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Role of Symmetry and Group Structure in Optics

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It is an honour and privilege to be invited to speak at this symposium celebrating the birth centenary of India's greatest scientist, C. V. Raman. Light in all its aspects and colours held a lifelong fascination for Raman, and so did symmetry and beauty in physical systems and nature in general. For these reasons it seems appropriate to describe here some of the ways in which group theory – the perfect language for the expression of symmetry in any physical system – illuminates our understanding of a variety of problems of importance and interest in optics. I wish to specially highlight the roles played by certain continuous groups, Lie groups, in problems involving the description of polarization, the actions of optical systems on beams of light, and the properties of some simple but practically useful kinds of beams. Both at the start and towards the end, I shall be concerned with problems of polarization, but in between I shall refer to some other aspects in the language of scalar optics. Concerning polarization it is interesting to recall in passing that in one of his beautiful essays, Louis de Broglie calls it "an aspect of fundamental symmetry appertaining to light waves".[1]

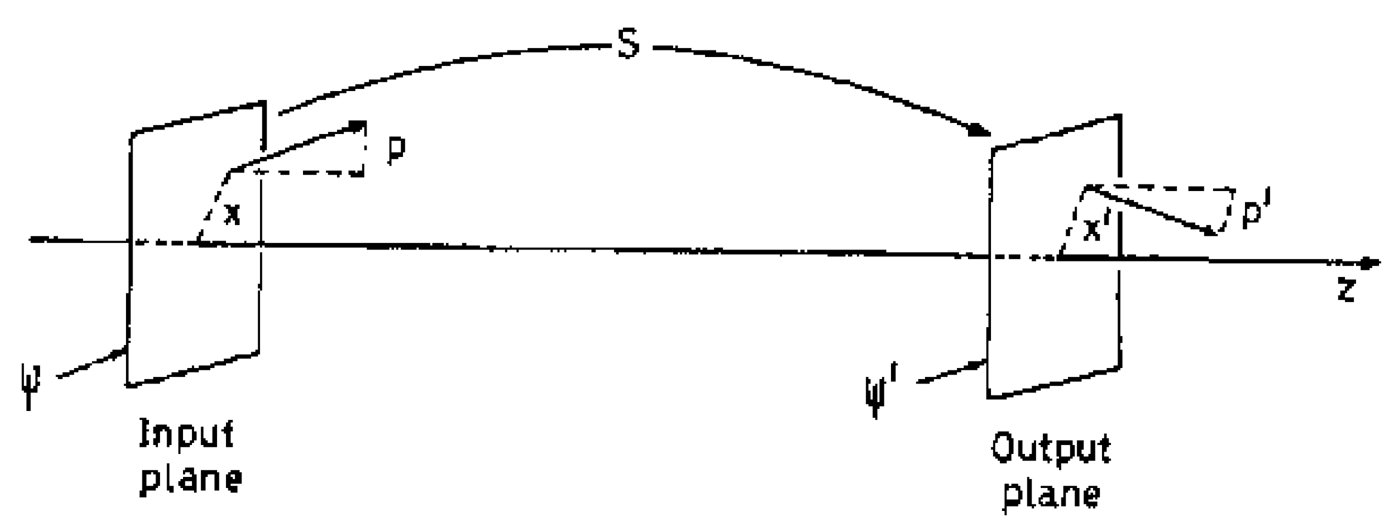

Figure 1. Ray variables and action of FOS on rays and waves.

In the ray limit, or short wavelength geometric limit, of wave optics, if one has a paraxial beam of light with a well-defined propagation direction or axis – say the positive z-axis – then on each transverse plane each ray can be described by its transverse two-dimensional position and wave vectors, say $\mathbf{x}$ and $\mathbf{p}$.[2] This is shown in figure 1. These are the ray parameters, or in the language of mechanics, the phase-space coordinates of a ray. For such beams, there is a particular class of optical systems which can be described naturally and economically with the help of group theory. These are the *Gaussian* or *First Order Systems* (FOS); they have the property of mapping each incoming ray into a definite single outgoing ray, with the ray coordinates undergoing a *linear* transformation. For the present let us restrict ourselves to axially symmetric systems, in which case the transverse x and y canonical pairs of ray variables behave in the same way; in other words, the problem essentially becomes one-dimensional and involves just one coordinate and one wave vector component. Any FOS is then completely described by its ray-transfer matrix, a real two-dimensional unimodular matrix. Its effect on the ray variables is given by[2]

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,R),\ ad-bc = 1:$$

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{x}' \\ \mathbf{p}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix}. \tag{1}$$

These are the equations of a linear canonical transformation – canonical because the behaviour of rays rests on Fermat's variational principle.

The significance of associating an element $S \in \mathrm{SL}(2,R)$ to each FOS is clearly this: successive action of optical systems in sequence corresponds to the product of group elements or matrices in that sequence. Some familiar FOS are the following:

Free propagation through a distance $D \geqslant 0$:

$$f(D) = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix} : x' = x + Dp,\ p' = p\,. \tag{2}$$

Lens of optical power g:

$$l(g) = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} : x' = x,\ p' = p - gx\,. \tag{3}$$

Magnifier:

$$m(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} : x' = e^{\eta/2}\, x,\ p' = e^{-\eta/2}\, p\,. \tag{4}$$

Fourier transformer:

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : x' = p,\ p' = -x\,. \tag{5}$$

Two interesting theorems may be mentioned at this point: a mathematical one, the Iwasawa decomposition,[3] according to which each FOS can be uniquely decomposed as a product of specific kinds of FOS:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = l(\xi)\ m(\eta)\ r(\zeta),$$

$$l(\xi) = \text{lens} = \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix},$$

$$m(\eta) = \text{magnifier} = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix},$$

$$r(\zeta) = \text{phase space rotator} = \begin{pmatrix} \cos \zeta/2 & \sin \zeta/2 \\ -\sin \zeta/2 & \cos \zeta/2 \end{pmatrix}. \tag{6}$$

This is a global decomposition, available and unique for each $S \in \mathrm{SL}(2,R)$; the expressions for the parameters ξ, η, ζ in terms of $a\ b\ c\ d$ are easy to obtain, but will be omitted. There are two ways of remembering this decomposition: lens – magnifier-rotator or left-middle-right. The physical theorem is that any FOS can be synthesized as the product of at most three lenses and three free propagations over positive distances: [4]

$$S = f(D_1)\; l(g_1)\; f(D_2)\; l(g_2)\; f(D_3)\; l(g_3)\ . \tag{7}$$

There are elements of SL(2,R) – certain FOS's – for which fewer factors would suffice, but there definitely are others for which all six are needed. The six parameters are of course not always unique.

This description of FOS's was by means of their actions on rays in the geometric optics limit. They can also be specified by their actions on a general paraxial scalar wave field. If as shown in figure 1 such a wave field is described on an input transverse plane by a wave amplitude $\psi(\mathbf{x})$, then the output wave amplitude $\psi'(\mathbf{x})$, is given by an integral kernel called the generalized Huyghens kernel: [5]

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,R):$$

$$\psi'(\mathbf{x}) = (\bar{U}(S)\ \psi)\ (\mathbf{x}) = \int d^2x'\ \langle\ \mathbf{x}|\bar{U}(S)|\mathbf{x}'\ \rangle\ \psi(\mathbf{x}'),$$

$$\langle \mathbf{x}|\bar{U}(S)|\mathbf{x}'\rangle = \begin{cases} \dfrac{-i}{2\pi b}\ \exp\left[\dfrac{i}{2b}\ (d\mathbf{x}^2 - 2\mathbf{x}\cdot\mathbf{x}' + a\mathbf{x}'^2)\right], & b \neq 0; \\ \exp\left(\dfrac{ic}{2a}\ \mathbf{x}^2\right)\cdot\dfrac{1}{a}\cdot\delta^{(2)}\left(\mathbf{x}' - \dfrac{1}{a}\mathbf{x}\right), & b = 0\ . \end{cases} \tag{8}$$

Here the notation of quantum mechanics has been used, and in fact $\bar{U}(S)$ is a unitary operator giving a unitary representation of SL(2,R). (The following delicate point must be mentioned: in a strictly one-dimensional case, the generalized Huyghens kernel is related to a unitary representation of the so-called *metaplectic group,* Mp(2), which is a double covering of SL(2,R);[6] but since in eq. (8) we are dealing with amplitudes defined over two-dimensional transverse planes, we have here a true representation of SL(2,R) itself). This action of FOS's on the scalar wave field ψ of course reduces to the simple action (1) on rays in the geometric limit. It is actually quite simple in its operator form too – that is most easily seen by exploiting the Iwasawa decomposition, which leads to:

$$\begin{aligned} \bar{U}(S) &= L(\xi)\ M(\eta)\ R(\zeta) \\ &= \exp\left(\frac{-i}{2}\ \xi\ \hat{\mathbf{x}}^2\right)\ \exp\left(\frac{-i\eta}{4}\ (\ \hat{\mathbf{x}}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\hat{\mathbf{x}})\right)\ \exp\left(\frac{-i\zeta}{4}\ (\ \hat{\mathbf{x}}^2 + \hat{\mathbf{p}}^2\ )\right), \\ [\hat{x}_j,\ \hat{p}_k] &= i\delta_{jk},\ j,\ k = 1,\ 2\ . \end{aligned} \tag{9}$$

We see that we have here exponentials of quadratic expressions in the quantum-mechanics type operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, which is why the group representation property holds. For lenses and magnifiers the integral kernel collapses and one has

$$\begin{aligned} L\ (\xi) &: \psi'(\mathbf{x}) = \exp\left(\frac{-i}{2}\ \xi\ \mathbf{x}^2\right)\ \psi(\mathbf{x}); \\ M(\eta) &: \psi'(\mathbf{x}) = e^{-\eta/2}\ \psi(e^{-\eta/2}\ \mathbf{x})\ . \end{aligned} \tag{10}$$

But free propagation for example remains non-trivial:

$$F(D) : \psi'(\mathbf{x}) = \frac{-i}{2\pi D}\int d^2x'\ \exp\left[\frac{i}{2D}\ (\mathbf{x} - \mathbf{x}')^2\right]\ \psi(\mathbf{x}')\ . \tag{11}$$

There are two interesting questions that now naturally arise:

(i) Can one define the action of FOS's on the wave field taking proper account of the polarization properties of light, especially in such a way that the association of optical systems with group elements is maintained? What replaces the generalized Huyghens kernel if this can be done?
(ii) In the wave theory, even in the approximation where polarization is ignored, can one generalise the notion of rays to get something that changes in the simple manner of rays in the geometric limit, i.e., via the matrix S associated with an FOS, without losing the specific wave features of interference and diffraction?

Both questions can be answered in the affirmative, though they of course involve rather different group-theoretical arguments and machinery. Let me outline the answers briefly, turning first to the question concerning the consistent treatment of polarization.

The complete description of electromagnetic waves uses the full set of Maxwell's equations, in place of the single d'Alembert wave equation of scalar optics. And all of Maxwell's equations are needed to treat polarization properly. We saw in the case of scalar optics that the generalized Huyghens kernel respects the group structure of FOS's because we are dealing with exponentials of quadratic expressions in "canonical" coordinate and momentum operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. Such hermitian quadratic expressions form a Lie algebra; and the key to all this is the Heisenberg type commutation relation between $\hat{x}$ and $\hat{p}$. To preserve this group structure we must search for replacements for $\hat{x}$ and $\hat{p}$, suitable for the transition from scalar to Maxwellian waves, such that the commutation relations are preserved. It turns out that the solution lies in the relativistic invariance of Maxwell's equations. Out of the mathematical expression of this invariance or symmetry, in a systematic way one can construct the replacements we are looking for [7]. One then finds the following: Any paraxial Maxwellian wave can be fully described by its electric vector; moreover all components of this vector cannot be independently specified over any transverse plane. Knowledge of the transverse x and y components fixes the longitudinal z component in the beam direction. If k is the longitudinal wave number, then to leading paraxial order

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \simeq \exp\left(\frac{i}{k}\,\mathbf{G}\cdot\hat{\mathbf{p}}\right)\begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\frac{\partial}{\partial \mathbf{x}},$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}. \tag{12}$$

And the action of any FOS on a paraxial Maxwellian wave is completely determined by its action in scalar wave optics by this rule of replacement within $\bar{U}(S)$:

$$\hat{\mathbf{x}} \to \hat{\mathbf{x}} + \frac{1}{k}\,\mathbf{G}, \;\; \hat{\mathbf{p}} \to \hat{\mathbf{p}} : \bar{U}(S) \to \bar{U}_M(S)\,. \tag{13}$$

This prescription follows unambiguously from the systematic exploitation of the relativistic invariance of Maxwell's equations in what is technically called the front form. It is important to stress that the structure of this rule guarantees that polarization will be handled properly and, equally important, the group properties of FOS's are preserved.

The replacement for the scalar Huyghens kernel is thus also unambiguously determined. However, rather than showing how the most general FOS would act on the electric vector, a few examples will be given to illustrate the situation[8]. For free propagation there is no difference between scalar and Maxwell waves:

$$F_M(D) : \mathbf{E}'(\mathbf{x}) = F(D)\,\mathbf{E}(\mathbf{x})$$

$$= \exp\left(\frac{-iD}{2k}\,\hat{\mathbf{p}}^2\right)\,\mathbf{E}(\mathbf{x})\,. \tag{14}$$

But for a lens we find an important difference, a matrix being needed in addition to the phase:

$$L_M(g) : \mathbf{E}'(\mathbf{x}) \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ gx & gy & 1 \end{pmatrix} L(g)\ \mathbf{E}(\mathbf{x})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ gx & gy & 1 \end{pmatrix} e^{-(ik/2)gx^2}\ \mathbf{E}(\mathbf{x}). \tag{15}$$

The need for and the correctness of this extra matrix can be checked by verifying that on passing through the lens, the Poynting vector of the Maxwell wave gets bent in just the proper way. Similarly the magnifier and the Fourier transformer do differ in important ways from their forms in the scalar theory:

$$M_M(\eta) : \mathbf{E}'(\mathbf{x}) \simeq \exp\left(\frac{i}{k}(1 - e^{\eta/2})\ \mathbf{G}\cdot\hat{\mathbf{p}}\right) M(\eta)\ \mathbf{E}(\mathbf{x})$$

$$= \exp\left(\frac{i}{k}(1 - e^{\eta/2})\ \mathbf{G}\cdot\hat{\mathbf{p}}\right) e^{-\eta/2}\ \mathbf{E}(\,e^{-\eta/2}\ \mathbf{x});$$

$$\mathcal{F}_M : \mathbf{E}'(\mathbf{x}) \simeq \exp\left[i\mathbf{G}\cdot\left(\frac{1}{k}\,\hat{\mathbf{p}} - k\,\hat{\mathbf{x}}\right)\right]\mathcal{F}\ \mathbf{E}(\mathbf{x})\,. \tag{16}$$

To leading paraxial order, this answers all questions on the actions of FOS's on light beams endowed with polarization. Of course, the extension to polarization sensitive optical *systems,* undefined in scalar theory, can also be carried out [9].

Let us now turn to the second question raised above. Here one must remember that all traditional classical optics experiments and theory deal only with intensity measurements and the so-called two-point correlation function.[10] The famous experiment of Hanbury Brown and Twiss was the first one to go beyond this framework. At the level of the two-point correlation function, the answer to our question is contained in the notion of *generalized rays* introduced by Sudarshan.[11] Ignoring polarization for simplicity, one sets up the so-called *Wolf function* in analogy with the Wigner-Moyal phase space distribution in quantum mechanics:

$$W(\mathbf{x},\ \mathbf{p}) = (2\pi)^{-2}\int d^2x'\,e^{i\mathbf{p}\cdot\mathbf{x}'}\,\langle\langle\psi\,(\mathbf{x} + \tfrac{1}{2}\mathbf{x}'\,)^*\ \psi(\mathbf{x} - \tfrac{1}{2}\,\mathbf{x}'\,)\rangle\rangle\,. \tag{17}$$

Here a stationary, monochromatic ensemble of wave fields has been assumed, and the double angular brackets denote the ensemble average. This Wolf function is real but – as for the Wigner function in quantum mechanics – it does not have the property of being pointwise non-negative. Therefore, when it is interpreted as the intensity distribution function for generalized rays at transverse position **x** with transverse wave vector **p**, we must allow for both "bright" and "dark" rays, corresponding to $W(\mathbf{x},\ \mathbf{p})$ being positive or negative. It is now a specific property of the relationship between the group of linear canonical transformations on $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, and the definition of the Wolf function, that under any FOS this function transforms in a very simple way:[12]

$$S \in \mathrm{SL}(2,R):$$
$$\psi(\mathbf{x}) \to \psi'(\mathbf{x}) = (\tilde{U}(S)\ \psi)\ (\mathbf{x}) \Rightarrow$$
$$W(Q) \to W'(Q) = W(S^{-1}Q)\ ,$$
$$\underset{\sim}{Q} = \begin{pmatrix} \mathbf{x} \\ \mathbf{p} \end{pmatrix} \tag{18}$$

It is this extremely simple behaviour of generalized rays on passage through any FOS – the same as for ordinary rays in the usual geometric optics limit but exact in the sense of wave theory – that makes them so useful in many discussions. A few examples, in the context of particular kinds of light beams, will be presented here. But it is worth mentioning that this notion of generalized rays can be and has been extended to both higher order correlation functions and to the quantum domain.[13] When this is done, one finds new non-local correlation properties among generalized rays reflecting the Bose nature of light.

A practically important class of light beams is the family of Gaussian Schell model (GSM) beams. To begin we consider the isotropic or axially symmetric case, the IGSM family, for which the two-point correlation function and the Wolf function are both Gaussian:[14]

IGSM Beams

$$\langle\langle\ \psi(\mathbf{x})^*\ \psi(\mathbf{x}')\ \rangle\rangle = [I(\mathbf{x})\ I(\mathbf{x}')]^{1/2}\ g(\mathbf{x},\ \mathbf{x}'),$$
$$I(\mathbf{x}) = (A/2\pi\sigma_I^2)\ \exp\ (-\mathbf{x}^2/2\sigma_I^2),$$
$$g(\mathbf{x},\ \mathbf{x}') = \exp\ [-|\mathbf{x} - \mathbf{x}'|^2/2\sigma_g^2 - ik(\mathbf{x}^2 - \mathbf{x}'^2)/2R]; \tag{a}$$
$$W(Q) = (A/\pi^2)\cdot \det \mathrm{G} \cdot \exp\ [-k\ Q^T G Q],$$
$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$
$$G_{11} = \frac{1}{2}\left(\frac{1}{k\sigma_I^2} + \frac{k\gamma^2}{R^2}\right),\ G_{12} = G_{21} = \frac{-k\gamma^2}{2R},\ G_{22} = \frac{k\gamma^2}{2}, \tag{b}$$
$$\frac{1}{\gamma^2} = \frac{1}{\sigma_g^2} + \frac{1}{4\sigma_I^2}\ . \tag{19}$$

Here I (x) is the intensity distribution, $g(\mathbf{x}, \mathbf{x}')$ the normalized (complex) degree of coherence, and σ_I, σ_g and R the associated widths and phase curvature. The important point is that everything about this two-point function is Gaussian, and so then is the Wolf function. The latter is completely specified by the 2×2 *real symmetric positive definite* matrix G – the *parameter matrix* of the IGSM beam.

It now happens that under passage of this beam through any FOS, the IGSM nature is retained, and all that happens is that the parameter matrix G changes in a simple way. This is quite clear from the general rule (18) for any Wolf function. In the present case we find:

$$S \in \mathrm{SL}(2,R) : G \to G' = (S^{-1})^T\ G\ S^{-1}. \tag{20}$$

This is a *symmetric symplectic* transformation. This behaviour can be pictorially represented in a very effective way.[14] By writing G in the form

$$G = x^0 - x^1\sigma_1 - x^2\sigma_3 = \begin{pmatrix} x^0 - x^2 & -x^1 \\ -x^1 & x^0 + x^2 \end{pmatrix}, \tag{21}$$

each IGSM beam (apart from the total intensity) can be represented by a point or vector x in a fictitious three-dimensional Minkowski space $M_{2,1}$, which is *timelike* and *positive*:

$$G \rightarrow x \in M_{2,1}\colon x^0 > 0,\ x.x \equiv (x^1)^2 + (x^2)^2 - (x^0)^2 < 0. \tag{22}$$

And the passage of such a beam through the FOS corresponding to $S \in \mathrm{SL}(2,R)$ has the effect of subjecting the vector x to a proper Lorentz transformation $\Lambda(S) \in \mathrm{SO}(2,1)$ in two space and one time dimension:

$$\begin{aligned} &S \in \mathrm{SL}(2, R)\colon G \rightarrow G' = (S^{-1})^T\, G.\, S^{-1} \Leftrightarrow \\ &x \rightarrow x' = \wedge(S)\, x, \\ &\wedge(S) \in S\,0\,(2,1) \end{aligned} \tag{23}$$

All this is shown in figure 2. This construction is much simpler than one might imagine at first sight. We recall that in the description of the polarization of a plane light wave, we can use the Poincaré sphere to depict various states of polarization. (We will in fact turn to that topic in the sequel). The Poincaré sphere is actually a "fictitious" one, constructed to conveniently exhibit the parameters of the polarization ellipse, which is really the object of physical interest and is located in the plane transverse to the direction of propagation of the wave. Nevertheless, the Poincaré sphere method is a very useful one, and moreover many polarization related *devices* act in such a way as to produce *rotations* on this sphere. In an analogous way, we have here a representation of the parameters of a particular class of (isotropic, scalar) light beams by means of time-like positive vectors in a three-dimensional *Minkowski* space (replacing the Poincaré sphere) and the action by FOS's as *Lorentz* rotations (replacing ordinary rotations) in this space. Thus under such action, the vector x representing a given IGSM beam moves on its own single-sheeted time-like hyperboloid Ω in $M_{2,1}$, preserving its Lorentz square, as depicted in figure 2. The Lorentz transformations corresponding to the actions of lens, free propagation, magnifier, Fourier transformer, and "phase space rotator" are of particular interest and are as follows:

Lens

$$\mathrm{l}(g) = \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix}:$$

$$\wedge(g) = \begin{pmatrix} 1 + g^2/2 & -g & +g^2/2 \\ -g & 1 & -g \\ -g^2/2 & g & 1-g^2/2 \end{pmatrix},\ x^0 - x^2 = \text{invariant.} \tag{a}$$

Free propagation

$$f(D) = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix}:$$

$$\Lambda(D) = \begin{pmatrix} 1 + D^2/2 & +D & -D^2/2 \\ +D & 1 & -D \\ +D^2/2 & D & 1 - D^2/2 \end{pmatrix},\ x^0 + x^2 = \text{invariant.} \tag{b}$$

Magnifier

$$m(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}:$$

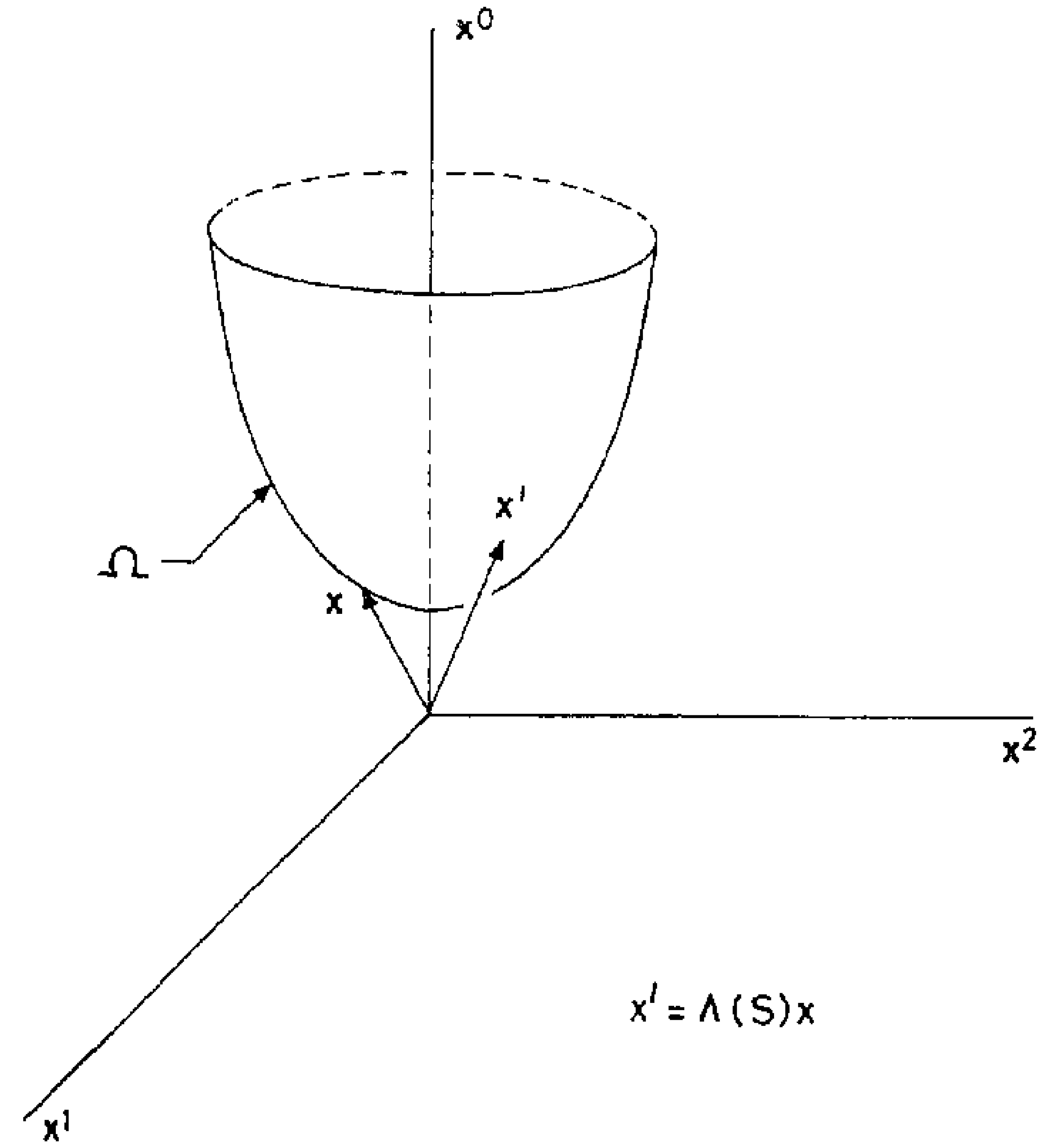

Figure 2. Minkowski space picture for IGSM beams and fluctuation matrices.

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & 0 & -\sinh\eta \\ 0 & 1 & 0 \\ -\sinh\eta & 0 & \cosh\eta \end{pmatrix}, \; x^1 = \text{invariant}. \tag{c}$$

Fourier transformer

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}:$$

$$\Lambda(\mathcal{F}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; x^0 = \text{invariant} \tag{d}$$

"Rotator"

$$r(\zeta) = \begin{pmatrix} \cos\zeta/2 & \sin\zeta/2 \\ -\sin\zeta/2 & \cos\zeta/2 \end{pmatrix},$$

$$\Lambda(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & -\sin\zeta \\ 0 & \sin\zeta & \cos\zeta \end{pmatrix}, \; x^0 = \text{invariant}. \tag{e}$$

These five cases can be respectively described as follows: x moves on the intersection of Ω and the null plane $x^0 - x^2$ constant; x moves on the intersection of Ω and the null plane $x^0 + x^2 =$ constant; x undergoes a pure Lorentz transformation in the $0 - 2$ plane and so moves on the intersection of Ω with the plane $x^1 =$ constant; x undergoes a "spatial reflection" or rotation by an angle π; finally x undergoes a "spatial rotation" by an angle ζ. (Naturally the Fourier transformer is a particular case of the rotator). Thus one sees how IGSM beams are naturally pictured as time-like vectors in $M_{2,1}$ and physically important optical systems act via Lorentz transformations on them.

This combined Minkowski-Lorentz picture is actually not restricted (in a sense) to Gaussian beams, but can be used to discuss the position and wave-vector spreads of *any* paraxial beam, and the effects of FOS's on these spreads.[15] It can also be used to discuss the physically very different but mathematically similar problem of squeezing.[16] If we continue to deal with axially symmetric beams, then as noted earlier we are concerned with just a single canonical pair of variables, x and p say. Then given a time-stationary monochromatic ensemble, the means and spreads in x and p are defined in the usual ways:

$$\begin{aligned}
x_0 &= \langle\langle\, (\psi\, |\hat{x}|\, \psi)\,\rangle\rangle = \int \mathrm{d}x\ \mathrm{d}p\cdot x\cdot W(x,p),\\
p_0 &= \langle\langle\, (\psi\, |\hat{p}|\, \psi)\,\rangle\rangle = \int \mathrm{d}x\ \mathrm{d}p\cdot p\cdot W(x,p);\\
(\Delta x)^2 &= \langle\langle\, (\psi\, |(\hat{x} - x_0)^2|\, \psi)\,\rangle\rangle = \int \mathrm{d}x\ \mathrm{d}p\ (x - x_0)^2\ W(x,p);\\
(\Delta p)^2 &= \langle\langle\, (\psi\, |(\hat{p} - p_0)^2|\, \psi)\,\rangle\rangle = \int \mathrm{d}x\ \mathrm{d}p\ (p - p_0)^2\ W(x,p);\\
\Delta(xp) &= \frac{1}{2}\,\langle\langle\, (\psi\, |\{\hat{x} - x_0,\ \hat{p} - p_0\}|\, \psi)\,\rangle\rangle\\
&= \int \mathrm{d}x\ \mathrm{d}p\ (x - x_0)\ (p - p_0)\ W(x,p).
\end{aligned} \tag{25}$$

Here in each case we first evaluate "quantum mechanical" type averages, then follow it up with an ensemble average. In addition to the usual spreads Δx and Δp, a new quantity $\Delta(xp)$ has also been defined. We can combine them to define a *fluctuation matrix* for *any* paraxial beam, once again a real symmetric positive definite matrix $\mathcal{G}$:

$$\mathcal{G} = \begin{pmatrix} (\Delta x)^2 & \Delta(xp) \\ \Delta(xp) & (\Delta p)^2 \end{pmatrix} \tag{26}$$

Somewhat like the parameter matrix G of an IGSM beam, each fluctuation matrix $\mathcal{G}$ determines a positive timelike vector x in $M_{2,1}$, which undergoes a Lorentz transformation when the beam goes through any FOS:

$$x^0 = \frac{1}{2}\,(\,(\Delta x)^2 + (\Delta p)^2\,),\ x^1 = \Delta(xp),$$

$$x^2 = \frac{1}{2}\,(\,(\Delta x)^2 - (\Delta p)^2\,); \tag{27a}$$

$$\begin{aligned} S \in \mathrm{SL}(2,R)\colon \mathcal{G} \to \mathcal{G}' &= S\ \mathcal{G}\ S^T \Leftrightarrow \\ x \to x' &= \Lambda\ (S)x \end{aligned} \tag{27b}$$

The usual statement of the uncertainty principle for any (isotropic) paraxial beam is

$$(\Delta x)^2\ (\Delta p)^2 \geqslant 1/4, \tag{28}$$

but a possibly more useful stronger version is

$$\det \mathcal{G} \equiv (\Delta x)^2\ (\Delta p)^2 - \Delta(xp)^2 \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 \geqslant 1/4\,. \tag{29}$$

In the application to squeezing, which is physically a different situation but shares the same mathematical structure, the squeezing transformation is the same as the "magnifier":

Squeezing

$$m(\eta) \sim e^{\frac{\eta}{4}(a^{+2} - a^2)} = e^{\frac{-i\eta}{4}(\hat{x}\hat{p} + \hat{p}\hat{x})}$$

$$\Delta x \to e^{-\eta/2}\, \Delta x,\ \Delta p \to e^{\eta/2}\, \Delta p,\ \Delta(xp) \to \Delta(xp);$$

$$x^0 \pm x^2 \to e^{\mp\eta}\, (x^0 \pm x^2),\ x^1 \to x^1. \tag{30}$$

Thus squeezing is realized in the Minkowski space picture as a pure Lorentz transformation in the 0–2 plane. Based on these geometric representations, one is motivated to replace the usual definition of minimum uncertainty states for a beam, and of a squeezed state in the context of squeezing, by new definitions reflecting the properties of the fluctuation matrix $\mathcal{G}$ in each case. The usual and the alternative definitions are summarized in the following way:

	Usual definitions and properties	*Alternative definitions and properties*
Uncertainty principle	$(\Delta x)^2 (\Delta p)^2 \geq 1/4$	$\det \mathcal{G} \equiv (\Delta x)^2 (\Delta p)^2 - \Delta(xp)^2 \geq 1/4$
	Invariant under magnifier action alone	Invariant under all FOS action
Minimum uncertainty states	$(\Delta x)^2 (\Delta p)^2 = 1/4$	$(\Delta x)^2 (\Delta p)^2 - \Delta(xp)^2 = 1/4$
	Invariant under magnifier action alone	Invariant under all FOS action
Squeezed states	$(\Delta x)^2$ or $(\Delta p)^2 < 1/2$	One eigenvalue of $\mathcal{G} < 1/2$.
	Not SO(2) invariant	SO(2), but not SL(2,R), invariant

Even with the suggested alternative definition of squeezed states, we can see that the transformation (30) does lead to squeezing: the diagonal elements of $\mathcal{G}$ are reciprocally scaled while the off-diagonal elements are left unaltered, so for a large enough $|\eta|$ one eigenvalue of $\mathcal{G}$ will be reduced to a value less than 1/2. One also can see a complementarity principle operating here. The motivation behind the alternative definitions is to increase their invariance properties; however, for a beam, for example, the new definitions of uncertainty and minimum uncertainty do not necessarily lead to the "best possible trajectory for a ray of light", this is achieved by the usual definitions. In any case one can see that the Minkowski space picture – quite similar in spirit to the use of the Poincaré sphere for polarization problems – is ideally suited for these discussions.

Geometric and group theoretic ideas like the above can be developed for higher dimensional situations and multimode problems as well. In the optics context we can consider beams which are not necessarily axially symmetric, and allow for anisotropy.[17] Then the class of FOS's gets enlarged since in them too anisotropy must be permitted; the family of such FOS's corresponds to elements of the four-dimensional real symplectic group Sp(4,R), which is locally isomorphic to the de Sitter group SO(3,2). Anisotropic Gaussian Schell Model (AGSM) beams turn out to be describable by parameter matrices G which are real four dimensional, and in addition to symmetry and positive definiteness have further specific properties. They can be pictured as special second rank antisymmetric tensors in the 3 + 2 de Sitter space, undergoing SO(3,2) transformations when FOS's act on the beam. One can carry such an analysis even further to systems with any number n of degrees of freedom. Here the concept of an FOS is that it is generated by any quadratic Hamiltonian. Such FOS's then correspond to elements of the symplectic group Sp($2n$,R). (Optical IGSM and AGSM beams thus correspond to n = 1, 2 respectively). As an example of states of such systems, we mention the pure Gaussian ones.[18] It can be shown that such states are in a one-one correspondence with points of the coset space $Sp(2n,R)/U(n)$.[19] This space is a special orbit with respect to the adjoint action of $Sp(2n,R)$ on its Lie algebra, and by a famous theorem due to Kostant, Kirillov and Souriau, this orbit carries a natural phase space structure.[20] Then the action of any FOS, i.e. any element of $Sp(2n,R)$, on a pure Gaussian state has the form of a canonical transformation on this orbit. Without entering into too much detail, the point to be emphasized is that these methods would be essential for analysis of squeezing in multimode systems. The key features of these higher dimensional generalizations can be summarized thus:

	Geometric Representation of state	*FOS's and their actions*
IGSM	Time-like positive $x \in M_{2,1}$	$S \in SL(2,R)$: Lorentz transformations in $M_{2,1}$
AGSM	Special second rank antisymmetric tensor in $M_{3,2}$	$S \in Sp(4,R)$: de Sitter transformations in $M_{3,2}$
Gaussian pure states for n degrees of freedom	Points of coset space $Sp(2n,R)/U(n)$ (adjoint orbit)	$S \in Sp(2n,R)$: Canonical transformations on orbit

We leave this discussion with the comment that there is one interesting result which unifies all these cases: it is an "*ABCD*-Law" which expresses the effect of any FOS on the parameter-matrix of a state as a fractional linear transformation on a suitably defined complex parameter which is a scalar if $n = 1$ but a matrix if $n \geq 2$.[19]

After these examples of the uses of generalized rays for describing certain kinds of beams and their passage through optical systems, leading to practically useful geometric methods, we return to the description of polarization and present some interesting new applications of a very old idea due to Hamilton.[21] This is his *theory of turns*, which is a geometric description of the group SU(2). As is well known and was mentioned earlier, the states of polarization of a plane electromagnetic wave can be represented in a one-to-one manner by the points on a sphere $\mathcal{P}$ in a fictitious three-dimensional space. This is called the Poincaré sphere.[22] As shown in figure 3, the north and south poles of this sphere represent right and left *circular* polarizations respectively; points on the equator correspond to various states of *linear* polarization; all other points represent general *elliptic* polarizations. If the transverse electric field vector is written in complex form and after removal of a harmonic time factor as

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \tag{31}$$

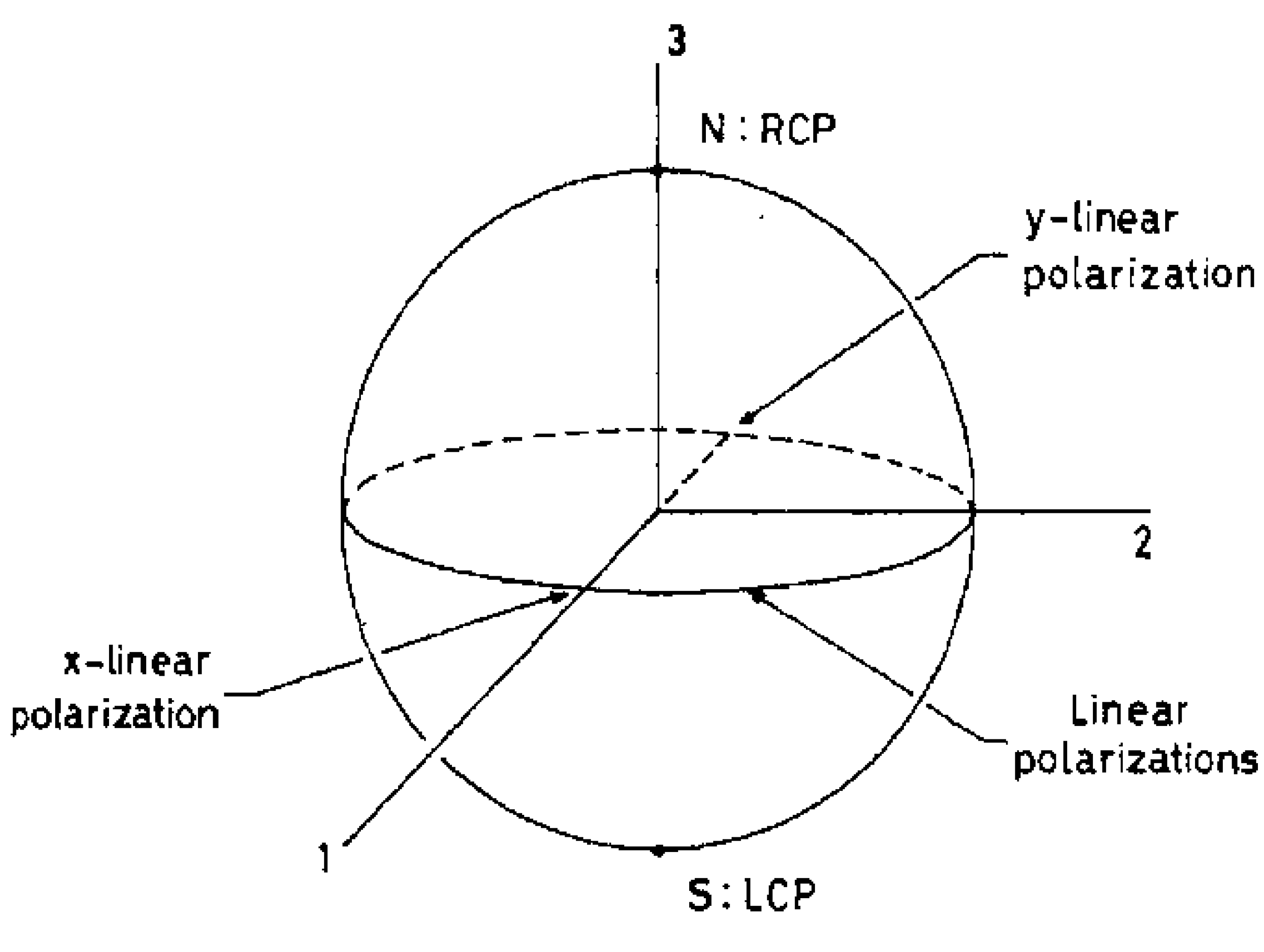

Figure 3. Poincaré sphere $\mathcal{P}$: Representation of polarization states.

then the components of the real Stokes vector

$$\vec{S} = E^{\dagger}\ (\sigma_3,\ \sigma_1,\ \sigma_2)E \tag{32}$$

determine the point on $\mathcal{P}$ representing this polarization state. There is now a class of optical systems – those which act linearly on E and preserve the intensity – which arise naturally. These systems correspond one-to-one to elements of the group U(2). A subgroup of such systems corresponds to elements of SU(2). The most familiar examples, also of obvious physical significance, are these:

Rotator

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{33}$$

Compensator

$$(C_\delta)_0 = \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix} \tag{34}$$

This is a birefringent plate with its fast axis along the spatial x-axis.

Compensator with general orientation

$$(C_\delta)_\theta = R_\theta\ (C_\delta)_0\ R_\theta^{-1} \tag{35}$$

Now the fast axis is at an angle θ to the spatial x-axis. Quarter wave plates are the systems $(C_{\pi/2})_\theta$, while half wave plates are $(C_\pi)_\theta$. In each of these cases, the SU(2) matrix acting on the electric field vector E of eq. (31) gives the effect of the corresponding "polarizing system". One can say that the basic building blocks of (intensity preserving) polarizing systems are *rotators* and *compensators* – just like lenses and free propagations for FOS's.

It is of course well known that the effect of any such "polarizing system", or any combination of them, on the polarization state of a plane wave is to produce an orthogonal SO(3) rotation on the Poincaré sphere $\mathcal{P}$. But while this *effect* can be displayed in geometric form, one has to describe the system themselves, and their combinations, in *algebraic* terms, i.e. as SU(2) matrices and their products. It is here that Hamilton's theory of turns provides a *geometrical* picture for "polarizing systems" and their combinations as well.[23] The value of this method is that it gives insight into problems of synthesis of desired "polarizing systems", because they can be visualized geometrically. For this purpose one has to invent another fictitious sphere $\mathcal{T}$ – the sphere of turns or the Hamilton sphere – on which elements of the group SU(2) can be represented. A turn $\vec{\tau}$ is any directed great circle arc on the sphere $\mathcal{T}$; an example is shown in figure 4. However two arcs are regarded as equivalent if by sliding on the great circle one can be made to coincide with the other; so a turn is actually an equivalence class of directed great circle arcs. Now it happens that each element of the group SU(2) corresponds precisely to a turn (one has to pay special attention to the two elements $\pm$ 1). And multiplication of elements of SU(2) corresponds to "vector addition of turns", as depicted in figure 4. Given two turns $\vec{\tau}_1$ and $\vec{\tau}_2$, to "add" $\vec{\tau}_1$ to $\vec{\tau}_2$ we slide them on their respective great circles till the head of $\vec{\tau}_1$ coincides with the tail of $\vec{\tau}_2$. Then the resultant turn runs from the tail of $\vec{\tau}_1$ to the head of $\vec{\tau}_2$. This is indeed non commutative, and correctly expresses the SU(2) composition law.

Some examples will explain the construction: We show in figure 5 the rotator R_θ as an equatorial arc of length θ; it can be placed anywhere on the equator. The compensator $(C_\delta)_0$ is an arc of length $\delta/2$ in the 2–3 plane; a general compensator $(C_\delta)_\theta$ is a meridional arc, and can again be placed anywhere on its great circle. General quarter wave plates and halfwave plates are meridional arcs of lengths $\pi/4$ and $\pi/2$ respectively.

Given a turn $\vec{\tau}$ on the sphere of turns $\mathcal{T}$, i.e. an element of SU(2), its effect on the Poincaré sphere $\mathcal{P}$ is geometrically determined in a simple way. It is a rotation about the axis perpendicular to the plane containing $\vec{\tau}$, by *twice* the angle or length of $\vec{\tau}$. This makes the well-known results, that any quarter wave plate takes each of the circular polarizations into some linear polarizations,

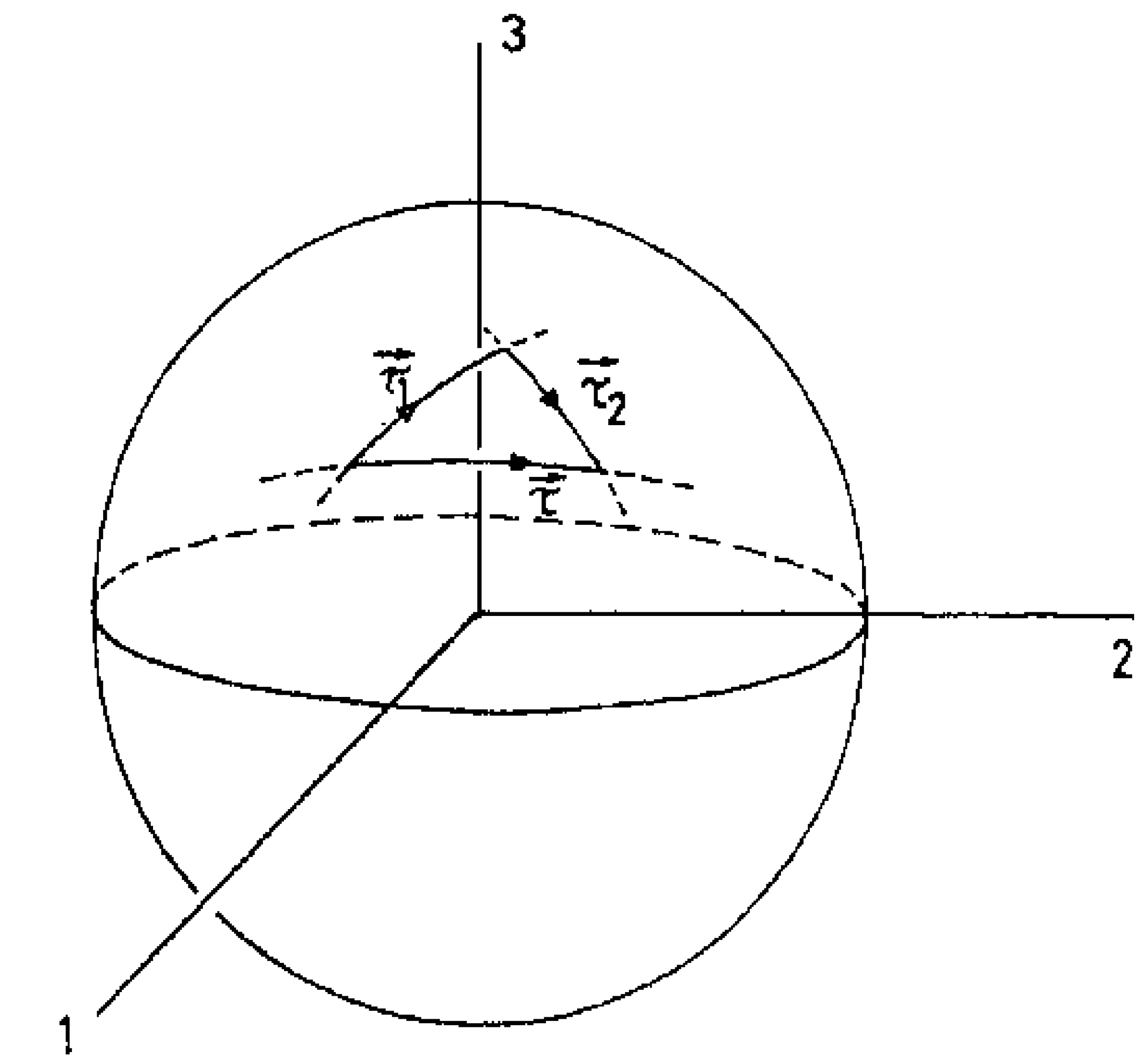

Figure 4. The sphere of turns $\mathcal{T}$.

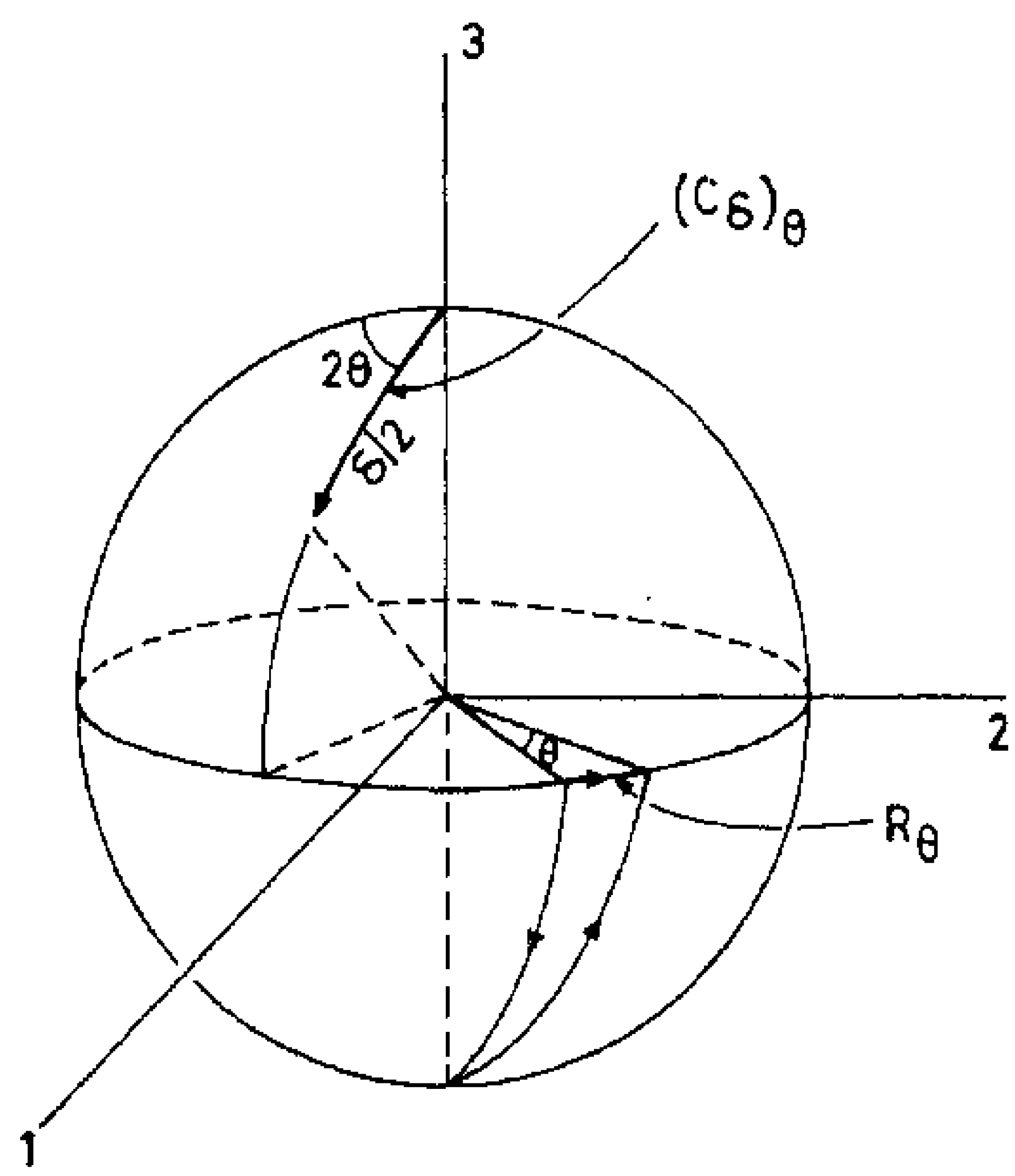

Figure 5. The rotator and compensator turns.

and that any halfwave plate interchanges the two circular polarizations, immediately and visually obvious. Equally obvious is the fact that the circular polarizations are "eigenstates" of any rotator.

As applications of such ideas, two examples may be mentioned: one obvious, the other requiring a little effort.[23] The first is that the rotator R_θ is realizable as a sequence of two half-wave plates, their fast axes making an angle θ with one another. The turns drawn in the lower hemisphere in figure 5 make this completely obvious. The second result is that *any* SU(2) polarizing system can be synthesized using *six fixed elements,* and only varying their relative orientations. This is indicated in figure 6, and it requires four half wave plates and two quarter wave plates to carry out the construction. The general turn AB in this figure is the "sum" of the rotator turn AC and the compensator turn (meridional arc) CB. The former has just been seen to be realizable using two half wave plates; it is the latter that requires the remaining two half wave plate and the two quarter wave plates.

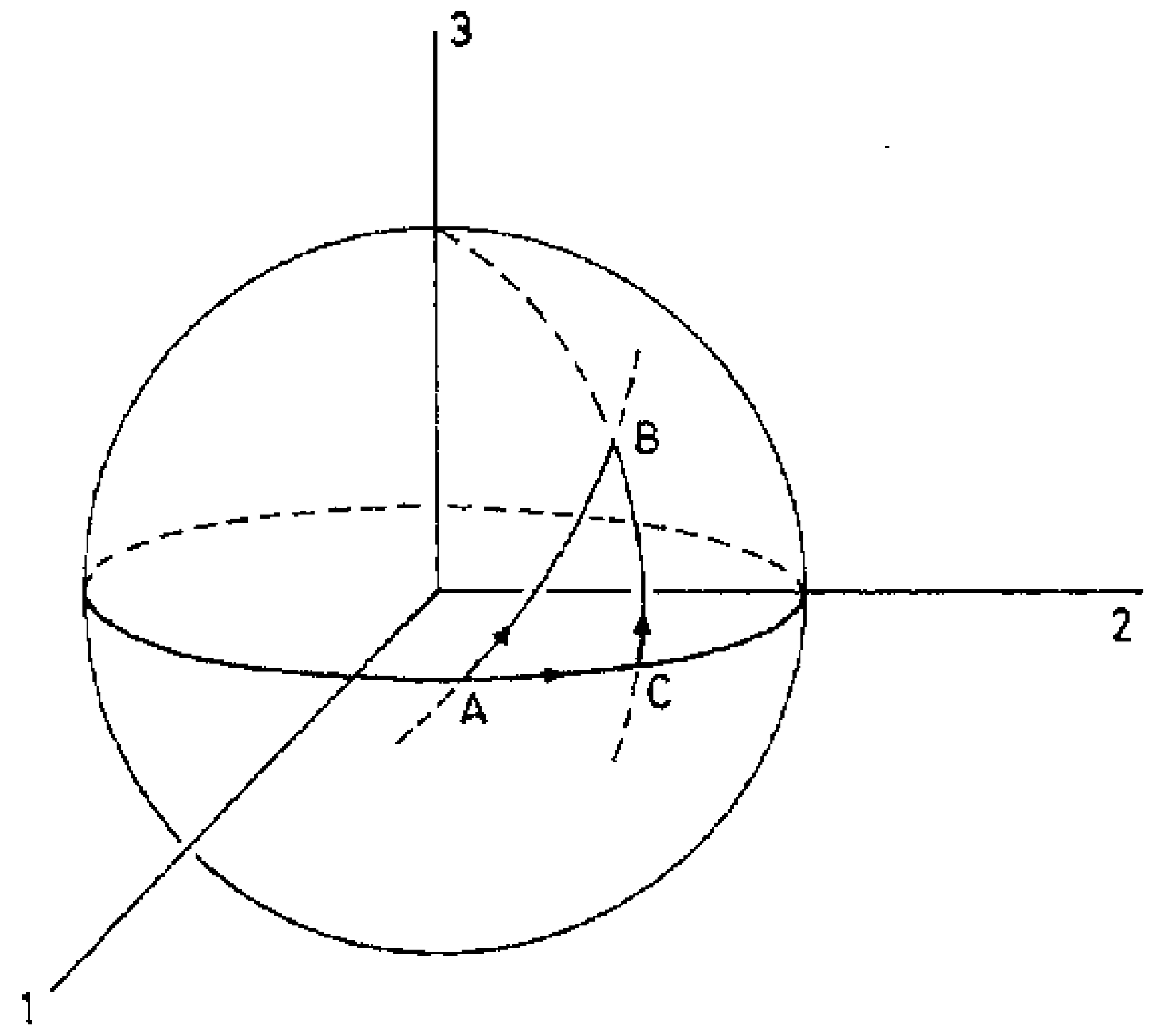

Figure 6. Towards a universal SU(2) gadget.

Some times fewer elements may suffice. This reminds us of the result mentioned earlier in eq. (7) that *every* FOS in SL(2,*R*) is *definitely* obtainable using at most three lenses (but of variable powers) and three free propagation sections (over variable distances).

It is in this way that Hamilton's method of turns allows both polarization states and polarizing systems, their actions and compositions, to be viewed and handled geometrically. This is of use in discussing geometrical phases in the context of polarization optics, as has been shown elsewhere.[23]

We come now to the last topic we wish to mention in this survey of group theoretical methods in the analysis of optical problems. It brings us back to the axially symmetric FOS's described by

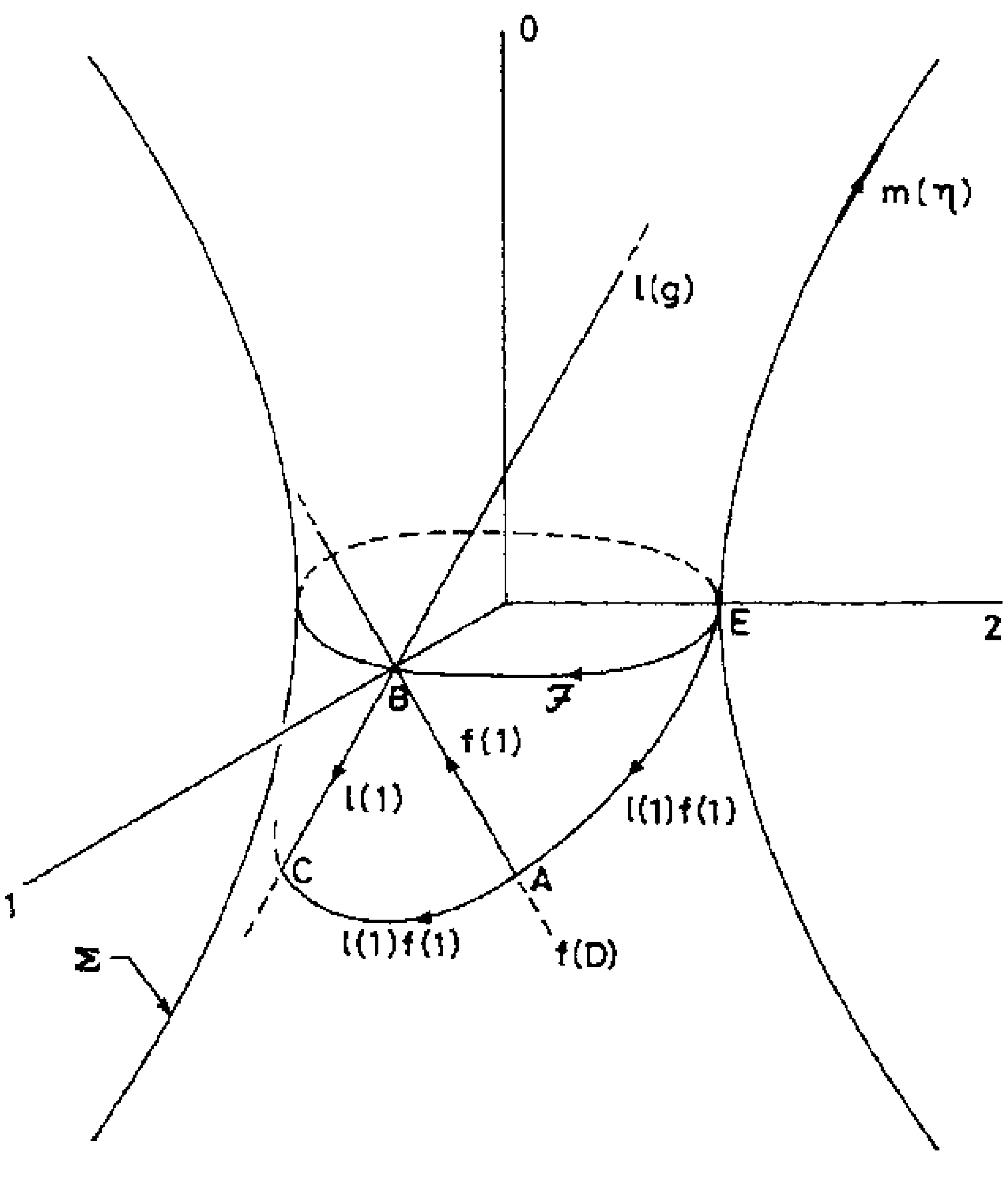

Figure 7. Geometric representation of FOS's.

elements of SL(2,R). If Hamilton's method of turns gives a pictorial way of handling the group SU(2), one would suspect that for the closely related group SL(2,R) something similar should be possible. This is indeed so, though there are some important and non-trivial differences. For polarization states and SU(2) systems we worked with the spheres $\mathcal{P}$ and $\mathcal{T}$ in three-dimensional Euclidean spaces; now our constructions have to be in the Minkowski space $M_{2,1}$. In this space, there are three distinct kinds of "spheres", time-like hyperboloids, the light cones, and spacelike hyperboloids. We already used the time-like hyperboloids Ω to represent IGSM beams, and the fluctuation matrices of arbitrary beams; thus such hyperboloids here play the role of the Poincaré sphere $\mathcal{P}$. It turns out that the analogue of the sphere of turns $\mathcal{T}$ is the unit (single sheeted) *spacelike hyperboloid* Σ in $M_{2,1}$. One can represent every element of SL(2,R), i.e., every FOS in optics, as (an equivalence class of) a "great circle arc" on Σ, i.e., a directed arc on the curve of intersection of Σ with some plane through the origin, accompanied by a "flag" or sign $\pm$ 1 which must be carried along.[24] This is one new feature in contrast to the SU(2) case. Another feature is this: just as there are three kinds of "spheres" in $M_{2,1}$, we have three types of "great circles" on Σ: finite ellipses which are the intersections with Σ of planes with time like normals; pairs of infinite parallel straight lines, i.e. generators of Σ, arising as the intersections of Σ with planes having light like normals; and branches of hyperbolas which are the intersections of Σ and planes with space like normals. These correspond to qualitatively distinct kinds of elements of SL(2,R), so of FOS's. Examples of all these are shown in figure 7.[25] Lenses and free propagations, the usual basic building blocks of FOS's, happen to be straightline "arcs" on Σ parallel to the 0–2 plane, so they are portions of two special generators of Σ in the sense of a ruled surface. Magnifiers are arcs along the hyperbolas in which Σ cuts the 0–2 plane. The rotator $r(\zeta)$ in phase space, and in particular the Fourier transformer, are arcs on the circle or waist of Σ in the 1–2 plane. The SL(2,R) composition law too takes a direct geometrical form, as for turns.[24] Some interesting results follow from this geometry: the well-known fact that the Fourier transformer is the focal plane to focal plane map of a thin lens of unit power is made trivially obvious, since in the product representation

$$\mathcal{F} = f(1)\ l(1)\ f(1) \tag{36}$$

we can represent the factors from the right to the left by the successive arcs AB, BC and AB. So the product $l(1)\ f(1)$ corresponds to the arc AC, which by sliding can be brought to the position EA. Now combining it with AB representing the remaining factor $f(1)$, we get the resulting arc EB which is the Fourier transformer. Another result is that *every* FOS is the combination of a suitable *rotator* (arc in the waist plane), followed by a rather unusual "graded index fibre" (arc in a vertical plane). This decomposition, shown in figure 8, is analogous to the SU(2) decomposition shown in figure 6, and in the language of Lorentz transformations it is just the statement that every $\Lambda \in$ SO(2,1) is a pure rotation followed by a pure boost. One other quite remarkable result is that *any* FOS is expressible in infinitely many ways as the product of two suitable *elliptic* type FOS's whose representative arcs on Σ are portions of ellipses and so represent stable systems.[24] This has implications both for the squeezing problem and for the stability criterion for laser resonators.

The action of an FOS realised geometrically as an arc plus flag on Σ, on an IGSM state or fluctuation matrix of a beam represented as a point on some Ω, can be determined geometrically. This is a natural generalisation of the kinds of constructions we encountered with turns and the Poincaré sphere. To distinguish the noncompact SL(2,R) from the compact SU(2), the term "screw" could be used for the former in place of "turn" for the latter. The geometrical picture shows easily for example that an IGSM beam can be invariant under an FOS only if the latter belongs to a compact subgroup of SL(2,R). In other words, the FOS must be representable as an arc on an ellipse on Σ. Other parabolic and hyperbolic type FOS's do not possess "eigenstates" among the IGSM family.

We have presented many areas in optics where geometry, symmetry and group theory play significant roles. What unifies them is the fact that the relevant mathematical techniques and geometrical constructions are closely related. In particular, one can say that FOS's are nothing but noncompact versions of polarizing systems, in that SU(2) and turns are replaced by SL(2,R) and screws! Some directions in which group theoretical methods can be further exploited are: general non paraxial beams; systematic analysis of aberrations; waves guided along fibres; etc.

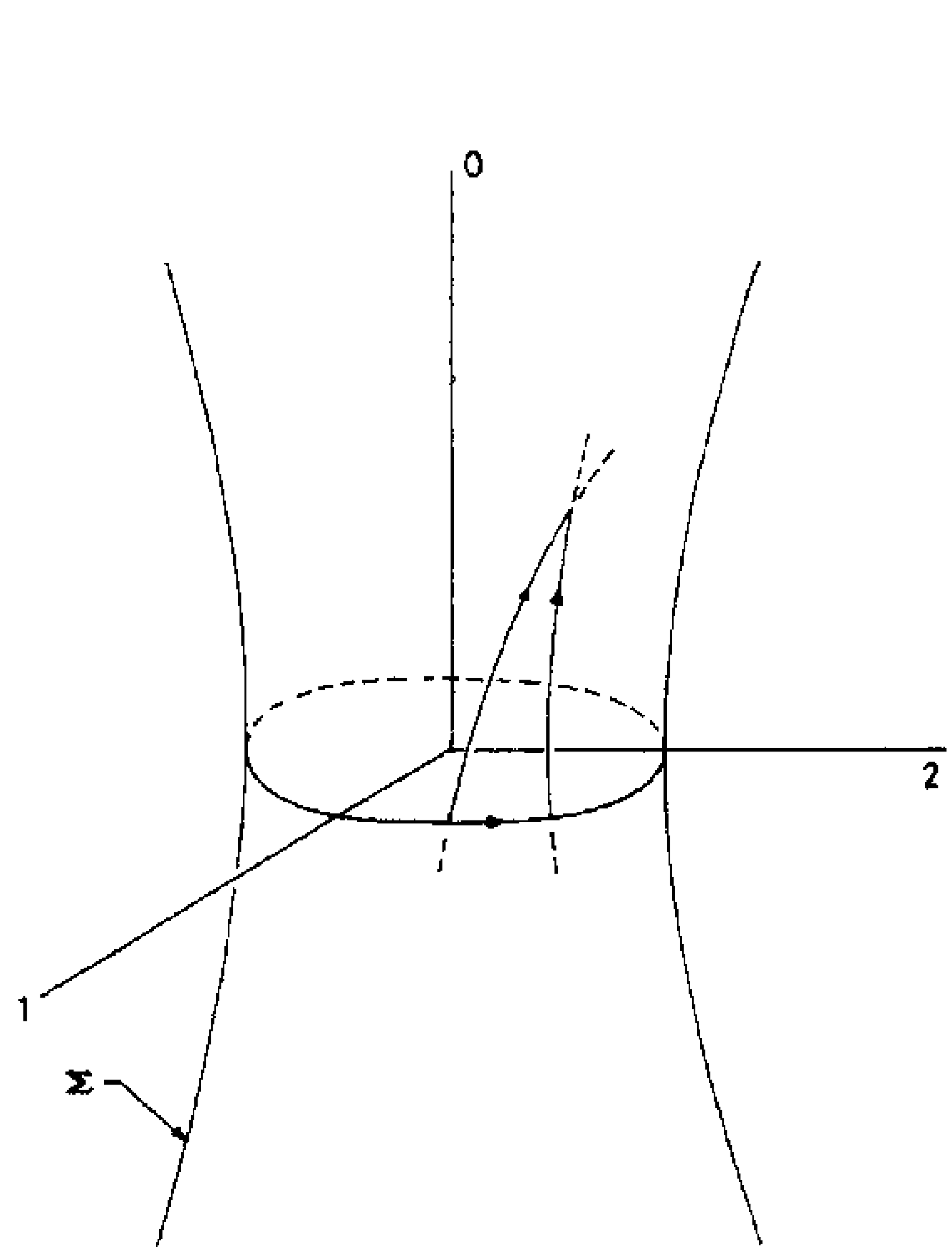

Figure 8. Two-element decomposition of general FOS.

In conclusion, I would like to mention two things. The first is that the developments described above, apart from the other general references given, are the results of work done jointly by R. Simon, E. C. G. Sudarshan and myself. The second is this: at the beginning I referred to de Broglie's eloquent way of describing the polarization of light. In his several beautiful essays on the history of the understanding of light and optics, he repeatedly and justifiably speaks with great pride of the fantastic French contributions and contributors to this branch of science: Descartes, Fermat, Fresnel, Foucault and Fizeau, to name but a few[26]. Of course to these we must add Poincaré. We in this country and this century have also seen many wonderful and deep contributions to the understanding of light – both of its own nature, and in its interaction with matter. Apart from the pre-eminent work of Raman, we have the discoveries and insights given to us by S. N. Bose on the indistinguishability of photons; S. Chandrasekhar on the theory of radiative transfer; G. N. Ramachandran and S. Ramaseshan on crystal optics; S. Pancharatnam on coherence and polarization; and E. C. G. Sudarshan on the fundamental theorems of quantum optics. Their work and achievement serve as inspiration and guidance to many of us who also are struck by the brilliance of light and are attracted to its study.

REFERENCES

1. de Broglie, Louis, "Old Ways and New Perspectives in the Theory of Light" in *"Matter and Light – The New Physics"*, Dover Publications, New York, 1946.
2. See, for instance, Stavroudis, O.N., "The Optics of Rays, Wave fronts and Caustics", Academic Press, New York, 1972.
3. See, for instance, Helgason, S., "Differential Geometry and Symmetric Space", Academic Press, New York, 1962.
4. Sudarshan, E.C.G., Mukunda, N., and Simon R., *Opt. Acta*, 1985, **32**, 855; see however the statements to the contrary in ref (2).
5. Nazarathy, M., and Shamir, J., *J. Opt. Soc. Am.*, 1982, **72**, 356; Wolf, K.B., "Integral Transforms in Science and Engineering", 1979, Plenum Press, New York.
6. Guillemin, V., and Sternberg, S., "Geometric Asymptotics", AMS, New York, 1977; Bacry, H., and Cadilhac, M., *Phys. Rev.*, **A23**, 2533, 1981.
7. Sudarshan, E.C.G., Simon, R., and Mukunda, N., *Phys. Rev.*, 1983, **A28**, 2921; Mukunda, N., Simon, R., and Sudarshan, E.C.G., *Phys. Rev.*, 1983, **A28**, 2933.

8. Mukunda, N., Simon, R., and Sudarshan, E.C.G., *J. Opt. Soc. Am.*, 1985, **A2**, 416.
9. Mukunda, N., Simon, R., and Sudarshan, E.C.G., *J. Opt. Soc. Am.*, 1985, **A2**, 1291.
10. Mandel, L., and Wolf, E., *Rev. Mod. Phys.*, 1965, **37**, 231; Klauder, J.R., and Sudarshan, E.C.G., "Fundamentals of Quantum Optics", Benjamin, New York, 1968.
11. Sudarshan, E.C.G., *Phys. Lett.*, 1979, **73A**, 269; *Phys. Rev.*, 1981, **A23**, 2802.
12. Simon, R., Sudarshan, E.C.G., and Mukunda, N., *Phys. Rev.*, 1987, **A36**, 3868.
13. Mukunda, N., Sudarshan, E.C.G., and Simon, R., *Found. Phys.*, 1988, **18**, 277.
14. Simon, R., Sudarshan, E.C.G., and Mukunda, N., *Phys. Rev.*, 1984, **A29**, 3273. References to earlier work can be found here.
15. Simon, R., Mukunda, N., and Sudarshan, E.C.G., *Opt. Commun.*, 1988, **65**, 322.
16. Walls, D.F., *Nature*, 1983, **306**, 141; for review and references, see Simon, R., "Symmetries in Science II", 1986 (eds. Gruber, B., and Lenczewski, R.), Plenum, New York.
17. Simon, R., Sudarshan, E.C.G., and Mukunda, N., *Phys. Rev.*, 1985, **A31**, 2419; *Phys. Lett.*, 1987, **A124**, 223; see also ref. (12) above.
18. Bialynicki - Birula, I., in *"Coherence, Cooperation and Fluctuations"*, 1986, (eds. Haake, F., Narducci, L. M., and Walls, D.F.), Cambridge University Press, London, 1986.
19. Simon, R., Sudarshan, E.C.G., and Mukunda, N., *Phys. Rev.*, 1988, **A37**, 3028.
20. See, for instance, Chu, B.Y., *Trans. Am. Math. Soc.*, 1974, **197**, 45.
21. Hamilton, W., "Lectures on Quaternions", Dublin, 1853; for a recent presentation, see Biedenharn, L.C., and Louck, J.D., "Angular Momentum in Quantum Physics", *Encyclopedia of Mathematics and its Applications*, 1981, **8** Addison-Wesley, Reading, Mass.
22. Poincaré, H., "Theorie Mathematique de la Lumiere", (ed. George Carré) Paris, 1892; Ramachandran, G.N., and Ramaseshan, S., *Handb. Physik* Vol. XXV/1, 1961, Springer-Verlag, Berlin; Born, M., and Wolf, E., "Principles of Optics", 1965, Pergamon Press.
23. Simon, R., Mukunda, N., and Sudarshan, E.C.G., "Hamilton's Theory of Turns and a New Geometrical Representation for Polarization Optics", preprint, Institute of Mathematical Sciences, Madras, India, 1988.
24. Simon, R., Mukunda, N., and Sudarshan, E.C.G., "The Theory of Screws – a New Geometric Representation for the group SU(1,1)", *J. Math. Phys.* (to appear).
25. Simon, R., Mukunda, N., and Sudarshan, E.C.G., "Hamilton's Theory of Turns Generalized to Sp (2, R)", preprint, Institute of Mathematical Sciences, Madras, India, 1988.
26. de Broglie, Louis, "A Survey of the History of Optics" in *"Matter and Light – The New Physics"*, 1946, Dover Publications, New York.