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Thermodynamics of Nonlinear Plasma Systems

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1. INTRODUCTION

A plasma is a classical many-body system with a large number of degrees of freedom. Its behaviour is dominated by collective phenomena which owe their origin to the long-range nature of the Coulomb force which urges groups of particles to move cooperatively. Furthermore a plasma is typically characterized by large deviations from thermodynamic equilibrium. It is typically born with enormous reservoirs of free energy in the form of pressure gradients ('confined' plasmas), magnetic field gradients (plasma currents), distribution function anisotropies etc.[1] Approach to thermodynamic equilibrium by conventional Coulomb 'collisions' in a hot plasma would take an extremely long time. The plasma therefore uses its ingenuity to *approach* thermodynamic equilibrium (of course never succeeding completely because of lots of *constraints*) by non-conventional or *anomalous* relaxation processes. In these processes, the free energy reservoirs are partially used up in driving up instabilities i.e. large amplitude collective modes of oscillation which drive the plasma into a nonlinear state – sometimes a very coherent one with periodic nonlinear waves and sometimes (more often) a stochastic one with seething turbulence[2]. A plasma is then typically caught up in an intermediate nonlinear state with large amplitude fluctuations which causes it to relax towards thermodynamic equilibrium at an anomalously fast rate. One often maintains a steady state wherein external drivers balance the relaxation phenomena and keep the plasma in a steady non-equilibrium configuration.

To describe the observed plasma state one follows one of the following two approaches. Traditionally, one assumes that the plasma starts from a state of mechanical equilibrium (where the volume forces are completely balanced even though the plasma may be far from thermodynamic equilibrium). It is then perturbed and one asks if the perturbation is unstable because of the available free energy sources. This analysis involves linear equations and many standard methods are available to carry out this task. If the plasma is unstable, the next task is to find the saturated spectrum of unstable waves in which the linear growth is modified by quasi-linear and nonlinear effects and also quenched by nonlinear absorption effects. This is usually a non-standard task and one often relies on one's intuition to make approximations which are not always fully justifiable. This is where the central difficulty of plasma physics lies and this is what makes a typical plasma so very unpredictable in its behaviour.

A second approach, which has achieved a certain measure of success over the past couple of decades is what may be called a thermodynamic approach to nonlinear plasma problems. In this approach, one simply asks, if one can understand the observed plasma behaviour to be due to minimization of some appropriately defined free energy for the plasma subject to some reasonably defined constraints. This leads one directly to preferred stationary states which are like local minima in which the plasma likes to reside (figure 1). Or it may give us critical information about fluctuation levels etc. without a detailed nonlinear analysis. This approach is particularly suited for systems which may be treated as 'isolated' e.g. valid for time-scales over which the diffusion (or relaxation) of heat, particles, magnetic fields etc. over the macroscopic length dimensions may be treated as

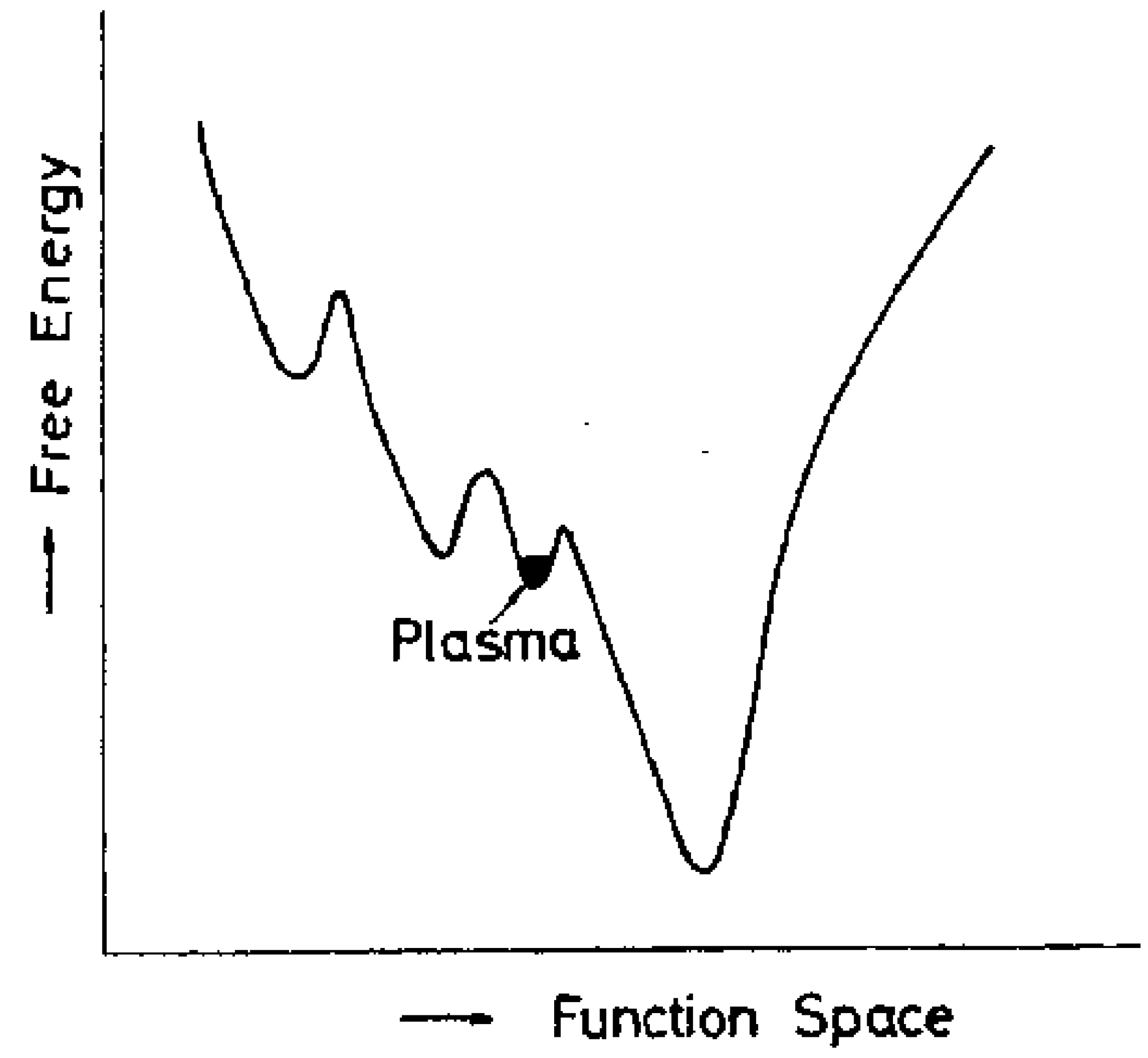

Figure 1. Schematic of the free energy functions in some general parameter space. The plasma gets trapped in a local energy minimum because of constraints on the minimization process.

negligible. In this review talk we shall describe the salient features of the nonlinear thermodynamic approach by illustrating it with some examples.

2. Thermodynamics of microscopic instabilities

One of the free energy sources in a plasma is the one associated with the details of the microscopic distribution function. e.g. one may have a distribution function with two beams, or one with a bump-on-tail or one with a net current. All these cases are known to be unstable and lead to the growth of collective plasma oscillations. The conventional theory starts with the Vlasov-Poisson set of equations (ions a smoothed out background, for simplicity)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m} \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \,, \tag{1a}$$

$$\nabla^2 \phi = 4\pi e \left[\int f \, \mathrm{d}^3 v - n_0 \right] . \tag{1b}$$

Linearized equations can be solved by the method of fourier analysis viz. $\phi_1, f_1, \sim \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$ so that

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) f_{1k} = \frac{-ie}{m} \phi_k \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \tag{2a}$$

$$k^2 \phi_{1k} = -4\pi e \int f_{1k} \, \mathrm{d}^3 v \tag{2b}$$

Elimination of f_{1k}, ϕ_{1k} leads to the dispersion relation

$$1 + \frac{4\pi e^2}{mk^2} \int \frac{\mathbf{k} \cdot \partial f_0 / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} = \mathrm{d}^3 v \tag{3}$$

which gives the relationship $\omega = \omega(k)$. The equation then demonstrates that certain k values give an ω with a positive imaginary part which corresponds to growing modes.

Next, if one wants to understand, how these unstable modes saturate, one writes down the nonlinear equations.

$$\frac{\partial f_k}{\partial t} + i\mathbf{k}\cdot\mathbf{v} f_k + \frac{ie}{m}\sum_{k'}\phi_{k'}\mathbf{k}'\cdot\frac{\partial f_{k-k'}}{\partial\mathbf{v}} + \frac{ie}{m}\mathbf{k}\phi_k\cdot\frac{\partial f_0}{\partial\mathbf{v}} = 0, \tag{4a}$$

$$k^2\phi_k = -4\pi e\int f_k\,\mathrm{d}^3\mathrm{v}, \tag{4b}$$

$$\frac{\partial f_0}{\partial t} = \frac{ie}{m}\sum_{k}\phi_k\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_{-k}\,. \tag{4c}$$

Analysis in which the equilibrium distribution function f_0 is altered but the nonlinear terms in equation (4a) are ignored are known as quasi-linear theories. One can often show that the neglected mode-coupling terms are as important as the kept terms. This has led to a large number of renormalized quasi-linear theories[2] with and without mode coupling and one can typically base the approximations on intuitive considerations only. In weak turbulence theories, one uses γ/ω_k as an expansion parameter so that one is considering the nonlinear interaction of quasi-modes (quasi-particles) in the plasma with some sort of a random phase approximation. However, if $\gamma/\omega_k \sim \Delta\omega/\omega_k$ or ~ 1, one can no longer ignore the strong turbulence effects. The entire spectrum of waves gets woven into a strongly correlated spectrum (sometimes coherent and sometimes not so) which may then be solved for. The theories of strong turbulence are still in their infancy. We now give a few examples of the application of thermodynamic methods to microscopic nonlinear plasma problems.

Bounds on Fluctuation Energy

The thermodynamic approach to microscopic problems was pioneered by Gardener, Newcomb and others[3]. Gardener argued that one can obtain an upper bound on fluctuation energy in a microscopically unstable plasma by very general arguments. He pointed out that in a microinstability, a plasma is merely converting kinetic energy of particles into electrostatic field fluctuations. Thus the plasma is attempting to minimise its free energy content, which is essentially the kinetic energy of its particles. Thus the field fluctuation energy $\mathcal{E}_F$ is limited by the condition

$$\mathcal{E}_F \leqslant W(0) - W(t),$$

$$W = \iint \mathrm{d}x\,\mathrm{dv}\,f(x,\ \mathrm{v},\ t)\ (\tfrac{1}{2}m\mathrm{v}^2). \tag{5}$$

However, in this minimization process, the plasma is constrained to obey the Vlasov equation $Df/Dt = 0$. This equation simply means that the motion of the phase space fluid is incompressible. Gardener next gave a prescription for finding a lower bound on $W(t)$. This is given by

$$W_1 = \tfrac{1}{2}m\iint \mathrm{v}^2 f_l(\mathrm{v})\,\mathrm{d}x\,\mathrm{dv} \tag{6}$$

where $f_l(\mathrm{v})$ is (i) a monotone decreasing function of v^2 and (ii) for any $\alpha > 0$, phase space volume of region where $f_l(\mathrm{v}) > \alpha$ is equal to phase space volume of region where $f_i(x,\mathrm{v}) > \alpha$.

In a way, the problem is very similar to that encountered when a heavy liquid sits on top of a light liquid, becomes unstable and rearranges itself taking account of the constraint of incompressibility. Using the above arguments, Gardener was able to obtain a bound on fluctuation energy.

$$\mathcal{E}_{F\max.} \simeq 4\mathcal{E}_{FN} \tag{7}$$

where $\mathcal{E}_{FN}$ is the result of a detailed nonlinear theory of the kind outlined above after equation (4). It is also clear that if one has better constraints than the one used above, one can get even better estimates of the nonlinear fluctuation level. Thus, we note in this example how a decent estimate of the fluctuation energy can be obtained by simple global thermodynamic arguments.

Cyclotron Maser, Gyrotron etc.

We now consider another example, where thermodynamic methods have yielded useful bounds on the performance of a given system using plasma instability. This is the well-known example of a cyclotron maser[4]. In a cyclotron maser or a gyratron with anisotropy, a relativistic electron beam i.e. different parallel and perpendicular spread is sent through a plasma with a magnetic field and used to generate intense electromagnetic waves at a frequency which is tunable. The basic physics is rather simple. The detailed nonlinear theory of this instability is essential in determining the efficiency of conversion of the kinetic energy of the relativistic electron beam into maser waves. The detailed nonlinear theory is likely to be quite complex as it involves nonlinear fluid/Vlasov equations etc. However, as Davidson[5] has recently shown one can get upper bounds on the efficiency of conversion by following arguments very similar to those of Gardener[9]. By minimizing kinetic energy of particles with constraints related to conservation of density, momentum, energy etc. he has been able to obtain bounds on the conversion efficiency. Some of the plots obtained by him showing the bounds on the efficiency of such microwave generation are shown herewith (figures 2 and 3).

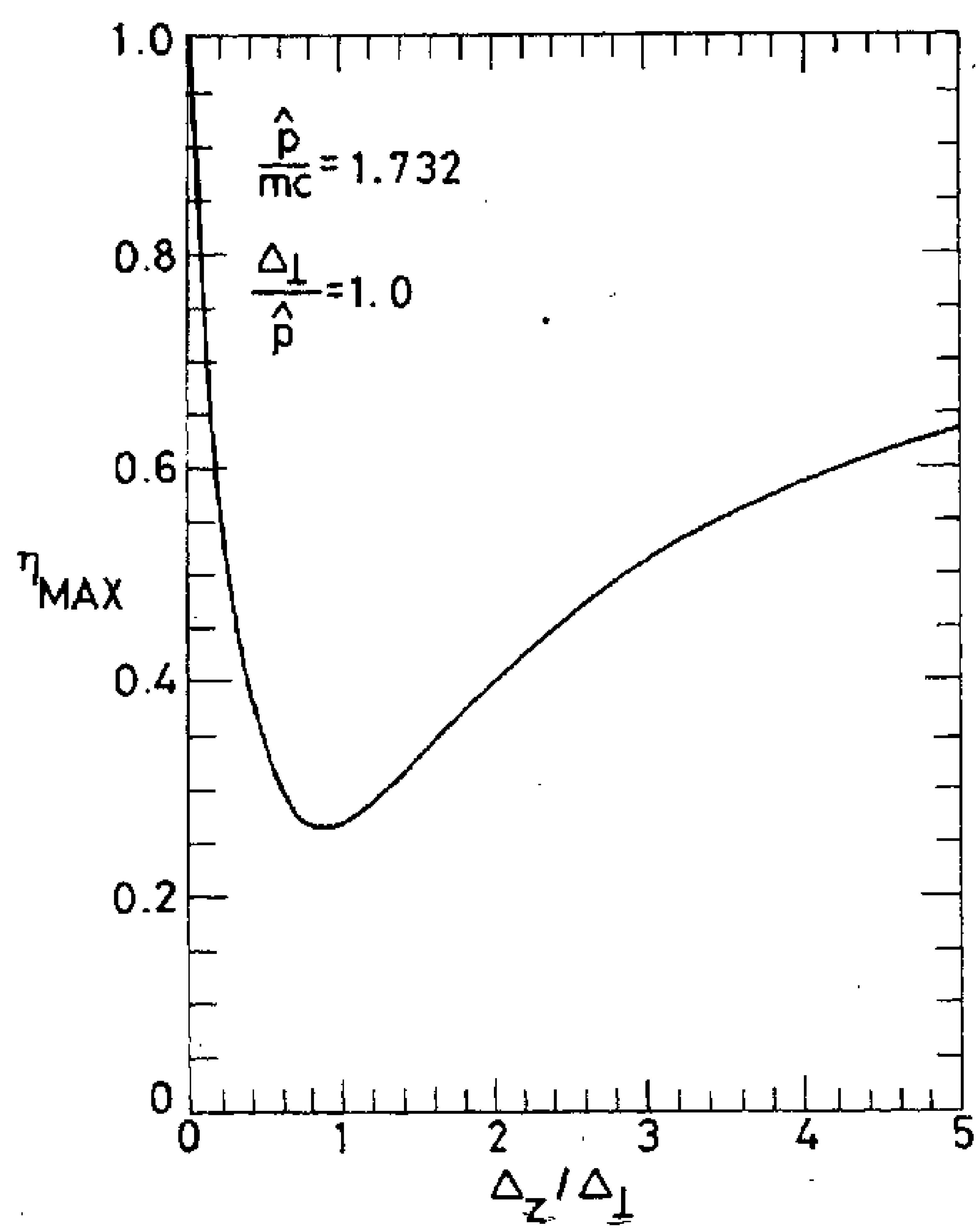

Figure 2. Lowest upper bound on efficiency of conversion from relativistic electron beam energy to EM wave energy as a function of parallel spread.

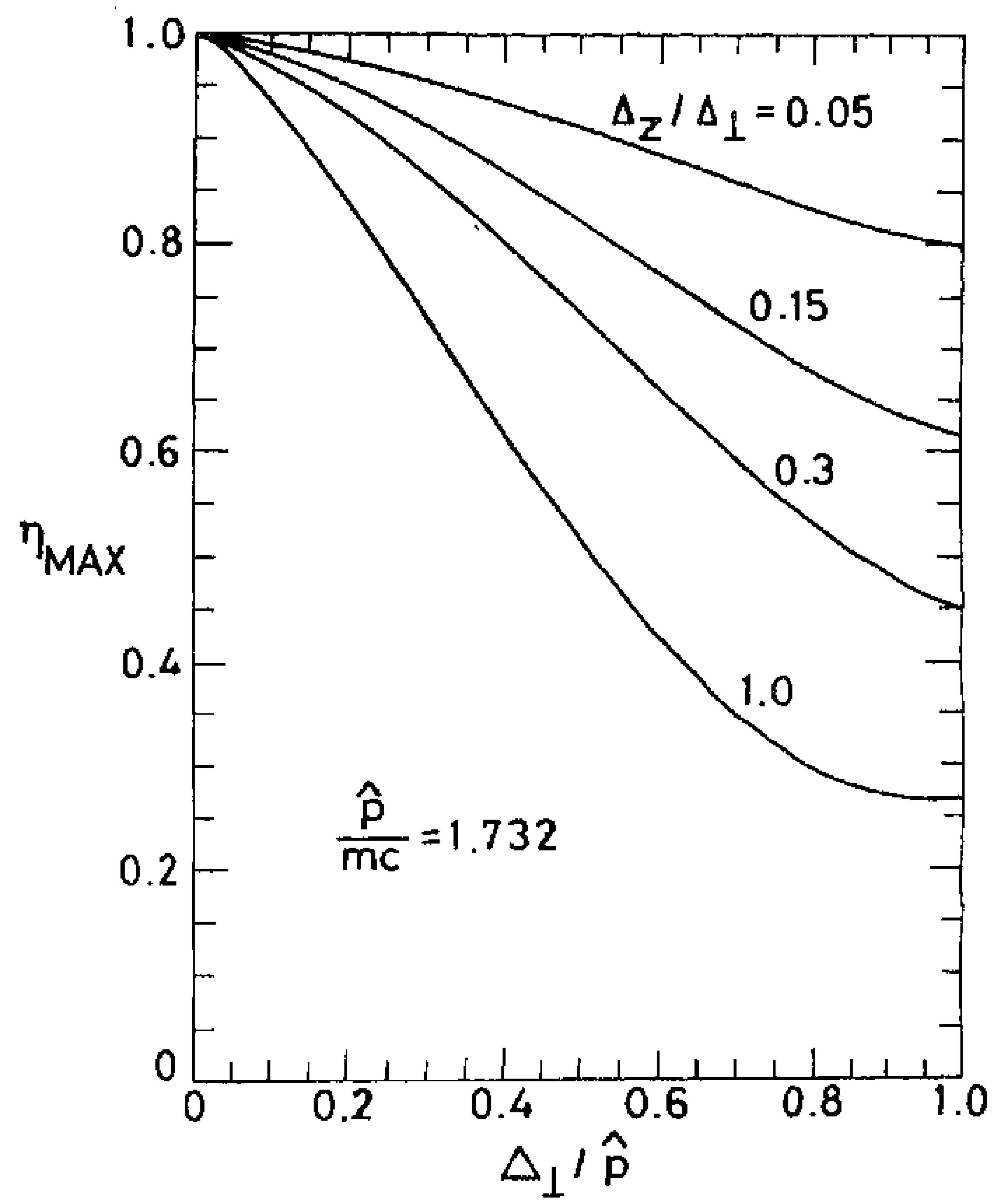

Figure 3. Lowest upper bound on efficiency of conversion from relativistic electron beam energy to EM wave energy as a function of perpendicular spread.

Two Stream Instability or Phase Space Holes etc.

To elucidate the basic nonlinear phenomena taking place in a two-stream instability, Berk, Nielsen and Roberts[6] carried out detailed computer simulations of an unstable 2-stream phase space fluid. They used the so-called water-bag model (figure 4) in which

$$f = 1 \qquad \tfrac{1}{2}v_0 < |v| < v_0 \tag{8}$$
$$= 0 \qquad \text{elsewhere}$$

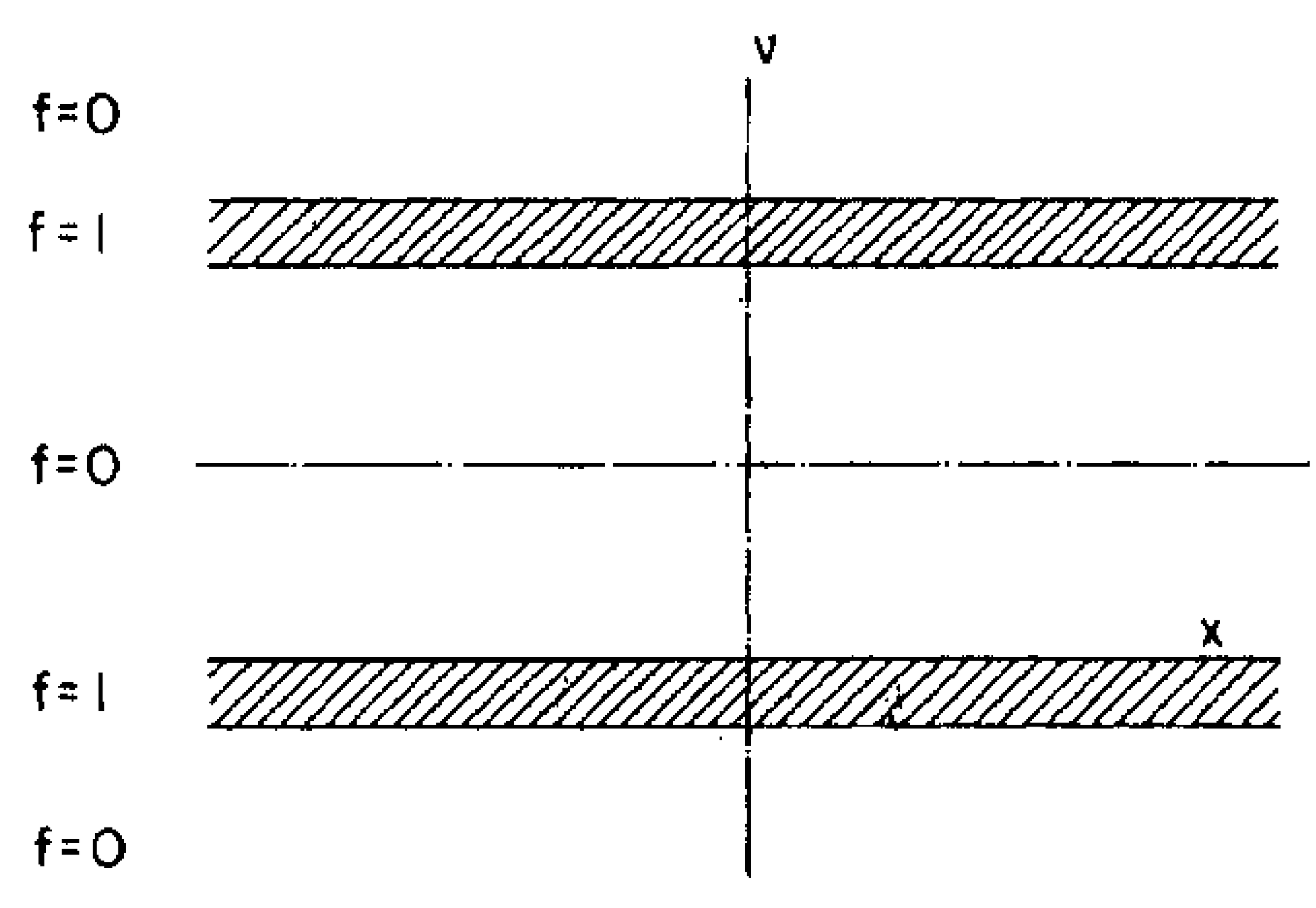

Figure 4. A two stream unstable water bag distribution function in phase space.

Such a system is completely described by the boundary curves, each point on which follows the equation of motion

$$dx/dt = v, \; dv/dt = -\partial\phi/\partial x, \tag{9}$$

ϕ is to be obtained from Poisson's equation with the charge density distribution coming from a geometrical construction. They used periodic boundary conditions and found numerically that the curves continually stretch so that representation by a fixed number of points is inaccurate. Therefore extra points were automatically inserted wherever needed. The results of their investigations are summarized in figure 5, for the following parameters:

$$v_0 \, \Delta t/\Delta x = 0{\cdot}25, \; \omega_p \, \Delta t = 1/20 \, , \; \Delta x = L/64 \, , \text{ linear } \gamma/\omega_p \simeq 0{\cdot}3.$$

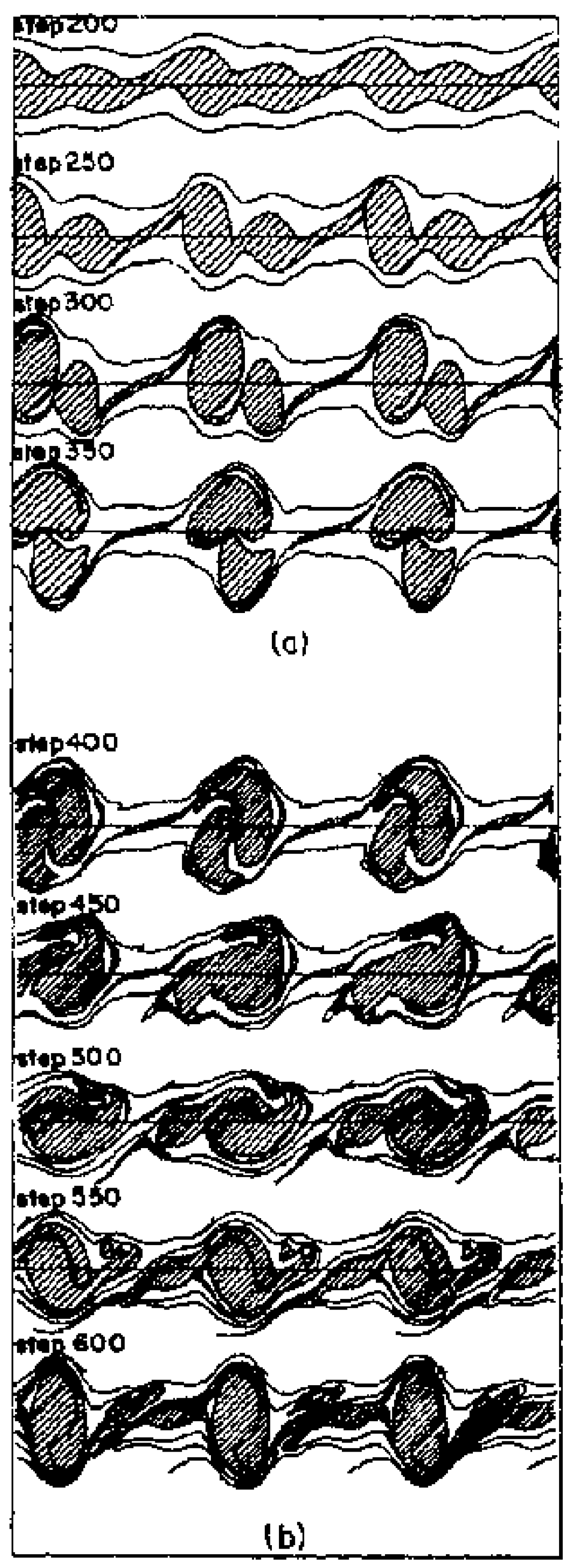

Figure 5. Nonlinear development of a two-stream instability. The inner $f = 0$ region has been shaded to emphasize the development and interactions of hole structures.

If one focusses attention on the dark regions ($f = 0$ cavities) one notes that it rolls up into *phase-space holes* of roughly elliptical shape. This hole preserves its area as it deforms. Furthermore, associated with the phase-space hole is a steady large amplitude electrostatic wave which is set up by the streaming instability. These phase-space holes are the major results of the instability. In one dimension, they seem to be fairly stable nonlinear entities which only have a tendency to attract and coalesce. In certain ways, they behave like positively charged bodies with negative mass which attract. The turbulent state seems to be describable in the phase space fluid, as a collection of weakly interacting phase-space holes superposed with minor disturbances such as background spray, tidal deformations etc.

What is a phase-space hole? In a way it is a coherent nonlinear solution of the Vlasov-Poisson set of equations. Such solutions were studied long back by Bernstein *et al.*[7] They discovered that in 1-d, if one goes to a moving frame and makes the ansatz of stationarity, then the Vlasov-Poisson equations take the form ($\xi = x - ut$)

$$(\mathrm{v} - u)\frac{\partial f}{\partial \xi} + \frac{e}{m}\frac{\partial \phi}{\partial \xi}\frac{\partial f}{\partial \mathrm{v}} = 0 \tag{10a}$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = -4\pi e\left[\int f \mathrm{dv} - n_0\right] \tag{10b}$$

Equation (10a) may be solved by the method of characteristics, giving

$$f = f(\mathrm{W}) \text{ where } \mathrm{W} = \tfrac{1}{2}\mathrm{m}(\mathrm{v} - u)^2 - e\phi$$

A phase-space hole is a region with deficiency of electrons $\tilde{f} = f - f_0 < 0$ *i.e.* one has an excess of positive charge due to the background sea of ions. Some electrons get 'trapped' in the resulting potential structure because their kinetic energy in the wave-frame is insufficient to overcome the potential hill, *i.e.* $\frac{1}{2}m(\Delta \mathrm{v})^2 < e\phi$. BGK[7] were able to demonstrate that arbitrary potential shapes in the moving frame can be sustained by an appropriate choice of trapped and untrapped particle distribution functions. Thus a phase-space hole is a special BGK type solution of the Vlasov-Poisson system of equations which is being created and sustained in the above 1-d computer simulation. It should be noted that a BGK solution is a highly nonlinear solution and cannot come out of any simple perturbative nonlinear theory. This is so because the orbits of 'trapped' particles are so drastically modified that they cannot be described by any perturbation theory. The main problem with BGK theory is that one may construct any potential shape by an appropriate choice of trapped and untrapped particle distributions, *i.e.* there is too much arbitrariness in the solution. However, as described below, certain arguments due to Dupree[8] may be used to uniquely fix the distribution functions and hence the final nonlinear structures.

An understanding of why the two-stream instability organizes itself into a collection of phase-space holes can be obtained by a thermodynamic argument due to Dupree[8]. He has recently given plausible arguments indicating that fluctuations in a turbulent plasma have a natural tendency to organize themselves into a collection of phase-space holes. He gives a maximal entropy argument to show that the most probable state for a fluctuation of given mass, momentum and energy is to be in the form of a phase-space hole.

From the Vlasov equation, we know that $df/dt = 0$ and hence that the entropy $\int f \ln f \, dx \, dv$ is truly conserved. However such a conservation of entropy is really meaningless because there is a great deal of stretching and twisting of the phase-space fluid with the result that there are localized regions of very high gradients, which are extremely small in size. A proper description should carry out 'coarse-graining' of these regions, which leads to entropy increase. Thus Dupree argues that the 'trapped' regions are phase mixed and have their entropy increased whereas the untrapped particles essentially have reversible behaviour with conservation of entropy. He thus maximizes the entropy

$$\sigma = n \iint_t dx \, d\mathrm{v} \, [f_t(\mathrm{v}) \ln f_t(\mathrm{v}) - f_0(u) \ln f_0(u)] \tag{11a}$$

with constancy of

$$[\mathrm{M,P,T}] = \iint dx \, d\mathrm{v} \, [f_t(x,\mathrm{v}) - f_0(u)] \, [m, m\mathrm{v}, (m\mathrm{v}^2/2) - e\phi] \tag{11b}$$

and $\phi(x)$ determined self-consistently by the equation

$$\left[\frac{\partial^2}{\partial x^2} - \omega_p^2 \, P\int \frac{d\mathrm{v}}{\mathrm{v}-u}\frac{\partial f_0}{\partial \mathrm{v}}\right]\phi = 4\pi \, \mathrm{en}\int_t d\mathrm{v}\left[f_t(\mathrm{v}) - f_0(u)\right]. \tag{11c}$$

Carrying out this maximization process, one finally gets

$$f_t = f_0(u) \exp\left[\frac{E - q\phi_m}{\tau}\right] \qquad q\,\phi(x) < E < q\,\phi_m \tag{12}$$

$$= f_0\,(u) \qquad q\phi_m < E < 0.$$

Such a phase-space hole (see figure 6 for a sketch of f) has many features similar to those observed in the 1-d numerical simulation of Berk *et al.*[6] One can also show that two such holes have a tendency to coalesce into a single hole, as observed.

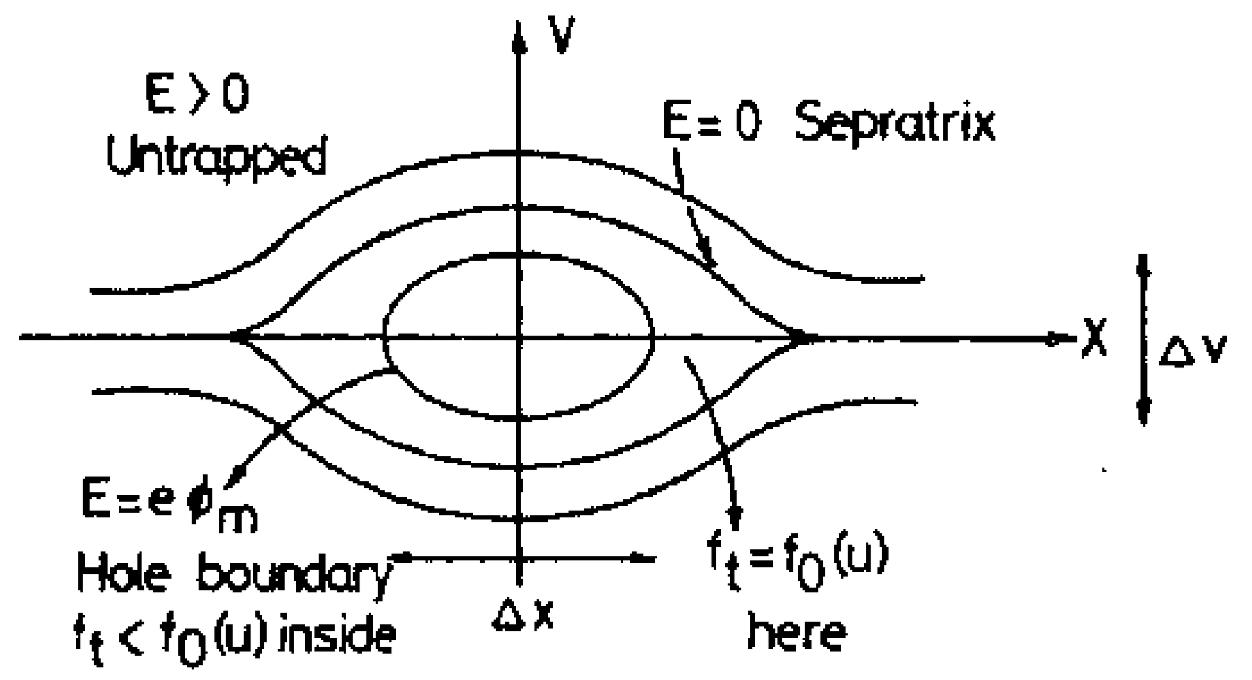

Figure 6. Schematic representation of a Dupree phase – space hole, a special BGK solution of the non-linear Vlasov-Poisson equations.

It is clear therefore that the observed nonlinear state in the numerical simulations is one where the plasma is maximizing the entropy of the 'trapped' parts of phase-space, subject to certain reasonable constraints. It may well be argued (as Dupree does) that the description of nonlinear states in real experiments with driven open systems in terms of a collection of weakly interacting nonlinear entities (like phase-space holes) is far superior to description in terms of strongly interacting Fourier modes (as the conventional theories demand). Again nonlinear thermodynamics has led to a greater insight into the final state than any conventional theories would have done.

Recently, the problem of microinstabilities in magnetically confined inhomogeneous plasma has also been looked at from the above refreshing point of view[9]. Rather than describe the final state in terms of nonlinearly saturated unstable waves which may go into a turbulent state, one instead works in terms of coherent nonlinear phase-space structures in a magnetized plasma. In these cases, one has not only the free energy sources associated with $\partial f_0 / \partial v$ but also those associated with $\partial f_0 / \partial x$. One then gets a coupling between the phase-space holes and $E \times B$ vortices in physical space. Such coupled vortex-phase-space structures in 2-dimensions are excellent candidates for describing the nonlinear turbulent state of an inhomogeneous confined plasma.

3. Thermodynamics of macroscopic phenomena

One of the outstanding problems of modern magnetically confined fusion systems is a description of the nonlinear state in which the plasma is typically trapped. Many fusion systems are essentially plasma current filaments which are kept in MHD equilibrium in external magnetic fields. These equilibria contain free energy sources in the form of poloidal magnetic field energy ($\mathcal{E}_F \sim B_\theta^2 \sim J^2$) and expansion free energy ($\mathcal{E}_F \sim \nabla p$). These free energy sources can in turn drive instabilities of natural MHD oscillations in the plasma, which then grow. Finally, the oscillations saturate at large amplitudes leaving the plasma in a nonlinear state. The description of this nonlinear state by conventional methods is quite complex and will now be described below:

The basic equations describing a plasma current filament in a magnetic field (such as in the laboratory or in a solar/astrophysical situation) are the macroscopic equations:

$$(\partial\rho/\partial t) + \nabla \cdot (\rho\mathbf{v}) = 0, \tag{13a}$$

$$\rho[(\partial/\partial t) + \mathbf{v} \cdot \nabla]\mathbf{v} = \mathbf{J} \times \mathbf{B} - \nabla p + \bar{\mu}\nabla^2\mathbf{v}, \tag{13b}$$

$$[(\partial/\partial t) + \mathbf{v} \cdot \nabla]\ [p/\rho^\gamma] = 0, \tag{13c}$$

$$(\partial/\partial t)\ \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta\ \nabla^2\ \mathbf{B} \tag{13d}$$

$$\mathrm{J} = \nabla \times \mathbf{B}. \tag{13e}$$

These equations are valid for a plasma which may be treated as a MHD fluid, a valid approximation when typical $\partial/\partial t \ll \omega_{ci}$ and typical scale lengths $L \gg r_{Li}$. The parameters $\bar{\mu}$ and η refer respectively to coefficients of viscosity and resistivity in the plasma and are responsible for the non-ideal nature of the plasma.

The basic plasma equilibrium is described by the equation[10].

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p. \tag{14}$$

The equation indicates $\mathbf{B} \cdot \nabla p = 0$ or that the pressure is constant along field lines. The fundamental concept of toroidal confinement systems is related to *integrability* of the associated field line trajectories.[10] It is well-known that field lines described by the equations

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \frac{B_x}{|B|}, \frac{\mathrm{d}y}{\mathrm{d}l} = \frac{B_y}{|B|}\ \frac{\mathrm{d}z}{\mathrm{d}l} = \frac{B_z}{|B|}, \tag{15}$$

essentially form a Hamiltonian flow. In general, the equations do not lead to integrable trajectories. However, if the field lines have additional invariants, then they form nested surfaces in Poincaré surface of section plots. In these cases, the field lines instead of wandering randomly in surface of section plots, (note that system is usually toroidal and hence periodic in the toroidal coordinate ϕ) actually reside on closed curves. These curves are known as magnetic surfaces (figure 7). They are densely traced out by a magnetic field line. A rigorous proof of the existence of magnetic surfaces can only be given for fields with external symmetries (Grad[10]) such as plasma equilibrium with no ϕ dependence (e.g. tokamaks and pinches).

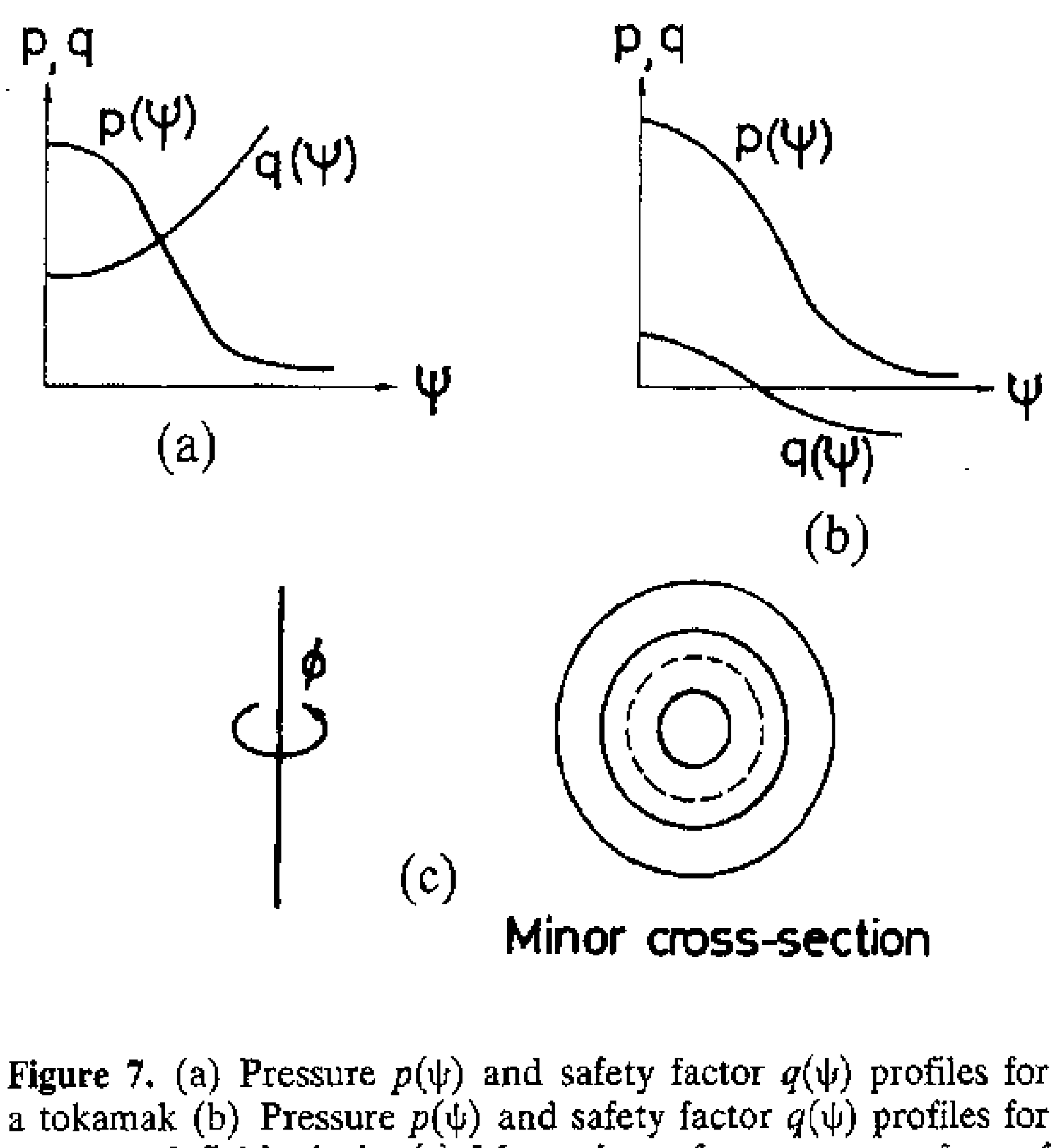

Figure 7. (a) Pressure $p(\psi)$ and safety factor $q(\psi)$ profiles for a tokamak (b) Pressure $p(\psi)$ and safety factor $q(\psi)$ profiles for a reversed field pinch. (c) Magnetic surfaces on a surface of section plot. The dotted curve shows how magnetic surfaces result from intersections of a helical field line with the minor cross-section as the field line wanders periodically in the toroidal direction.

A typical toroidal plasma equilibrium thus consists of a toroidal field, B_T and a poloidal field, B_P the latter being created by plasma currents:

$$\mathbf{B} = B_T\, \hat{e}_\varphi + B_P\, \hat{e}_\theta\,, \tag{16a}$$

$$B_P = \frac{1}{R}\, \nabla\psi \times \hat{e}_\varphi \tag{16b}$$

$$q = rB_\varphi \,/\, RB_P,\; q \equiv q\;(\psi), \tag{16c}$$

$$p = p(\psi) \tag{16d}$$

A pressure gradient and a current profile basically determine (or characterize) a given plasma equilibrium; the latter is usually given in terms of the safety factor profile $q(\psi)$ (figure 7).

Experiments have demonstrated that for axisymmetric plasma toroidal equilibria sustained by currents, only two kinds of equilibria give reasonably quiescent discharges viz. tokamaks and reversed field pinches. Both have a confined pressure profile with p peaked on the inside but differ in the shape of the q-profiles. One of the outstanding problems of plasma MHD physics is an understanding of why these two profiles give interesting stationary equilibria and why they differ from each other (figure 7).

Let us first start an analysis of the affairs with the conventional methods. We realise that p', J' and κ (viz. field line curvature or effective gravitational fields associated with centrifugal forces on ions following curved field lines) act as sources of free energy.

In ideal MHD theory, the stability of arbitrary equilibria is analysed either by a normal mode analysis (similar to exp $i(\mathbf{k} \cdot \mathbf{r} - \omega t)$ analysis of the linearised Vlasov equation) or by the use of an energy principle.[11] The energy principle looks for changes in the potential energy δW caused by a linearized perturbation on a given equilibrium: δW is given by

$$\delta W = \tfrac{1}{2}\int \mathrm{d}^3r\; [|B_\perp|^2 + \gamma p_0\, |(\nabla \cdot \xi)|^2 + B_0^2\, |\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa|^2$$

$$- 2\,(\xi_\perp \cdot \nabla p_0)\,(\kappa \cdot \xi_\perp^*) - \frac{\mathbf{J}_0 \cdot \mathbf{B}_0}{B_0^2}\,(\xi_\perp^* \times \mathbf{B}_0) \cdot B_\perp], \tag{17a}$$

$$\mathbf{B}_\perp = \nabla \times (\xi \times \mathbf{B}_0),\; \mathbf{v} = \mathrm{d}\,\xi/\mathrm{d}t,\; \kappa = (\hat{b} \cdot \nabla)\hat{b}\; \cdot \tag{17b}$$

The first three terms are positive definite and correspond to the expenditure of energy in bending field lines and compressing the plasma. The last two terms give negative contribution associated with release of expansion/gravitational field energies and the release of energy associated with $J_\parallel$ or poloidal magnetic fields. These negative contributions can thus lead to instabilities. Ideal instabilities are to be avoided at all costs since they destroy the plasma configuration on a very fast time scale viz. the time taken by an Alfven wave to transit the plasma. This limits the parameter space in which confinement systems can operate.

Ideal MHD equations are a highly constrained system. This is evident from the basic equations which can be used to prove an infinite number of conservation laws.

$$(\mathrm{d}/\mathrm{d}t)K_\psi = 0 \tag{18a}$$

$$K_\psi = \int_\psi (\mathbf{A} \cdot \mathbf{B})\; \mathrm{d}\tau, \tag{18b}$$

where K_ψ is the helicity of field lines associated with each closed flux tube and more generally with each magnetic surface. This infinity of constraints actually freezes the topology of field lines in an ideal MHD fluid (note similar constraints in constancy of f and entropy in the Vlasov equation). This statement is often said to mean that field lines are 'frozen' in an ideal MHD fluid.[11] This also means that ideal instabilities cannot change the basic topology of the magnetic surfaces, however much these surfaces may get twisted and stretched.

It is also well-known that in a plasma with finite resistivity ($\eta \neq 0$), many new instabilities arise.[12] Examples of these are the tearing instabilities which filament a given current sheet, the rippling instabilities which are driven by resistivity gradients, resistive g-modes etc. These modes need not satisfy the 'frozen-field' constraints of ideal MHD and can actually lead to drastic changes in the magnetic field topologies. It is indeed because of the magnetic topology changes that resistive instabilities arise in an equilibrium which is stable to ideal instabilities. Basically, the release of ideal MHD constraints permits an access to lower energy states, not originally possible. In contrast to ideal instabilities these non-ideal instabilities are typically slow since they depend on resistivity which for a hot plasma is a small parameter. They often go as a fractional power of resistivity such as $\eta^{1/3}$, $\eta^{3/5}$ etc. One can therefore hope to live with these instabilities and it becomes interesting to enquire as to the non-linear fate of the plasma in the presence of such instabilities. In a tearing instability in a tokamak, for example, the plasma may form magnetic islands on the $q = 2$ surface, *i.e.* the plasma current filament ultimately breaks up into two crescent-shaped helical filaments (as shown in figure 8a). If the plasma is unstable to tearing modes on only one surface, the final fate is generally not complex and is a coherent island formation. However, if several magnetic surfaces are unstable to tearing modes, the plasma loses all symmetries. The field line mapping now does not lead to nice magnetic islands but instead leads to a turbulent state where the field lines stochastically wander over a volume (figure 8b). In this case, the plasma loses all confinement and the current filament violently disrupts. The calculation of the final fate of the plasma current filament and the associated stochastic field lines in a resistive MHD plasma is an extremely complex task and can often be done only numerically. In view of the extremely complex nature of these calculations, it is a miracle that any significant progress has been made at all in a description of the observed nonlinear states.

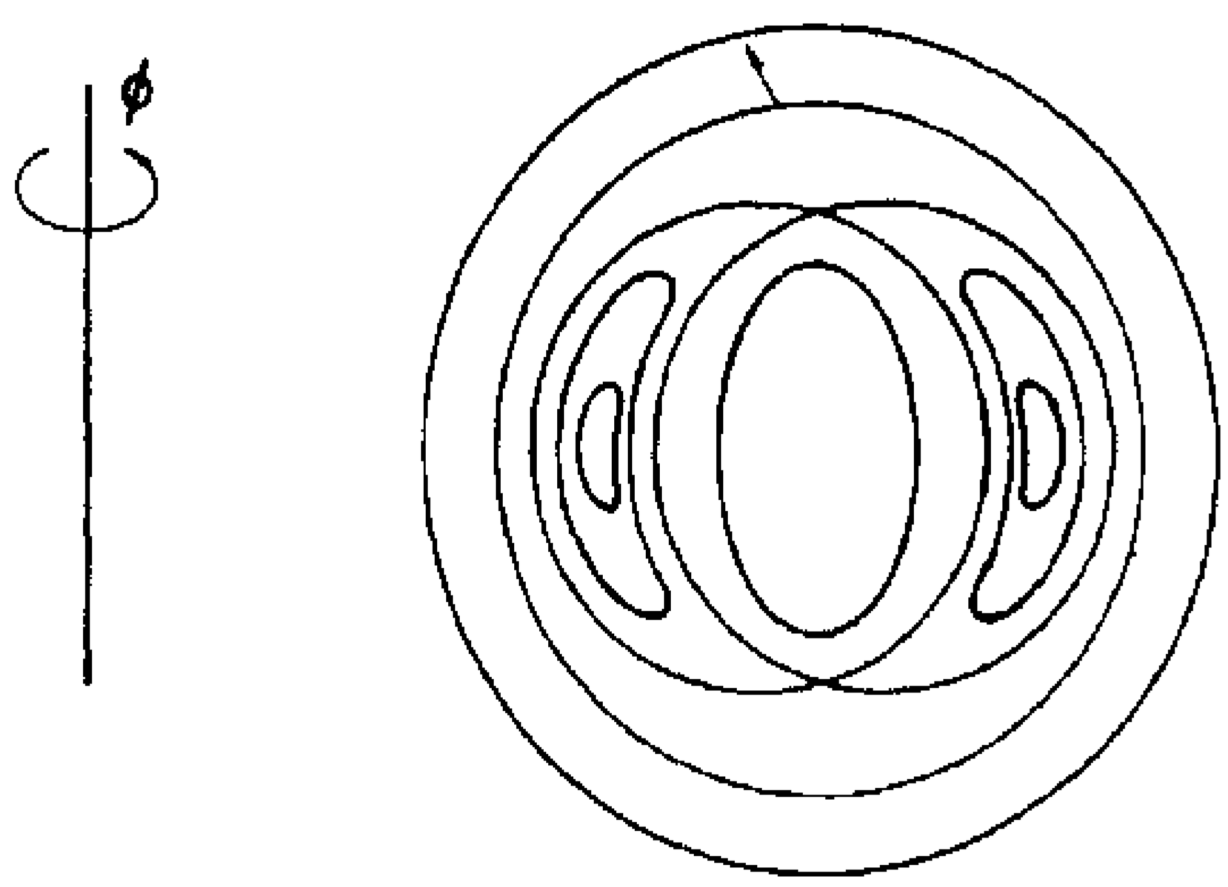

Figure 8. (a) Formation of a coherent $m = 2$ island structure due to tearing instability.

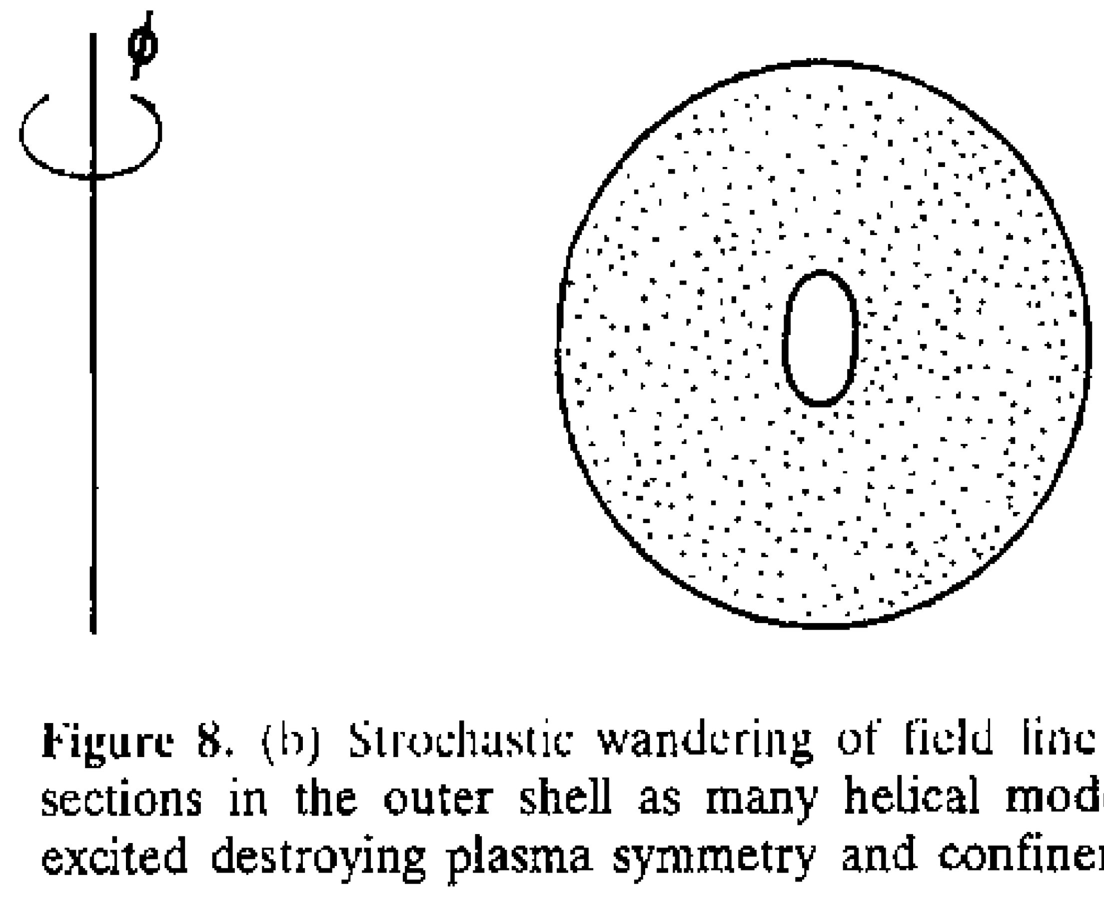

Figure 8. (b) Strochastic wandering of field line intersections in the outer shell as many helical modes are excited destroying plasma symmetry and confinement.

A thermodynamic approach to the above complex set of problems was initiated by Taylor[13] about 15 years ago. He was attempting to interpret a large amount of data that had been gathered on RFP's (reversed field pinches). He argued that the most important source of free energy in these phenomena is that associated with the poloidal magnetic field energy.

$$W = \int_V (B_P^2/8\pi)\, d\tau. \tag{19}$$

The plasma is attempting to get rid of this free energy and if it were allowed to do so in an unconstrained manner, it would reach a state with $B_P = 0$, *i.e.* get rid of all plasma current. However, as the plasma moves around, it is constrained by ideal MHD equations. If ideal MHD were exactly valid, one would have an infinite number of constraints associated with each magnetic surface. As it is, the plasma resistivity η is finite but not too large. The result is that certain magnetic surfaces break-up and reconnect forming magnetic islands on several surfaces. Under these conditions, conservation of K_ψ for each surface loses its meaning. However an overall conservation law for the total volume is still meaningful because the plasma is surrounded by a conducting shell. That is, we can use,

$$K = \int_V \mathbf{A} \cdot \mathbf{B}\, d\tau \tag{20}$$

as an overall constraint on the plasma motions. This kind of ansatz has also been verified in numerical computation (Montgomery *et al*)[14]. It is possible to give the following interpretation of the above thermodynamic process:

$$\dot{W} \sim -\eta \int J^2\, d^3r - \bar{\mu} \int |\nabla \mathbf{v}|^2\, d^3r, \tag{21}$$

$$\dot{K} \sim -2\eta \int \mathbf{J} \cdot \mathbf{B}\, d^3r. \tag{22}$$

If by virtue of plasma motions, sharp gradients of ψ (current layers) are created, they may lead to considerable decrease of energy with a corresponding minimal decrease of K. Conceptually $J \sim \eta^{-p}$ such that $\int \eta^{1-2p}\, d^3r \to$ finite while $\int \eta^{1-p} d^3r \to 0$ as $\eta \to 0$. Thus we may argue that instability processes are attempting to minimize the poloidal field energy while keeping K nearly invariant. This will always be valid as long as the overall relaxation is taking place on time scales much faster than resistive diffusion time scales such as η etc. Under these conditions, the system may still be treated as an isolated system.

Following Taylor[13] we now minimize W subject to the constancy of K. The result is an equilibrium configuration in which

$$\nabla \times \mathbf{B} = \mu \mathbf{B}. \tag{23}$$

This is a force-free configuration in which μ is entirely determined by K_0 and the toroidal flux ψ_T. Thus, the solution of the above equilibrium equation is

$$B_r = 0,\ B_\theta = \alpha\, J_1(\mu r),\ B_\varphi = \alpha\, J_0(\mu r), \tag{24a}$$

where μ and α are determined by

$$\frac{K_0}{\psi_T^2} = \frac{R}{aJ_1^2(\mu a)} [\mu a\{J_0^2(\mu a) + J_1^2(\mu a)\} - 2J_0(\mu a)\, J_1(\mu a)] \tag{24b}$$

$$\text{and } \psi_T = 2\pi \int_0^a B_\varphi\, r dr. \tag{24c}$$

Thus, given a value for K_0 and ψ_T, the final equilibrium state is completely determined. Experimentally, the results on reversed fields equilibria are expressed in terms of $F - \theta$ plots where $F \equiv B_\varphi\ (r = a)/\bar{B}_\varphi$ and $\theta \equiv \mu a/2$.

The above theory gives an excellent description of observations on reversed field pinches. The experiments indicate that after a violent initial phase, the pinch settles into a quiescent state with minimum fluctuations. Furthermore, the mean magnetic field profiles in the quiescent state are independent of particular experiment and the previous history and only depend on one parameter

$$\theta = (2I_\varphi / B_{\theta 0})\ (1/q_a).$$

Lastly for $\theta > \theta_c$, the toroidal field is observed to reverse in the outer region of the plasma, as described by the above solutions ($J_0(\mu r)$ etc.). Figure 9 and 10 give a very brief view of the excellent

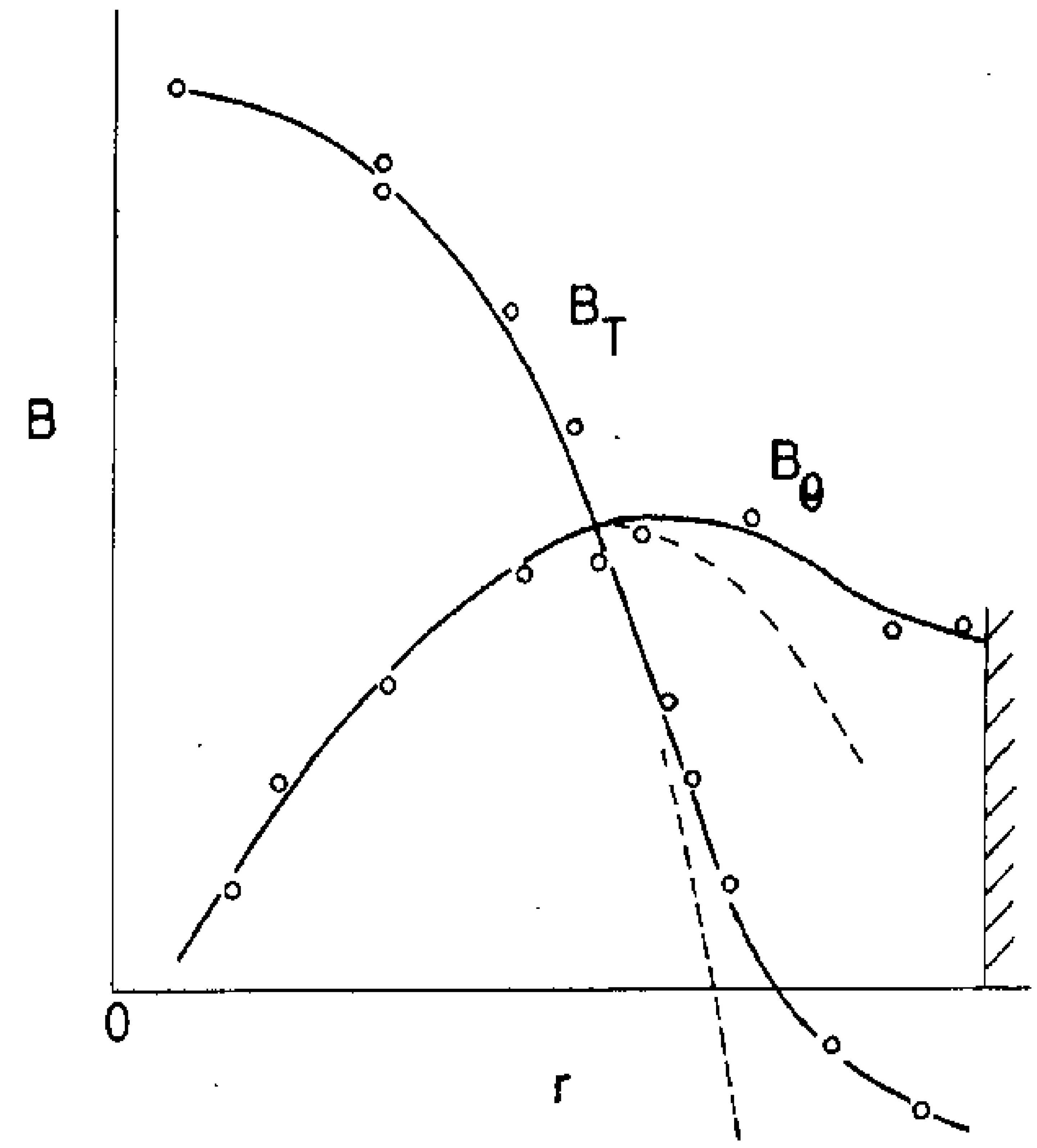

Figure 9. B_θ, B_T profiles in the reversed field pinch; note that B_T reverses in outer region. Dotted lines are the theoretical predictions.

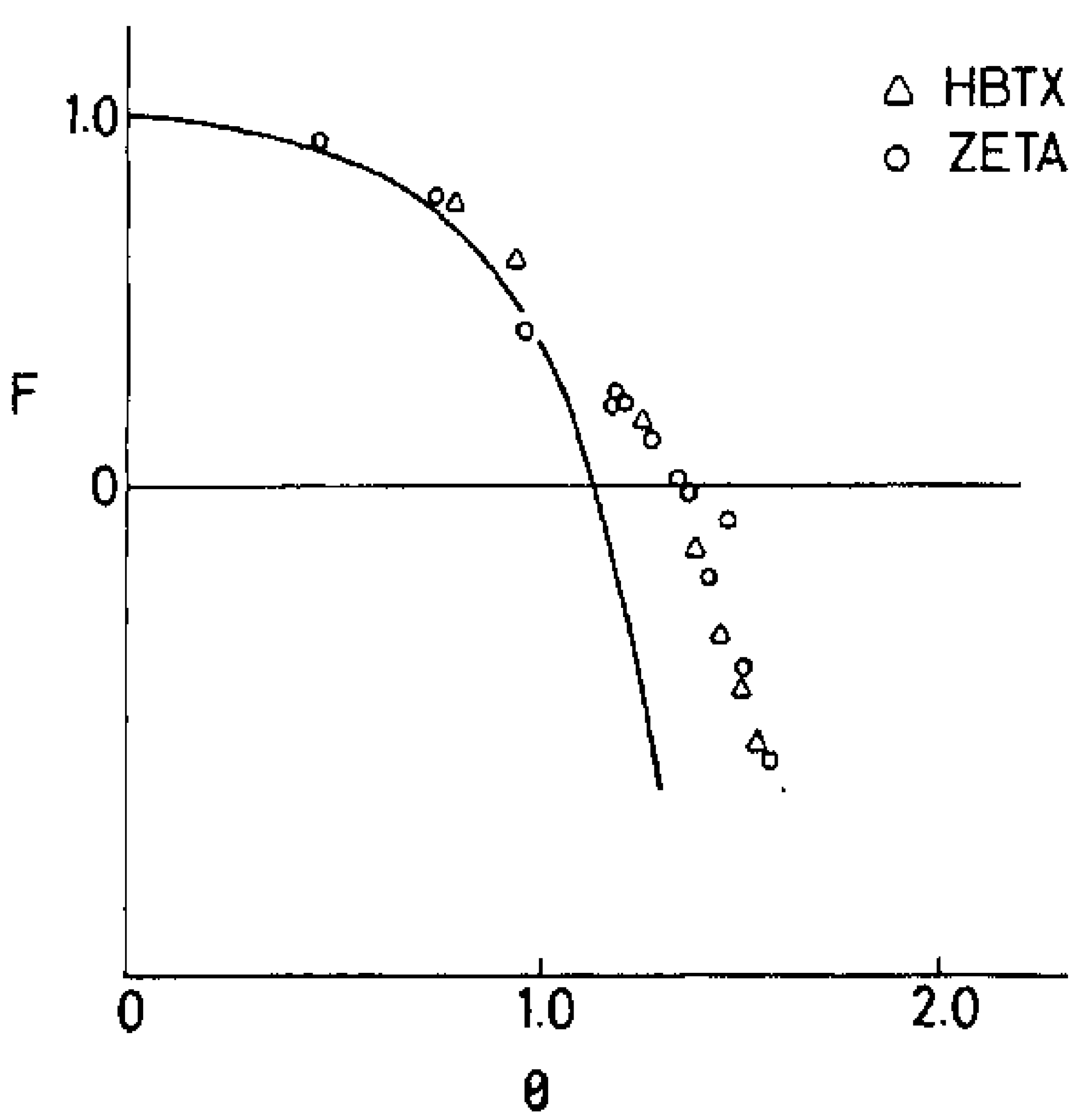

Figure 10. Comparison of the $F - \theta$ curves from theory and experiments (circles and triangles).