Laplace process

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The infinite divisibility of the Laplace distribution allows definition of a stochastic process with stationary independent increments. Here I show that the Laplace process is not scale-invariant and is neither strictly stationary nor covariance-stationary. I also show that the Laplace process is subordinated to the standard Wiener process.

THE Wiener process or Brownian motion has been thoroughly investigated and has also found many applications in areas like economics, biology, statistical mechanics, etc. In this paper we introduce a process called Laplace process and some of its properties and contrast it with the standard Wiener process. A connection between the two is also established.

The infinite divisibility of the Laplace distribution enables us to propose the following definition. A stochastic process $\{X(t),\ t\geqslant 0\}$ with stationary independent increments is called a Laplace process if $X(0)=0$ and $X(1)$ has characteristic function $\phi(u,1)=1/(1+u^2)$.

The definition suggests that the increment $X(t+p)-X(t)$, $p>0$ has characteristic function $\phi(u,p)=(1+u^2)^{-p}$. It can be easily seen that for the Laplace process the mean of an increment $X(t+p)-X(t)$ of magnitude $p>0$ is zero, variance is $2p$ and covariance is $E\,X(s)X(t)=2s$, $s<t$. Some properties of the Laplace process are established below.

P1. The Laplace process is neither strictly stationary nor covariance-stationary.

Proof: Since the process has independent increments it cannot be strictly stationary unless it is degenerate. Since $E\,X^2(t)=2t$ depends on the epoch t it is also not covariance-stationary.

P2. The Laplace process is not scale-invariant. That is for each $b>0$, $\{X^*(t)=bX(t/b^2),\ t\geqslant 0\}$ is not a Laplace process when $X(t)$ is.

Proof: Since $\phi[bu,(t/b^2)]\neq\phi(u,t)$, where $\phi(u,t)$ is the characteristic function of $X(t)$, the Laplace process is not scale-invariant.

Since the Laplace process is symmetric and has stationary independent increments it possesses the differential and time reversal properties.

P3. For the Laplace process $X(t)$, $\{Z_\lambda(t),\ \mathscr{F}_t,\ t\geqslant 0\}$ is a positive martingale for λ in $(-1,1)$, where

$$Z_\lambda(t)=(1-\lambda^2)^t\exp(\lambda X(t)),\quad \mathscr{F}_t=\sigma\{X(s);\ 0\leqslant s\leqslant t\}.$$

Proof: Here $\{\mathscr{F}_t,\ t\geqslant 0\}$ is a nondecreasing family of sigma fields with $\mathscr{F}_0=\{\Omega,\phi\}$, Ω being the sample space. Thus $\{Z_\lambda(t),\ \mathscr{F}_t,\ t\geqslant 0\}$ is an adapted family. For each $t\geqslant 0$, $E(Z_\lambda(t))=1$. Thus, for $s<t$,

$$\begin{aligned}E(Z_\lambda(t)/\mathscr{F}_s)&=(1-\lambda^2)^t E(\exp\lambda(X(s)+X(t)-X(s))/\mathscr{F}_s)\\&=(1-\lambda^2)^t(1-\lambda^2)^{s-t}\exp(\lambda X(s))\\&=Z_\lambda(s).\end{aligned}$$

P4. For the Laplace process $X(t)$, $\{X^2(t)-t,\ \mathscr{F}_t,\ t\geqslant 0\}$ is a submartingale.

Proof: As before one can see that $\{X^2(t)-t,\ \mathscr{F}_t,\ t\geqslant 0\}$ is an adapted family. Now, for $s<t$,

$$\begin{aligned}E(X^2(t)-t/\mathscr{F}_s)&=E((X(s)+X(t)-X(s))^2-t/\mathscr{F}_s)\\&=X^2(s)-t+2(t-s)\\&>X^2(s)-s.\end{aligned}$$

P5. The Laplace process is subordinated to the standard Wiener process.

Proof: Let $W_s(x)$ denote the distribution of the standard Wiener process $\{Y(s),\ s\geqslant 0\}$. Let $F_t(x)$ be the distribution of the process obtained by randomizing the time parameter s in $Y(s)$ by $G_t(x)$, the gamma distribution with parameter t. That is

$$F_t(x)=\int_0^\infty W_s(x)\,\mathrm{d}G_t(s).$$

Taking Fourier transforms on both sides we get

$$\begin{aligned}\phi(u,t)&=\int_0^\infty \exp(-su^2/2)\,\mathrm{d}G_t(s)\\&=(1+u^2/2)^{-t},\end{aligned}$$

which is the characteristic function of an increment of magnitude t in the Laplace process (scale changed).

We may note that the standard Wiener process is scale-invariant. Also, for the standard Wiener process $Y(t)$, $\{Y^2(t)-t,\ \mathscr{F}_t,\ t\geqslant 0\}$ is a martingale.

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Optical phase conjugation in an absorbing dye by degenerate four-wave mixing using a broad-band laser

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Using a Nd : glass broad-band laser we have observed optical phase conjugation (OPC) signals in Kodak 14015 dye. Signal intensity of the OPC beam was enhanced using the laser in Q-switched mode. We suggest that generation of OPC in the dye may be due to strong singlet–triplet crossover.

OPTICAL phase conjugation (OPC) is an emerging area of coherent optics involving the reversal of optical wavefronts, both in direction and phase. OPC using degenerate four-wave mixing (DFWM) has been achieved. Most workers used pulsed and continuous-wave lasers of small bandwidth and very few studies using broad-band lasers have been reported. Tocho *et al.*[1] reported OPC using a bandwidth of 2.3 nm. A theoretical analysis of OPC using broad-band lasers has been reported by Alber *et al.*[2] In this communication, we report the observation of OPC signals in an absorbing dye using a Nd:glass laser. The bandwidth of the Nd:glass laser is of the order of 22 to 29 nm[3], which is an order of magnitude more than that used by Tocho *et al.*[1]

The experimental set-up was as shown in Figure 1. The laser source is a home-made Nd:glass laser[4] which delivers 20-ns pulses at 100 mJ in the Q-switched mode. Laser power can be varied by changing the lamp power. The retroreflecting mirror M is a gold-coated plane glass plate kept at a distance of about one metre from the output mirror of the laser. The probe beam was incident on the sample at an angle of about 20° with respect to the pump beam. Kodak 14015 dye [*bis*(4-dimethylaminodithiobenzil), $C_{32}H_{30}NiS_4$] dissolved in 1,2-dichloroethane was used as the nonlinear medium. The solution was taken in a thin cell of 100 μm formed by cementing two glass plates together and was kept close to the mirror M. Initial adjustments were made using a He–Ne laser. The direction of the OPC beam was located by retroreflecting the probe beam and was detected using a PIN photodiode (HP 4207). Noise due to off-axis modes of the laser cavity was reduced using proper spatial filters.

In the free-running mode of the laser, the OPC beam was found to be very weak. Intensity of the signal was enhanced using the laser in the Q-switched mode. Signal and pump beams were monitored using two 100 MHz storage oscilloscopes. Figure 2 shows the dependence of OPC signal on pump power.

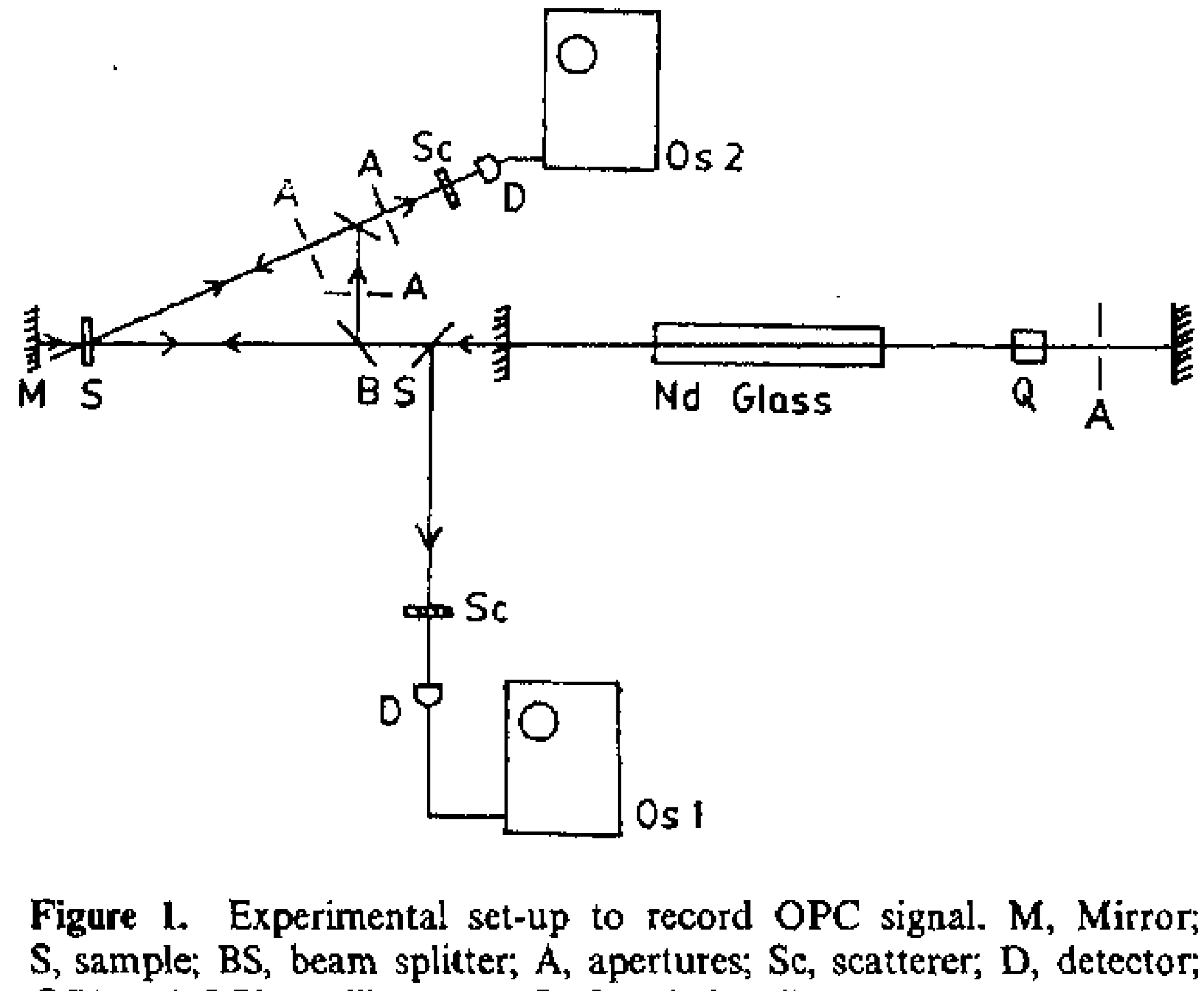

Figure 1. Experimental set-up to record OPC signal. M, Mirror; S, sample; BS, beam splitter; A, apertures; Sc, scatterer; D, detector; OS1 and OS2, oscilloscopes; Q, Q-switch cell.

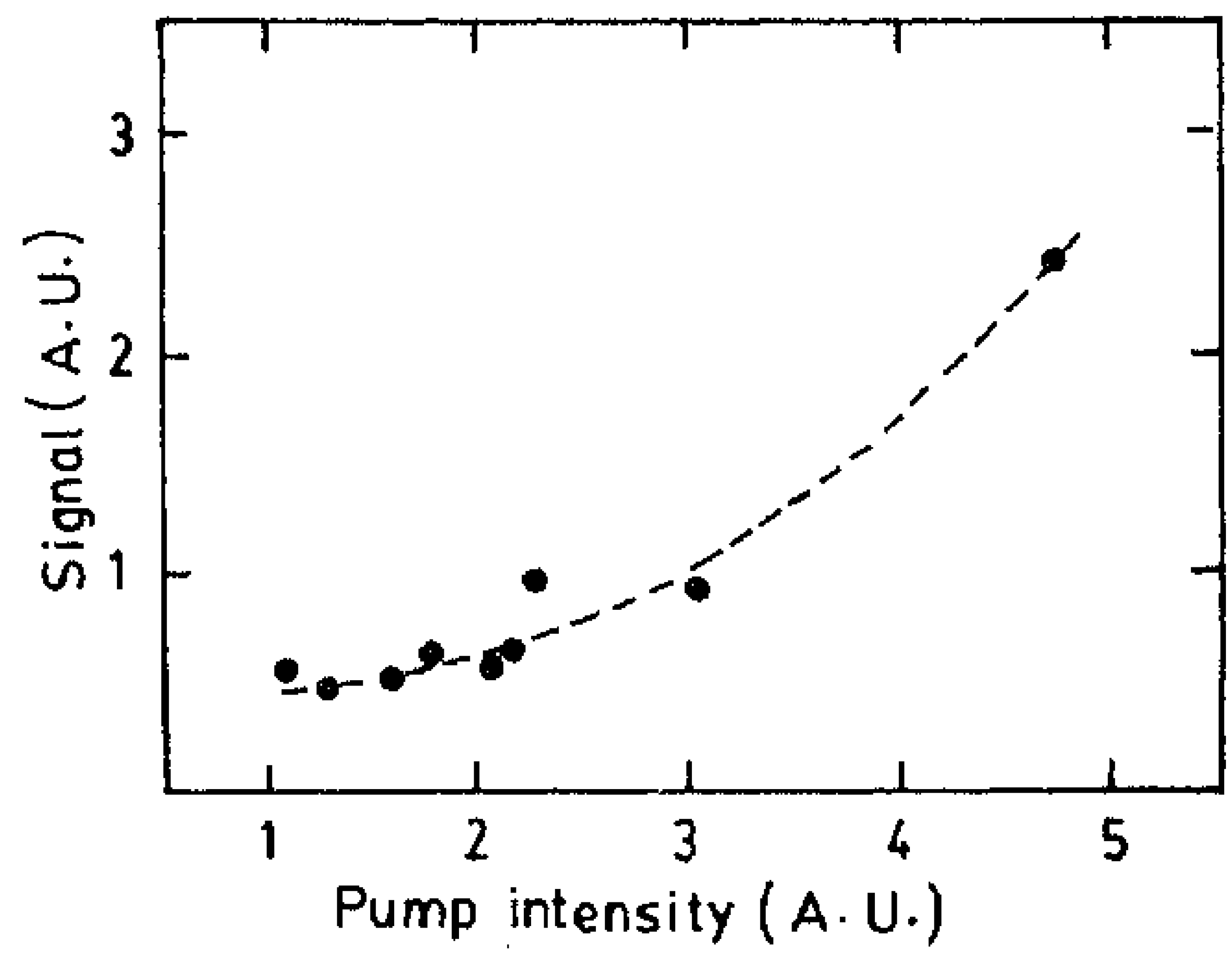

Figure 2. Dependence of OPC signal on pump intensity.

The OPC character of the beam was confirmed by the fact that its direction was not affected when the sample cell was rotated and the signal disappeared when any one of the probe and pump beams was blocked. Measurements of OPC reflectivity were not carried out owing to noise limitations. It was found that OPC signal has a decay time of a few milliseconds. The generation of OPC in the absorbing dye may be due to strong singlet–triplet crossover. A similar mechanism for OPC generation in saturable dyes has been suggested by various authors[5–7].

In conclusion, this paper describes a successful scheme for generating OPC of a broad-band laser in DFWM mode.

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