FRACTIONAL VOTING SYSTEM: A VOTING SYSTEM TO CIRCUMVENT ARROW'S PARADOX

K. K. NAMBIAR

School of Computer and System Sciences, Jawaharlal Nehru University, New Delhi 110 067, India

ABSTRACT

Making use of information theory it is shown that fractional voting system (FVS) can be used to circumvent Arrow's Paradox. A crucial fact about FVS is that the input to the voting system is the preference distribution of the voters instead of the usual preference orders. In the FVS it is possible to associate a preference distribution for the society as a whole and it is unique if the axioms of unanimity and independence are to be satisfied. An interesting fact is that the same unique distribution results when injustice, as defined here, is minimized.

INTRODUCTION

In 1951 Arrow showed that if the input to a voting system is the preference order of each voter for the candidates, then there is no reasonable way to assign a *preference order* appropriate for the entire society[1-3]. It is shown here that the situation alters drastically if the input to the voting system is changed to the *preference distribution* of each voter and it becomes not only possible but also natural to associate a unique preference distribution for the body of voters. The voting system proposed here is called the fractional voting system (FVS) and it is further demonstrated that information theory is an efficient tool to facilitate poll analysis.

In FVS each voter has at his disposal not just one vote but a number of votes. In figure 1, N_j is the number of votes available to voter b_j.

$$\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_n \end{array} \\ \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{m1} & k_{m2} & k_{m3} & \cdots & k_{mn} \end{bmatrix} \\ \begin{array}{ccccc} N_1 & N_2 & N_3 & \cdots & N_n \end{array} \end{array} \begin{array}{c} \\ M_1 \\ M_2 \\ \vdots \\ M_m \\ N \end{array}$$

Figure 1. Matrix $K=[k_{ij}]$.

This voter can distribute his N_j votes to the different candidates in any manner he pleases. The matrix $K=[k_{ij}]$ is called a *voting pattern*. The integer k_{ij} is the number of votes given by the voter b_j to the candidate a_i. Thus

$$\sum_{i=1}^{m} k_{ij}=N_j.$$

The total number of votes collected by the candidate a_i is given by

$$\sum_{j=1}^{n} k_{ij}=M_i.$$

Thus

$$\sum_{i=1}^{m} M_i=N$$

and $$\sum_{j=1}^{n} N_j=N.$$

After collecting the voting pattern as usual, the candidate who collects the maximum number of votes is declared as the winner by FVS.

$$\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots & b_n \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mn} \end{bmatrix} \\ \begin{array}{ccccc} r_1 & r_2 & r_3 & \cdots & r_n \end{array} \end{array} \begin{array}{c} \\ q_1 \\ q_2 \\ \vdots \\ q_m \\ \\ \end{array}$$

Figure 2. Matrix $P=[p_{ij}]$.

In figure 2,

$$p_{ij}=k_{ij}/N, \quad q_i=M_i/N, \quad r_j=N_j/N.$$

A matrix with non-negative elements which add up to one is called a *distribution*. An element of such a matrix is called a *mass*. The matrix $P=[p_{ij}]$ gives the preference distribution of the electorate for the candidates. The matrix $[r_j]$ gives the *prominence* *distribution* of the voters within the society. If

$$q_{ij} = p_{ij}/r_j$$

then each column of $Q = [q_{ij}]$ gives the preference distribution of an individual voter for the candidates. If

$$r_{ij} = p_{ij}/q_i$$

then each row of $R = [r_{ij}]$ gives the *affinity distribution* of a candidate for the voters.

We can now state the difference between the usual voting systems (UVS) and FVS. In UVS the input to the voting system is the *preference order* of a voter and each voter has a single vote[1-3]. In FVS the input is the *preference distribution* of a voter for the candidates and the *prominence distribution* of the voters within the society. Arrow[1] has shown that in the case of UVS, it is impossible for the society to have a reasonable preference order for the candidates if the axioms of unanimity and independence are to be satisfied. The main purpose of this paper is to show that in the case of FVS, it is possible for the society to have a preference distribution for the candidates satisfying both unanimity and independence as defined here. Further, it is shown that this distribution is unique.

INFORMATION THEORY AND PSEPHOLOGY

With FVS it is possible to carry out a thorough analysis of the poll. For the purpose we take some concepts from information theory and make up some definitions. All logarithms mentioned here are to the base 2. In the following A represents the candidates and B represents voters.

Definitions

1. *Voter hesitance*

$$H_j(A) = -\sum_{i=1}^{m} q_{ij} \log q_{ij}.$$

2. *Voter preference*

$$I_j(A) = \log m - H_j(A).$$

3. *Conditional hesitance*

$$H(A\bar{B}) = \sum_{j=1}^{n} r_j H_j(A).$$

4. *Conditional preference*

$$I(A\bar{B}) = \log m - H(A\bar{B}).$$

5. *Panel homogeneity*

$$H(A) = -\sum_{i=1}^{m} q_i \log q_i.$$

6. *Panel heterogeneity*

$$I(A) = \log m - H(A).$$

7. *Clan uniformity*

$$H_i(B) = -\sum_{j=1}^{n} r_{ij} \log r_{ij}.$$

8. *Clan affinity*

$$I_i(B) = \log n - H_i(B).$$

9. *Conditional uniformity*

$$H(\bar{A}B) = \sum_{i=1}^{m} q_i H_i(B).$$

10. *Conditional affinity*

$$I(\bar{A}B) = \log n - H(A\bar{B}).$$

11. *Electorate homogeneity*

$$H(B) = -\sum_{j=1}^{n} r_j \log r_j.$$

12. *Electorate heterogeneity*

$$I(B) = \log n - H(B).$$

13. *Societal homogeneity*

$$\begin{aligned} H(A+B) &= \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \log p_{ij} \\ &= H(A) + H(\bar{A}B) \\ &= H(A\bar{B}) + H(B). \end{aligned}$$

14. *Societal heterogeneity*

$$\begin{aligned} I(A+B) &= \log mn - H(A+B) \\ &= I(A) + I(\bar{A}B) \\ &= I(A\bar{B}) + I(B). \end{aligned}$$

15. *Election campaign*

$$\begin{aligned} H(AB) &= H(A) + H(B) - H(A+B) \\ &= H(A) - H(A\bar{B}) \\ &= -H(\bar{A}B) + H(B). \end{aligned}$$

16. *Election propaganda*

$$\begin{aligned} I(AB) &= -H(AB) \\ &= I(A)+I(B)-I(A+B) \\ &= I(A)-I(A\bar{B}) \\ &= -I(\bar{A}B)+I(B). \end{aligned}$$

17. *Popularity of a candidate*

$$P_i = \log mq_i.$$

18. a_i is a *popular candidate* if $P_i \geq 0$.
19. a_i is an *eminent candidate* if he is the only popular candidate.
20. a_i is a *favourite candidate* if $P_i \geq I(A)$.
21. a_i is an *outstanding candidate* if he is the only favourite candidate.
22. a_i is a *charismatic candidate* if he collects all the votes without exception.
23. *Prominence* of a voter

$$Q_j = \log nr_j.$$

24. b_j is a *significant voter* if $Q_j \geq I(B)$.
25. b_j is a *dominant voter* if he is the only significant voter.
26. b_j is a *dictator* if he has all the votes at his disposal without exception.
27. An election is a *passive election* if $I(AB)=0$.
28. An election is called a *dictatorial election* if

$$I(A+B) = \log mn.$$

29. An election is called a *positive election* if there is an eminent candidate.
30. An election is called a *definite election* if there is an outstanding candidate.
31. *Societal preference distribution* $[s_i]$ is the preference chosen by the voting system for the candidates.
32. *Societal injustice* to a candidate

$$J_i = \log (q_i/s_i).$$

33. Societal injustice

$$J = \sum_{i=1}^{m} q_i \log (q_i/s_i).$$

It can be shown that J can never be negative. Note that H is a function of distributions and

$$H[p_1 p_2 \cdots p_k] = -\sum_{i=1}^{k} p_i \log p_i.$$

H is called entropy in information theory and it is well known that there is no alternative but to have this definition[4] if H is to satisfy certain mild conditions. A possible set of such conditions is

i) H is an analytic function with $H[.5\ \ .5]=1$
ii) $H[p_1 p_2 \cdots p_k 0] = H[p_1 p_2 \cdots p_k]$
iii) $H[p_{ij}] = H[r_j] + \sum_{j=1}^{n} r_j H_j[q_{ij}]$.

We mention this as an aside only to indicate how powerful the concept of entropy can be in psephology. It can be shown[4] that H is non-negative and attains a maximum when all the masses are equal and it is zero when one of the masses equals one. The following lemmas can be easily proved making use of these facts and hence the proofs are omitted.

Lemmas

1. $P_i=0$ if and only if a_i gets exactly the average number of votes. P_i is positive or negative depending upon whether a_i collects above or below the average number of votes. $P_i = \log m$, if and only if a_i is a charismatic candidate. The range of P_i is $[-\infty, \log m]$.
2. The hierarchy of the candidates is: charismatic, eminent, outstanding, favourite and popular, i.e. each of these classes implies the classes that follow.
3. In any election there is at least one favourite candidate.
4. A positive election is always a definite election.
5. $I(A)=0$ if and only if all the candidates collect equal votes. $I(A)=\log m$ *if and only if there is a* charismatic candidate.
6. $Q_j=0$ if and only if b_j has exactly the average number of votes at his disposal. Q_j is positive or negative depending upon whether b_j has above or below the average number of votes. $Q_j=\log n$ if and only if b_j is a dictator. The range of Q_j is $[-\infty, \log n]$.
7. The hierarchy of voters is: dictator, dominant and significant.
8. In any election there is at least one significant voter.
9. $I(B)=0$ if and only if all the voters have equal votes. $I(B)=\log n$ if and only if there is a dictator.
10. $I(A\bar{B})=0$ if and only if each individual voter has given equal votes to all the candidates. $I(A\bar{B})=\log m$ if and only if each individual voter has given all his votes to a single candidate. The range of $I(A\bar{B})$ is $[0, \log m]$.

11. $I(A\bar{B})=0$ if and only if every candidate has received the same number of votes from each voter. $I(A\bar{B})=\log n$ if and only if every candidate has got all his votes from a single voter. The range of $I(A\bar{B})$ is $[0, \log n]$.
12. $I(A+B)=0$ if and only if each candidate has received the same number of votes from each voter. $I(A+B)=\log mn$ if and only if there is charismatic candidate and a dictator. The range of $I(A+B)$ is $[0, \log mn]$.
13. Let $\min\{m,n\}=m$. $H(AB)=\log m$, if and only if, each voter has given all his votes to one candidate and all candidates have collected equal votes.
14. Let $\min\{m,n\}=n$. $H(AB)=\log n$, if and only if, all voters have equal votes and each candidate has collected all votes from a single voter.

The above two lemmas can be proved if we recognize that $H(AB)$ cannot be greater than $\min\{H(A), H(B)\}$.

15. $H(AB)=0$, if and only if, all the voters have exactly the same preference distribution for the candidates. The range of $H(AB)$ is $[0, \log h]$ where $h=\min\{m,n\}$.
16. $J_i=0$, if and only if, $s_i=q_i$. J_i is positive or negative, depending upon whether q_i is greater or less than s_i. The range of J_i is $[-\infty, \infty]$.

POSSIBILITY AND JUSTICE THEOREMS

Axiom of unanimity: $(q_{11}=q_{12}=\ldots=q_{1n}=q)\Rightarrow(s_1=q)$. In words, if each individual voter gives the same preference mass to candidate a_1, so does the voting system.

Axiom of independence: $s_i=f(p_{i1}, p_{i2}, \ldots, p_{in})$, i.e. each s_i is the same function of the corresponding row of $[p_{ij}]$. In other words the voting system does not discriminate between the candidates.

Possibility theorem: In FVS it is possible to satisfy the axioms of unanimity and independence and to have a societal preference distribution. Further, this distribution is unique.

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \dfrac{p_{11}(1-q)}{q(m-1)} & \dfrac{p_{12}(1-q)}{q(m-1)} & \cdots & \dfrac{p_{1n}(1-q)}{q(m-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{p_{11}(1-q)}{q(m-1)} & \dfrac{p_{12}(1-q)}{q(m-1)} & \cdots & \dfrac{p_{1n}(1-q)}{q(m-1)} \end{bmatrix}$$

Figure 3. Modified P matrix.

Proof: From the given matrix $[p_{ij}]$, construct another P matrix as given in figure 3, where

$$q=q_1=\sum_{j=1}^{n} p_{1j}.$$

It is easy to see that for the Q matrix corresponding to this new P matrix, we have

$$q_{11}=q_{12}=\ldots=q_{1n}=q.$$

Hence from the axiom of unanimity we conclude that

$$s_1=q=q_1=\sum_{j=1}^{n} p_{1j}.$$

From the axiom of independence we conclude that

$$s_i=q_i=\sum_{j=1}^{n} p_{ij}.$$

Justice theorem: J attains the minimum value zero, if and only if, $[s_i]=[q_i]$, i.e. the only way to make sure that no injustice is done to the candidates is to choose $[q_i]$ as the societal preference distribution.

Proof: We use the method of Langrange multipliers. Consider

$$U=\sum_{i=1}^{m} q_i\log(q_i/s_i)+\lambda\sum_{i=1}^{m} s_i.$$

$$[\partial U/\partial s_i=-(q_i/s_i)\log e+\lambda=0]\Rightarrow$$
$$[(q_i/s_i)=(\lambda/\log e)]\Rightarrow$$

$$\left[\sum_{i=1}^{m} q_i=(\lambda/\log e)\sum_{i=1}^{m} s_i\right]\Rightarrow$$

$$[(\lambda/\log e)=1]\Rightarrow$$
$$s_i=q_i.$$

We have shown the uniqueness of the societal preference distribution through minimization of societal injustice.

Corollary: The range of societal injustice J is $[0, \infty]$. $J=0$ when $[s_i]=[q_i]$. $J=\infty$ when one $s_i=0$

and the corresponding $q_i \neq 0$, i.e. when the society has unreasonably chosen to ignore a candidate totally.

IMPLEMENTATION OF FVS

In FVS it is useful to consider the last candidate a_m as fictitious and the candidate may be named *anarchy*. All the votes of a voter who protests over the election itself will go to the anarchy candidate. If any voter has utilized some but not all his votes, the unutilized votes will go to anarchy. Any voter who absents himself without protesting against the election will be totally ignored by FVS. If one or more candidates get disqualified after the voting has taken place, FVS will delete their names from the contest and consider the marginal preference distributions of individual voters with respect to the remaining candidates. If anarchy wins the election, it is an indication of the existence of a substantial group of disgruntled citizens who do not want to participate in the democratic process and the breakdown of democracy. It is interesting to note that FVS caters even to this group of people and FVS is generous enough to listen to everybody and repeat the well-known Russel's paradox: What should a true democrat do when the majority says that they do not want democracy.

AN ILLUSTRATIVE EXAMPLE

The matrix shown in figure 4 shows an example in which $N=32$, $m=4$ and $n=8$.

$$[p_{ij}]=\frac{1}{32}\begin{bmatrix}1&2&0&0&0&0&0&1\\2&0&4&0&8&2&0&0\\1&0&0&4&0&1&0&2\\0&0&0&0&0&1&2&1\end{bmatrix}\quad [q_i]=\frac{1}{32}\begin{bmatrix}4\\16\\8\\4\end{bmatrix}$$

$$[r_j]=\frac{1}{32}\ [4\ \ 2\ \ 4\ \ 4\ \ 8\ \ 4\ \ 2\ \ 4]$$

Figure 4. An example.

Each of the voters b_6 and b_8 used three of their votes even though each of them had four votes at their disposal, hence their unused votes have gone to anarchy a_4. Voter b_7 had protested against the election and hence his two votes have been given to anarchy. In this election a_2 gets the highest number of votes, namely 16, and hence he gets elected.

Voter preference

$I_1(A)=1/2,\ I_2(A)=2,\ I_3(A)=2,\ I_4(A)=2,$
$I_5(A)=2,\ I_6(A)=1/2,\ I_7(A)=2,\ I_8(A)=1/2.$

Conditional preference

$I(A\bar{B})=23/16.$

Panel heterogeneity

$I(A)=1/4.$

Candidate popularity

$P_1=-1,\ P_2=1,\ P_3=0,\ P_4=-1.$

a_2 and a_3 are popular candidates and a_2 is an outstanding candidate.

Clan affinity

$I_1(B)=3/2,\ I_2(B)=5/4,\ I_3(B)=5/4,\ I_4(B)=3/4.$

Conditional affinity

$I(AB)=21/16.$

Electorate heterogeneity

$I(B)=1/8.$

Voter prominence

$Q_1=0,\ Q_2=-1,\ Q_3=0,\ Q_4=0,\ Q_5=1,\ Q_6=0,\ Q_7=-1,\ Q_8=0$; b_5 is a dominant voter.

Societal heterogeneity

$I(A+B)=25/16.$

Election campaign

$H(AB)=19/16.$

CONCLUSION AND RECOMMENDATION

Consider the Presidential election system in India where the single transferable vote (STV) is used at present. As a hypothetical case assume that the total value of the votes of all the Members of Parliament is 1001. Imagine an election in which 501 votes are in favour of the preference order ABCDE and the rest 500 votes in favour of the preference order BCDEA. If this situation arises, STV will choose *A* as the President, which is obviously a wrong choice especially because half the electorate hate *A*. Hence it is recommended that for the Presidential election the present system of STV be discarded in favour of the FVS proposed here. From the possibility and justice theorems given earlier it should be clear that anomalous situations cannot occur with FVS.

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2. Nambiar, K. K., *Curr. Sci.*, 1987, **56**, 1085.
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4. Shannon, C. E. and Weaver, W., *The mathematical theory of communication*, University of Illinois Press, Urbana, 1949.

ANNOUNCEMENTS

National Symposium on Bioactive Compounds from Marine Organisms

Place: Parangipettai
Date: December 1989
Contact: Prof. A. Subramanian
Centre of Advanced Study in Marine Biology
Parangipettai 608 502

National Symposium on Conservation and Management of Living Resources

Place: Bangalore
Date: 10–12 January 1990
Contact: Dr Shakunthala Sridhara
Organizing Secretary
National Symposium on Conservation and Management of Living Resources
University of Agricultural Sciences
GKVK, Bangalore 560 065

National Symposium on Environmental Influences on Seed and Germination Mechanism—Recent Advances in Research and Technology

Place: Jodhpur
Date: 27–29 January 1990
Contact: Prof. David N. Sen
Department of Botany
University of Jodhpur
Jodhpur 342 001

First International Conference on Vibration Problems of Mathematical Elasticity and Physics

Place: Jalpaiguri, India
Date: 20–23 October 1990
Contact: Dr M. M. Banerjee/Dr P. Biswas
Vibration Conference Secretariat
Department of Mathematics
A.C. College
Jalpaiguri 735 101
India
Abstracts by 1 February 1990

XXI National Seminar on Crystallography

Place: Bombay
Date: 27–29 December 1989
Contact: The Secretary
XXI National Seminar on Crystallography
Neutron Physics Division
Bhabha Atomic Research Centre
Bombay 400 085

National Seminar on Aquatic Pollution

Place: Trivandrum
Date: 18–20 December 1989
Contact: Dr P. K. Abdul Azis
Organizing Secretary
National Seminar on Aquatic Pollution
Department of Aquatic Biology and Fisheries
Trivandrum 695 007