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The Poincaré Sphere and The Pancharatnam Phase – Some Historical Remarks

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1. THE POINCARÉ SPHERE

In the early forties, I was interested in the effect of birefringence on measurements of magneto-optic rotation. My research supervisor C. V. Raman brought to my notice a paper in which Becquerel[1] had measured the Faraday Rotation in a birefringent crystal in a direction away from the optic axis[2]. It was while reading this paper along with my colleague V. Chandrasekharan (who is now known for his work on Brillouin scattering in anisotropic crystals) that I first came across the Poincaré representation of polarised light. It is my view that the Poincaré Sphere gave an impetus and a new dimension to optics research in India (see references).

We require two parameters to describe a general (elliptic) state of polarisation – (a) the azimuth i.e. the orientation of the major axis of the ellipse, and (b) the "ellipticity", i.e. the ratio of major to the minor axes $b/a (b < a)$, $\omega = \tan^{-1} b/a$ – one uses positive and negative ω's for left and right rotating ellipses (figure 1a).

Poincaré (1892) took the remarkable and imaginative step of representing these states on the surface of a sphere by using 2ω and 2λ as the latitude and longitude (this also makes it obvious that the sphere has the right topology for representing polarised light). Figure 1b represents the Poincaré sphere, L and R, the poles represent the left and right circular vibrations and the

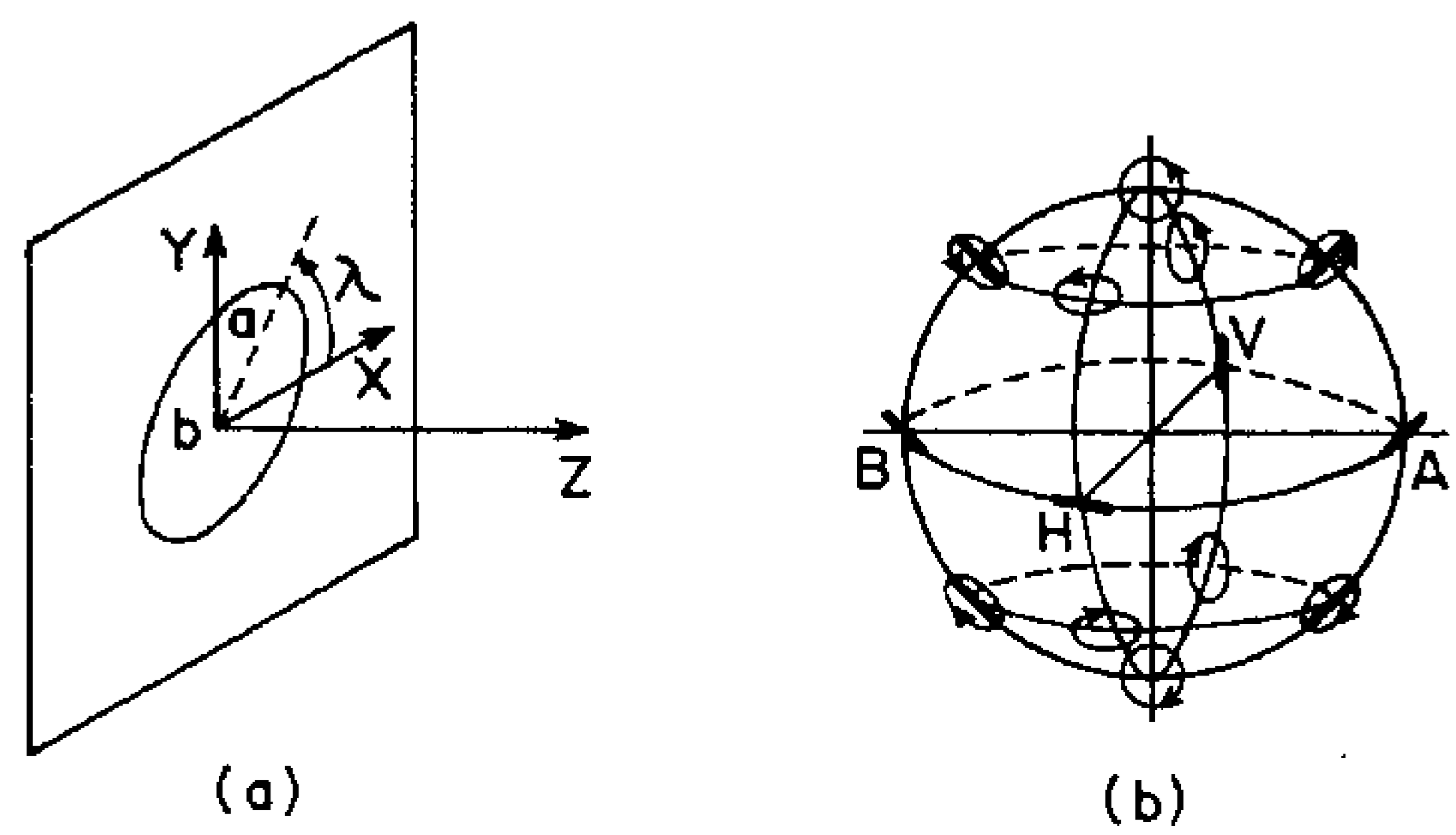

Figure 1(a). Elliptically polarised light. Two parameters are required to describe a general state: azimuth λ, $0 \leq \lambda \leq \pi$ and ellipticity b/a, $-1 \leq b/a \leq 1$ and $-\pi/4 \leq \omega = \tan^{-1} b/a \leq \pi/4$. **(b)** The Poincaré Sphere: A point (P) of longitude 2 λ anf latitude 2 ω represents an elliptic vibration of azimuth λ and ellipticity ω. The points on the equator represent plane polarised light; north pole and south pole represent left handed and right handed circular light.

equator HCVD represents linear vibrations, H and V being horizontal and vertical linear polarised light. Every other point on the sphere represents an elliptic vibration.

The Poincaré representation is ideally suited to deal with the case of the state of polarisation of a beam of light traversing an anisotropic medium. When there is no absorption the beam splits up into two orthogonal opposite states, (two linear in the case of a linear birefringence, two circular in the case of optical activity, two opposite (orthogonal) elliptic states in the case of birefringence and optical activity). The coherent superposition of these two orthogonal states gives points on a great circle joining them. Changing the relative phase is equivalent to a rotation of the sphere about the axis joining the two opposite states.

When S. Pancharatnam at the age of 19 came to Bangalore in 1953 to work as a research student, we are told that C. V. Raman gave him specimens of two minerals from his collection – iolite and amethystine quartz. Iolite is an absorbing biaxial crystal while amethyst is an absorbing, birefringent, optically active crystal. Raman told him that the optics of these crystals was little understood, and they were sure to exhibit many new unexpected phenomena. "The understanding of the propagation of light in these crystals will not only advance the science of crystal optics but

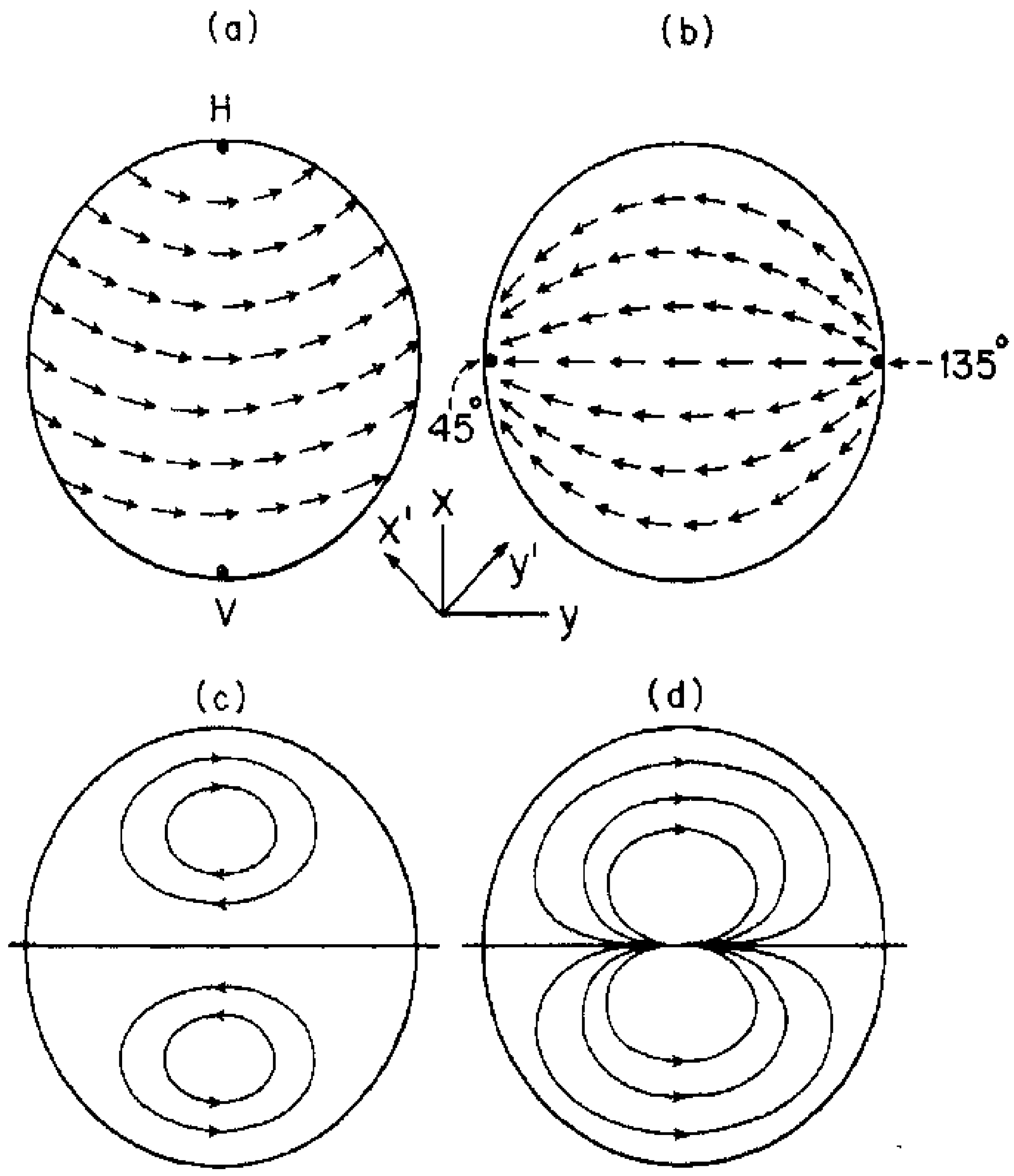

Figure 2. Motion of the state of polarisation on the Poincaré Sphere – four cases. **(a)** Linear birefringence, i.e. phase difference between H and V. The motion is a simple rotation (unitary i.e. intensity preserving transformation). **(b)** Linear dichroism i.e. differential absorption between two orthogonal linear polarisations, inclined at 45° and 135°. **(c)** Superposition of (a) and (b) with dichroism weaker than birefringence. Note that the two states which are preserved by the combined operation are non-orthogonal. **(d)** The singular case which results when the dichroism equals the birefringence and has principal axes inclined by 45°. Only one (circular) state is left unaltered and there is a dipole-like flow pattern resulting from the coalescence of the two normal modes.

Cases **(b)**, **(c)** and **(d)** all correspond to nonunitary transformations. When the two modes have unequal coefficients of absorption (as when the angle is not 45°) the motion of the representative point is a spiral expanding from the more absorbed state and converging on the less absorbed one.

will go a long way in revealing some aspects of the properties of light itself". Many years later Pancharatnam was to quote this to indicate the intuitive feel Raman had for physical problems.

A few months after he joined the Raman Research Institute we happened to talk of our research problems. I discussed with him my specific interest on the differential absorption of X-rays with wavelength in a crystal. He told me that the problem he was tackling was slightly different, i.e. differential absorption with polarisation and the changes in the states of polarisation due to this in a biaxial crystal (with and without optical activity). I knew little about this field but referred him to Perrin's paper* wherein he points out the advantages of using the Stokes' parameters for representing polarised light. I was particularly impressed with the simple relationship Perrin brought out between these parameters and the Poincaré Sphere which had been of so much use to me in my studies of magneto-optic rotation and birefringence. I could see that Pancharatnam was struck by the power of this method but he just asked me "why are these beautiful ideas kept out of standard treatises?". Almost immediately Pancharatnam put the Poincaré Sphere to use to design an achromatic quarter wave plate (1955)[5] for his experimental study on amethyst†.

The beauty of the Poincaré Sphere is that changes in polarisation as light travels through a crystal, acquire a simple geometrical significance. Thus if the vibration along X and that along Y have different refractive indices one can depict it as a rotation of the sphere. Figure (2a) shows this operation as a vector field. Pancharatnam[6] introduced the effect of differential absorption into the Poincaré Sphere. The vector field due to a differential absorption between X' and Y' (45° to X and Y) is as shown in figure 2b (note that it is not a rotation!) Pancharatnam discovered that by superposing the two examples given here, one could get remarkable effects (figure 2c & d). In crystal optics (or even in quantum mechanics) one is normally familiar only with orthogonal modes. But when absorption comes in, these modes are no more orthogonal and more dramatic effects can also occur. These modes may even coalesce to only one state (figure 2c) depending on the relative magnitudes of the absorption and birefringence. With the right balance between absorption and birefringence one can get a single circular state which is left unaltered, and all other states flow to it along a dipole-like pattern.

PANCHARATNAM AND THE GENERALISED THEORY OF INTERFERENCE

Pancharatnam next went on to consider the generalised theory of interference. This deals with the cases when light is in two distinct modes which can interfere *without the need for an analyser.* When these vibrations interfere, how does one define the phase difference between them? Pancharatnam gave an experimental definition: the two modes are in the same phase when the intensity of the superposed state reaches a maximum. The meaning of this can be pictured easily in the case of a circular and linear vibration. For example, a circular state is in phase with a linear state when the directions of the E vectors agree at the maximum amplitude. Likewise two linear states making an acute angle are in the same phase if they reach the maximum at the same time (figure 3b). (The cases of an ellipse and a linear vibration or two ellipses are slightly more difficult to visualise.)

Pancharatnam discovered that in traversing a closed circuit (i.e. say a state A goes to B and then to C, and finally returns to A, figure 3a), *there is a phase deficit which is half the spherical excess of the triangle ABC* (i.e. the solid angle subtended by ABC at the centre). When A is circular, B and C linear (acute to each other this deficit can easily be understood from figure 3b). This was a new result in optics. I know that when Pancharatnam discovered it he was first surprised (and elated). He also felt that there was something of much deeper significance to physics in this. However, he dealt with it in a most pragmatic manner as a direct and simple result from classical optics. He used it in his own experiments on the interference of polarised light traversing absorbing biaxial crystals (without and with optical activity), and showed that this extra phase term does exist

* My knowledge of Stokes' parameters came from reading Francis Perrin[3,4], who was perhaps amongst the earliest to call them by that name. The parameters M, C and S are the rectangular coordinates of the point P on the Poincaré Sphere.

† Of this work, G. W. Series[7] has said "The achromatic quarter wave plate and circular polariser may well find use in contemporary laser technology". Commercial versions of these are made in Germany for use in optical astronomy.

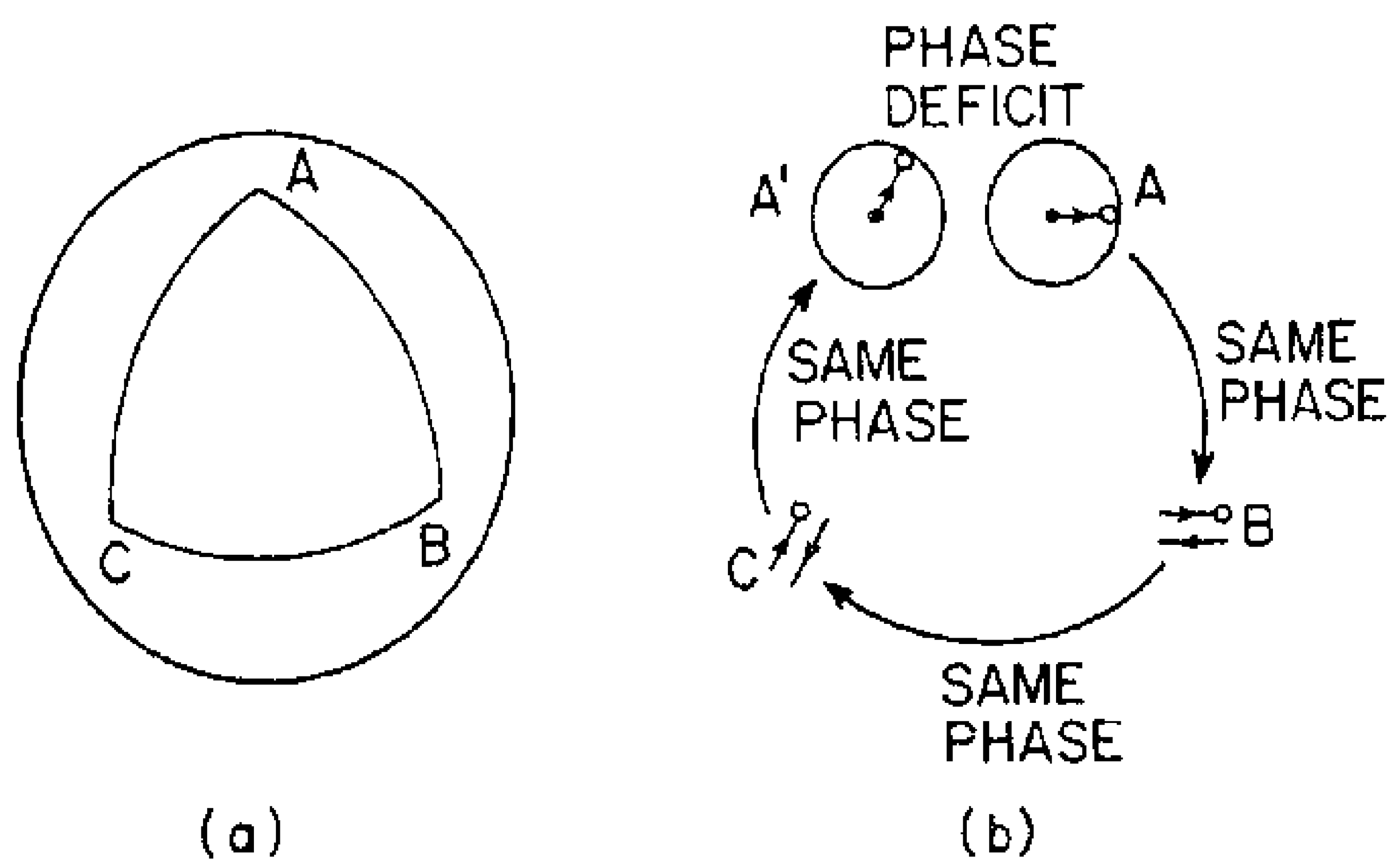

Figure 3. The Pancharatnam Phase. (a) When a state A is analysed along B, B along C, and then C along A, there is a phase change equal to half the solid angle subtended by the spherical triangle ABC at the centre. (b) A simple illustration of the Pancharatnam Phase when A represents circular polarisation, B and C linear polarisations at some angle to each other. The open circles on the path of the electric vector represent a given phase of the oscillation.

with the calculated magnitude, and is essential to explain the observed phenomena (Pancharatnam[8])*.

The Berry's Phase and the Pancharatnam Phase

I religiously attend the Journal Club meetings of the Raman Research Institute (and am also known to snooze there). At one of these meetings, Berry's Phase[9] was discussed by two speakers (Joseph Samuel and Chandrakant Shukre). When I heard these talks I got the (*deja vu*) feeling that all this had been said thirty years before in another context by Pancharatnam. I mentioned to Rajaram Nityananda and V. Radhakrishnan that Berry's Phase and the spherical excess theorem Pancharatnam talked of in his generalised theory of interference appeared to me to be identical. I also felt that the adiabaticity condition of Berry may not be an essential one since it is not essential in Pancharatnam's derivation.

Once he had overcome some initial scepticism, it did not take Rajaram long to verify the relationship between Berry's Phase and the solid angle theorem and this was written up in *Current Science* [Ramaseshan and Nityananda[11]].

Thanks to Berry's own efforts,[12] the connection between Pancharatnam's optical work in the fifties and the more recent flood of papers on "Berry's Phase" is widely recognised among workers in this field†.

* Since then many elegant theorems on the Poincaré Sphere have been proved. One such is that it is impossible to define a continuous phase convention which is globally valid over the entire Poincaré Sphere [Nityananda[10]]. For example, Pancharatnam's convention breaks down at the antipode of the state used as phase reference, since the interference of two such orthogonal states gives the same intensity regardless of the phase.

† It started with Berry's detailed paper entitled "The adiabatic phase and Pancharatnam's Phase for polarised light (1987). I had the pleasure of meeting Prof. Michael Berry again in 1987 when he came to Bangalore and of presenting him with my personal copy of Pancharatnam's "Collected Works".

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