SHORT COMMUNICATIONS

SPHERICALLY SYMMETRIC LINE ELEMENT IN A GENERALIZED CARTESIAN COORDINATE SYSTEM

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NOTING the importance of spherical symmetry, Takeno[1] devoted a research memoir for its rigorous study. In 4-dimensions he defines that a geometric object G is spherically symmetric (SS) when it satisfies

$$L_{(K)}(G) = 0, \tag{1}$$

where $L_{(K)}$ denotes the Lie derivative with respect to the Killing vector K and

$$K = \overset{a(i)}{K} \qquad (a = 1,2,3).$$

The line element

$$ds^2 = g_{ij}\, dx^i dx^j \tag{2}$$

is said to be SS if g_{ij} are SS. Extending the definition of SS given in (1) for the line element (2), (1) takes the form of the Killing equation

$$K_{i;j} + K_{j;i} = 0,$$

where (;) denotes the covariant derivative. Further, this Killing equation after straightforward calculation becomes

$$K^h \frac{\partial g_{ij}}{\partial x^h} + g_{hj} \frac{\partial k^h}{\partial x^i} + g_{ih} \frac{\partial k^h}{\partial x^j} = 0. \tag{3}$$

In spherical polar coordinate system Takeno[1] obtained the general form of a SS line element

$$ds^2 = -B(d\theta^2 + \sin^2\theta d\phi^2) - A dr^2 + C dt^2 + 2D dr dt, \tag{4}$$

where A, B, C and D are functions of r and t. He further obtained a SS line element

$$ds^2 = h_1(dx^2 + dy^2 + dz^2) + h_3 dt^2 + r^2 h_2 dr^2 + 2r h_4 dr\, dt \tag{5}$$

in a generalized cartesian coordinate system x^i from (4) by applying a coordinate transformation

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta. \tag{6}$$

Instead of using (6), we have obtained with the help of (3), the general form of a SS line element in a generalized cartesian coordinate system ($x^i = x, y, z, t$). However, we have obtained the form (5) of Takeno[1] from our results.

Killing vectors

Following Takeno[1], the space-time (2) is SS if it admits the Killing vectors

$$\begin{aligned} \overset{1(i)}{K} &= (0,\ z,\ -y,\ 0), \\ \overset{2(i)}{K} &= (-z,\ 0,\ x,\ 0), \\ \overset{3(i)}{K} &= (y,\ -x,\ 0,\ 0). \end{aligned} \tag{7}$$

in a generalized cartesian coordinate system. Employing (7) in (3), we obtain

$$\begin{aligned} &z(\partial g_{11}/\partial y) - y(\partial g_{11}/\partial z) = 0,\ z(\partial g_{12}/\partial y) - y(\partial g_{12}/\partial z) - g_{13} = 0, \\ &z(\partial g_{13}/\partial y) - y(\partial g_{13}/\partial z) + g_{12} = 0, \\ &z(\partial g_{14}/\partial y) - y(\partial g_{14}/\partial z) = 0, \\ &z(\partial g_{22}/\partial y) - y(\partial g_{22}/\partial z) - 2g_{32} = 0, \\ &z(\partial g_{23}/\partial y) - y(\partial g_{23}/\partial z) - g_{33} + g_{22} = 0, \\ &z(\partial g_{24}/\partial y) - y(\partial g_{24}/\partial z) - g_{34} = 0, \\ &z(\partial g_{33}/\partial y) - y(\partial g_{33}/\partial z) + 2g_{23} = 0, \\ &z(\partial g_{34}/\partial y) - y(\partial g_{34}/\partial z) + g_{24} = 0, \\ &z(\partial g_{44}/\partial y) - y(\partial g_{44}/\partial z) = 0. \end{aligned}$$

for the first Killing vector $\overset{1}{K}{}^{(i)}$. Similarly one can get other two sets each of ten equations by inserting the values of $\overset{2}{K}{}^{(i)}$ and $\overset{3}{K}{}^{(i)}$ from (7) in Killing equations (3). Thus, we obtain thirty equations which can be further reduced to

$$AG = 0,$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & y & 0 \\ 0 & z & 0 & 0 & 0 & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & -x & 0 \\ 0 & 0 & -y & 0 & 0 & x & 0 & 0 & 0 & 0 \\ z & 0 & -x & 0 & -z & y & 0 & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & -z & 0 & y & 0 & 0 \end{bmatrix}_{6\times 10}$$

and $G = [g_{11}, g_{12}, \ldots, g_{44}]^T_{10\times 1}$. The rank of A is 6. Then from the theory of matrix algebra we

can select $10-6=4$ independent components of g_{ij}, say, g_{12}, g_{22}, g_{24} and g_{44}. The other components of g_{ij} can then be expressed in terms of g_{12}, g_{22}, g_{24}, g_{44}. Interestingly this is not the situation in the polar coordinate system[1] or the cylindrical coordinate system[2].

In the case of a polar coordinate system[1] $g_{12}=g_{13}=g_{23}=g_{24}=g_{34}=0$ and g_{11}, g_{22}, g_{44}, g_{14} are functions of r and t, $g_{33}=g_{22}\sin^2\theta$.

For the cylindrical coordinate system $g_{12}=g_{23}=g_{24}=0$ and assume $g_{13}=0\Longrightarrow g_{11}=g_{33}=g_{22}/r^2=A$ and $g_{44}=B$, $g_{14}=C$.

We then obtain (2) with

$$g_{24}=(y/z)C,\ g_{14}=(x/z)C,\ g_{23}=(z/x)B,$$
$$g_{13}=(z/y)B,\ g_{11}=[(x/y)-(y/x)]B+A,$$
$$g_{33}=[(z^2-y^2)/xy]B+A,$$

where $g_{22}=A$, $g_{12}=B$, $g_{34}=C$ and $g_{44}=D$. After some adjustment the line element assumes the form

$$ds^2=[A-(y/x)B](dx^2+dy^2+dz^2)+Dd t^2 + 2C/z\ (xdx+ydy+zdz)dt + [(B/xy)(xdx+ydy+zdz)^2],$$

which is the form of Takeno[1] with

$h_1=[A-(y/x)B]$, $h_2=B/xy$, $h_3=D$, $h_4=C/z$.

The authors are thankful to Prof. T. M. Karade for guidance and encouragement. Thanks are also due to the referee for suggestions.

18 October 1986; Revised 21 February 1987

1. Takeno, H., *The theory of spherically symmetric space-time*, *Sci. Rep. Res. Inst. Theor. Phys.*, Hiroshima University, 1966, p. 4.
2. Karade, T. M. and Tulsani, V., *Acta Cien. Ind.*, 1978, **4**, 215.

STUDIES IN NUCLEOPHILIC SUBSTITUTIONS : II. REACTIONS BETWEEN ARYL ALKYL ETHERS AND HYDROBROMIC ACID

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THE acid cleavage of alkyl ethers is a reaction that can proceed by a A_2 or A_1 pathway depending on the nature of the alkyl substituent[1,2]. While there have been a number of kinetic studies on acid-catalysed hydrolysis of dialkyl ethers, only a few investigations exist on aryl alkyl ethers. This system, with suitable modifications of the aryl system is eminently suited to study the structure-reactivity relationship in a displacement reaction. The results of such a kinetic study of the reaction between a number of substituted phenyl methyl ethers (anisoles) and HBr are now reported.

The kinetics of these reactions were followed by argentometric determination of unreacted HBr. The reaction between HBr and anisole is essentially one that follows the rate law

$$\frac{-d[HBr]}{dt}=k_2\ [\text{anisole}]\ [HBr].$$

All the other subtituted anisoles also obey a similar rate law. However, it is interesting to note that whatever be the substituent in the phenyl ring, each of the substituted anisoles reacts faster than the parent compound (table 1). The kinetic data, when subjected to a Hammett LFER analysis, yield a concave-up type of curve (figure 1). To our knowledge, this is the first instance of such a behaviour in the cleavage of aryl alkyl ethers. The splitting of halogen-substituted phenoxyacetic acid with hydriodic acid showed only a small substituent effect[3], but certainly not such a dichotomy.

The biphasic plot is clearly indicative of competing mechanisms for the cleavage of these aryl methyl ethers. Although ethers cleavage can take place either by an A_1 or A_2 mechanism, the observed rate law and the nature of the alkyl ethers (primary) would rule out the unimolecular pathway. Therefore, the incidence of the biphasic plot has to be explained on the basis of subtle variations in one and the same mechanism. The following explanations are offered to explain the observed trend in the Hammett Plot.

i) $R-C_6H_4-OCH_3+H^+ \overset{\text{fast}}{\rightleftharpoons} R-C_6H_4-\overset{+}{O}(H)-CH_3$

$R-C_6H_4-\overset{+}{O}(H)-CH_3+Br^- \xrightarrow{\text{slow}} R-C_6H_4-OH+CH_3Br$

ii) $R-C_6H_4-OCH_3+Br^- \xrightarrow{\text{slow}} R-C_6H_4-O^-+CH_3Br$

$R-C_6H_4-O^-+H^+ \overset{\text{fast}}{\rightleftharpoons} R-C_6H_4-OH$

The first of these two pathways would account for the left part of the plot (where the conjugate acid formation is facilitated by the electron-releasing substituents in the phenyl ring) and the second