2g(\overline{Y}/\overline{Z})\sigma_{z_h}(\rho_{2h}\sigma_{y_h} - \rho_{3h}\sigma_{x_h})\Big]. \qquad (27)$$

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DISPERSIVE OPTICAL BISTABILITY IN PLASMAS

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ABSTRACT

A model for the dispersive optical bistability in plasmas is suggested. Using the theory of coherent wave–wave interactions in plasmas, and the Vlasov–Maxwell equations, the conditions for the existence of optical bistability, satisfying the constraints of perturbative expansions, are derived for the general case of a cómplex coupling constant between the driven and the driving modes.

INTRODUCTION

Optical bistability (OB) has been extensively studied in recent years. Mean field approximation has been used in the theory of absorptive[1,2] and dispersive[3] OB. Bonifacio and Lugiato gave the theory of absorptive and dispersive OB incorporating the incident radiation[4]. The photon statistics of absorptive[5] and dispersive[6] OB have also been found. An anharmonic oscillator model[7] was proposed to describe dispersive OB and the switching characteristics studied[8], in the case of the incident radiation being damped either very slowly or very quickly with respect to the oscillator.

This paper studies the occurrence of OB in an electron-ion plasma using the anharmonic oscillator model. The non-linear processes in the plasma are assumed to give rise to the anharmonicity in the model. The nonlinear theory of laser-plasma interaction is studied in the kinetic picture. The most general conditions for the existence of OB in a dispersive medium have been derived, the special cases of which agree with those in literature.

Model

Consider a warm, fully ionized electron-ion plasma with a plane-polarised laser radiation incident on it. The plasma acts as a dispersive medium. Let the low frequency ion-acoustic waves of frequency ω_a be maintained at a constant amplitude by an external agent. The incident laser radiation has a frequency $\omega_L = \omega_p + O(\omega_a)$, where ω_p is the electron plasma frequency. [$O(\omega_a)$ means that, since ω_L and $\omega_p \gg \omega_a$,