WAVES and SYMMETRY
Raman Centenary Symposium,
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Beyond Rainbows

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1. INTRODUCTION

A superstitious person might say that my giving this lecture was predestined. Although I never met Raman, whom we are honouring at this meeting, my whole career in physics has centred on his themes of waves and symmetry. As an ignorant youngster my first piece of independent research was on the diffraction of light by ultrasound in a liquid: the light *waves* were influenced by the *symmetry* of the periodic environment provided by the *wave* of ultrasound. My supervisor told me to read a paper about the theory of the phenomenon. It was written by two people I had never heard of – C. V. Raman and N. S. Nath – from an exotic place – Bangalore – and published in an obscure journal – Proceedings of the Indian Academy of Sciences. It was so difficult to understand; this was not mathematics as I had learned it at school, progressing in logical steps: it was physicists' mathematics – the kind I do now – where the argument took great leaps, supported by intuitions – mostly geometric – which I did not share.

To celebrate the occasion I have chosen not to speak about my latest researches – although these are always one's favourite children – but to describe some earlier work in the science of light, that is optics, because that was what Raman studied all his life. I hope he would have liked this lecture, because much of it deals with light as it can be seen with the bare eyes – in nature or with the simplest apparatus – and that was his style too.

Our branch of optics is the *focusing of light* as it occurs in nature. It has the unusual feature that its modern development (over the last fifteen years or so) was almost entirely driven by a discovery in mathematics, in geometry to be precise. This is the so-called *catastrophe theory* of René Thom and Vladimir Arnold. The emphasis on mathematics is something Raman might not have liked. I suspect he had a healthy distrust of theorists like me, believing us to be too easily bemused by mathematics. But in what I'll describe here mathematics really has proved its value, opening doors to previously unexplored corners of a very old subject, leading us to discover many new things.

2. STABLE FOCUSING

As a way to introduce the exotic mathematics in a non-technical way, we begin with something familiar: the reflection of sunlight on wavy water (figure 1). Each brilliant point of the sparkling image comes from a place where the water surface has the right slope to reflect light into the eye (figure 2a). In mathematical language, rays of light define a *map* between the surface and the eye, and this map is many-valued. The paths that are rays are those where the angles of incidence and reflection are the same. Another way to say this is through Fermat's principle: in a graph (figure 2b) on which the travel time of paths from the sun to the eye is plotted against the place where the path hits the surface, rays are the *critical points,* that is places where the graph has zero slope.

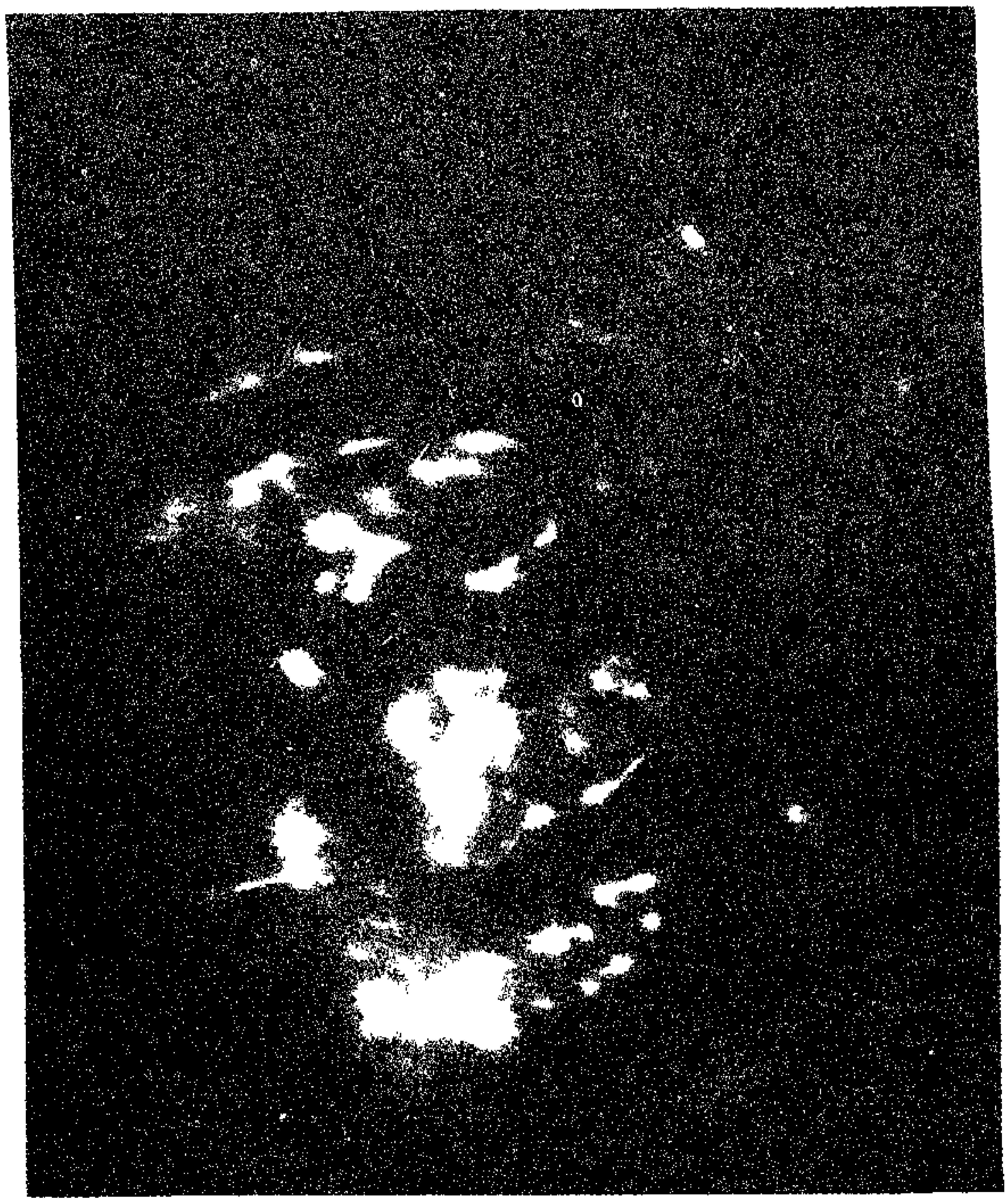

Figure 1.

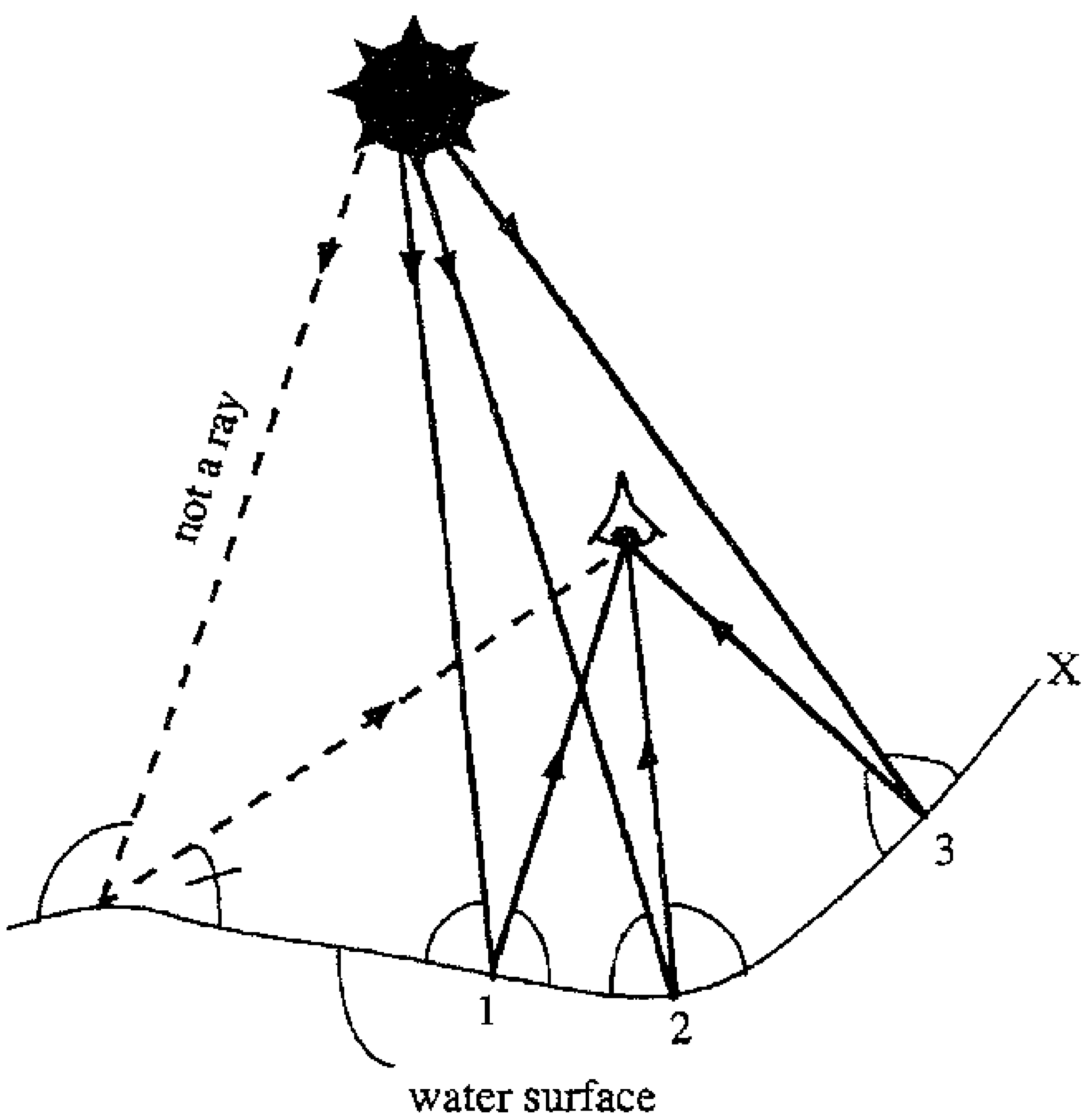

Figure 2a

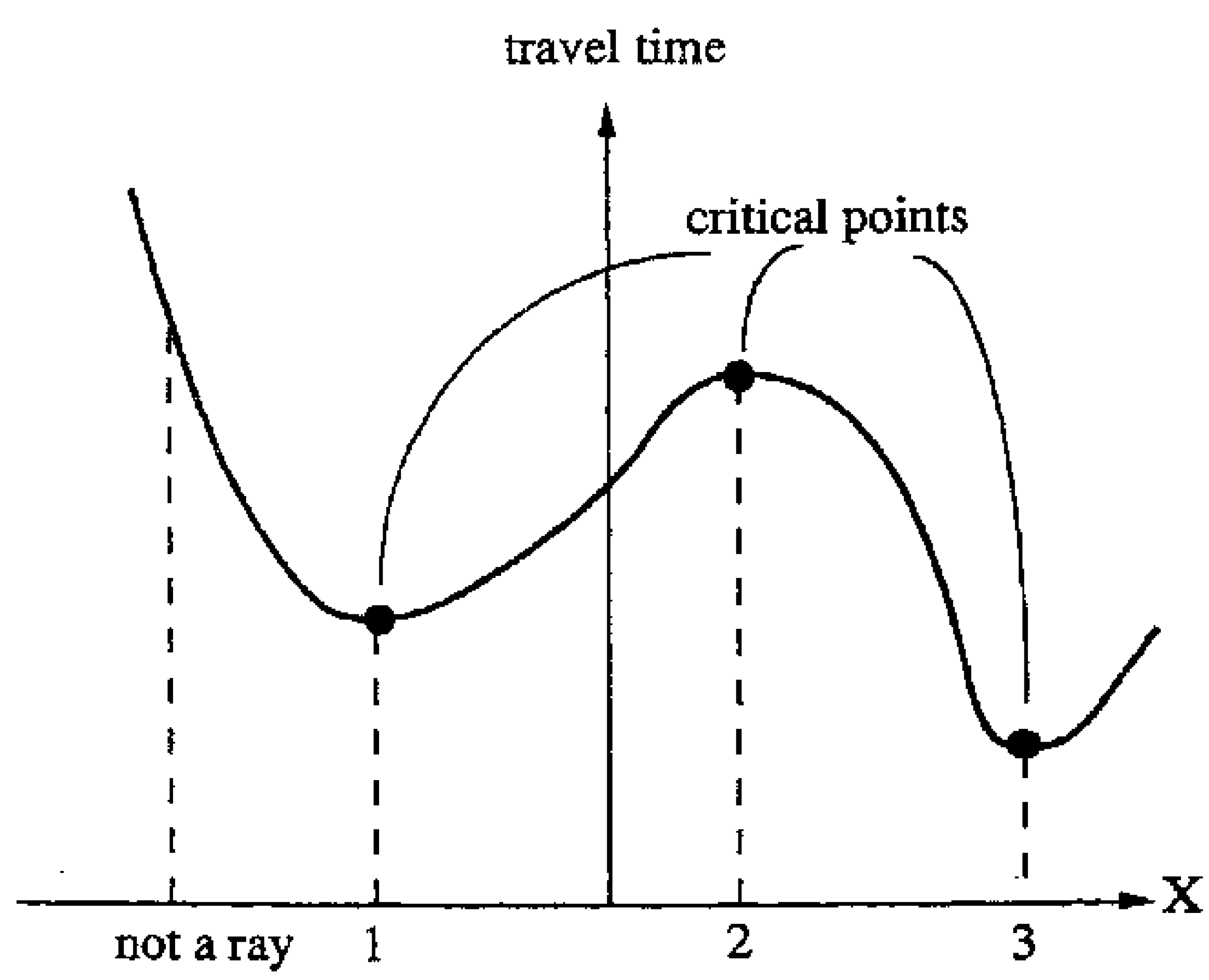

Figure 2b

Catastrophe theory deals with the critical points of maps, so with rays we are in the right mathematical world to apply it.

Now let time pass. Because the water surface changes, the brilliant points move about. They can collide and annihilate, or (the reverse) be born in pairs. It is the rapid repetition of such events – called "twinkles" – that gives the image its sparkling appearance. Mathematically, a twinkle is a coalescence of critical points, and it is precisely such coalescences that catastrope theory is about. The physical situation at a twinkle is that the water surface has not only the right slope to reflect light into the eye but also the right curvature to *focus* it there. Therefore catastrophe theory is about *focusing*.

To see this more clearly, it helps to think of the pattern of rays at a fixed time (figure 3), rather than the rays through a given point (the eye) at different times. Space is partitioned into regions illuminated by different numbers of rays (e.g. A and B in figure 3). The boundaries of these regions are surfaces on which focusing occurs; in two dimensions (as in figure 3) the boundaries are focal curves. These generalized focal surfaces and curves are called *caustics* (from the Greek word for burning, referring to the focusing patterns behind magnifying glasses in the sun).

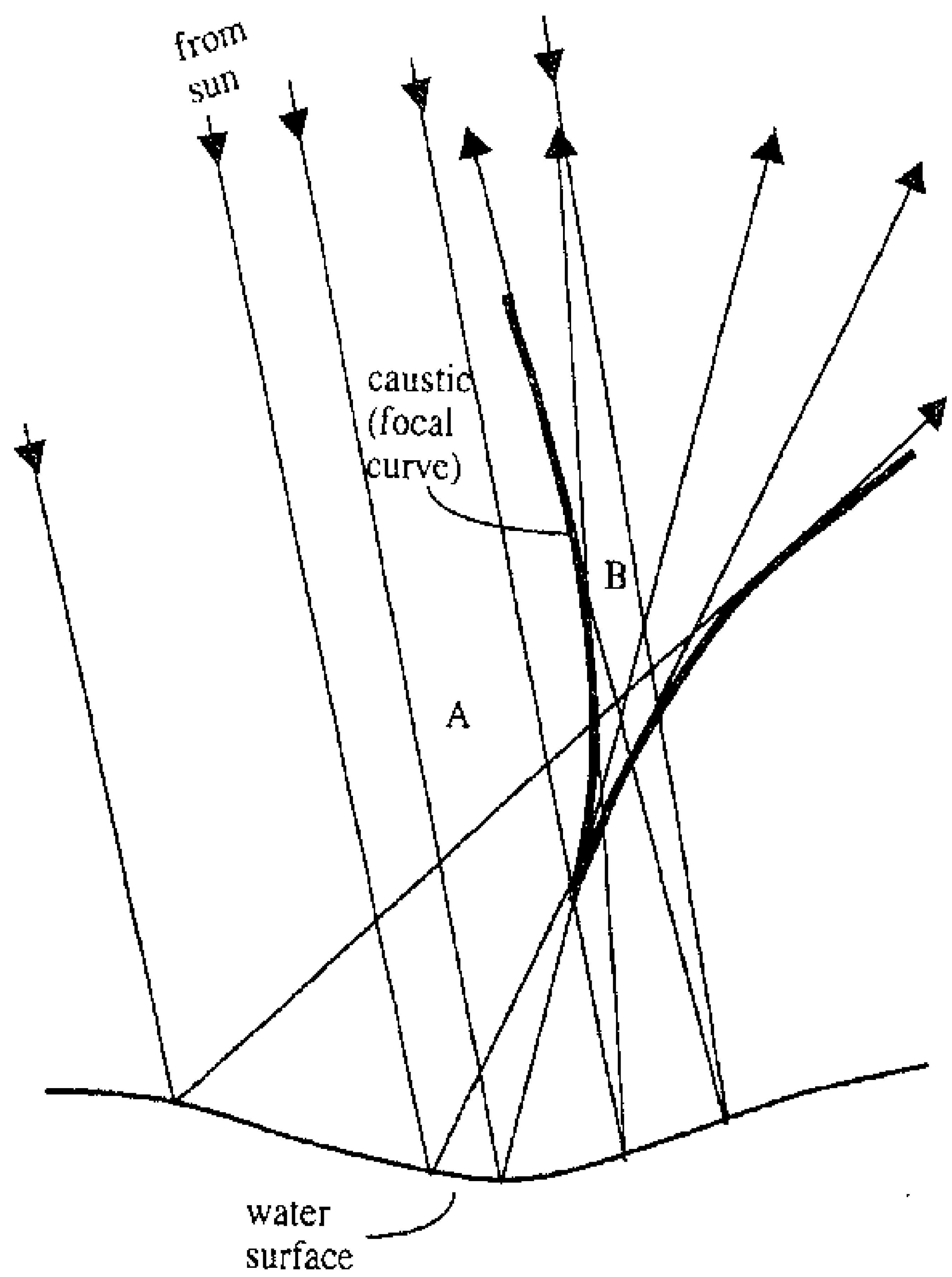

Figure 3.

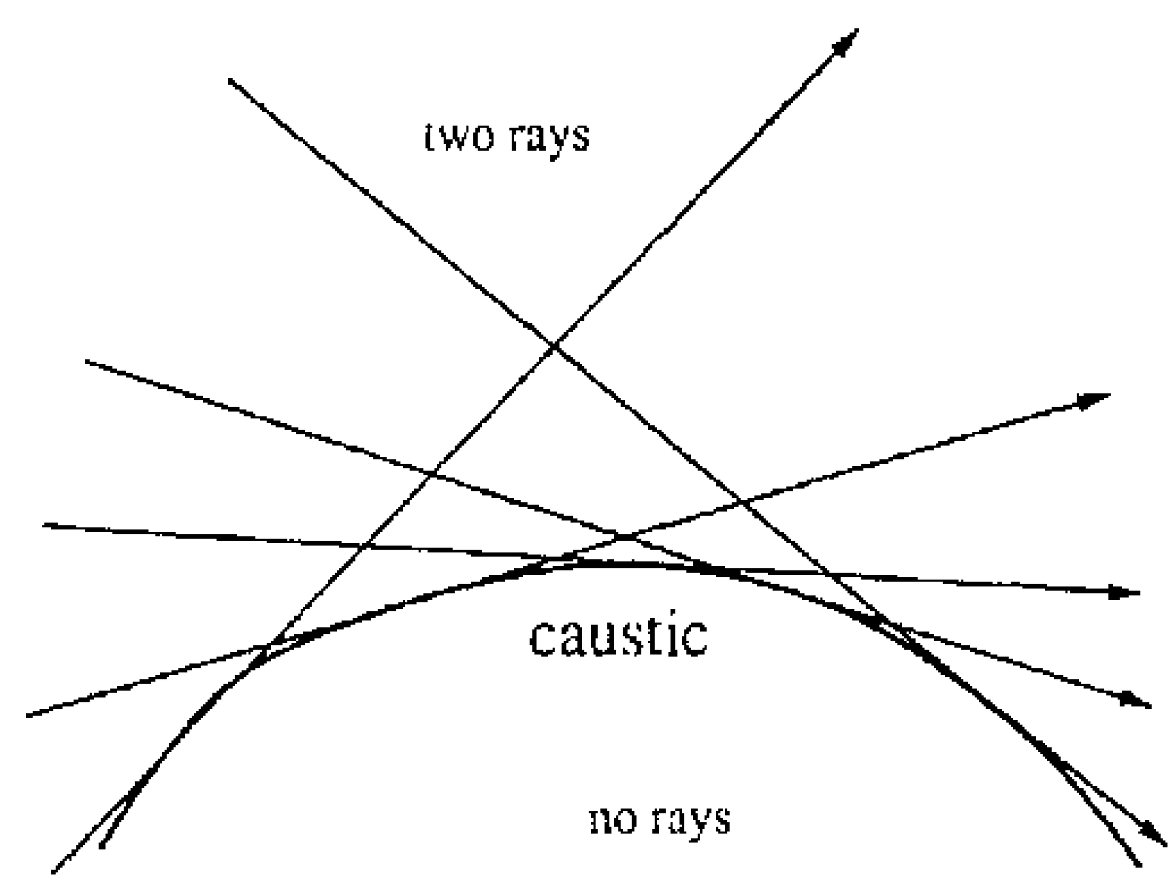

Figure 4.

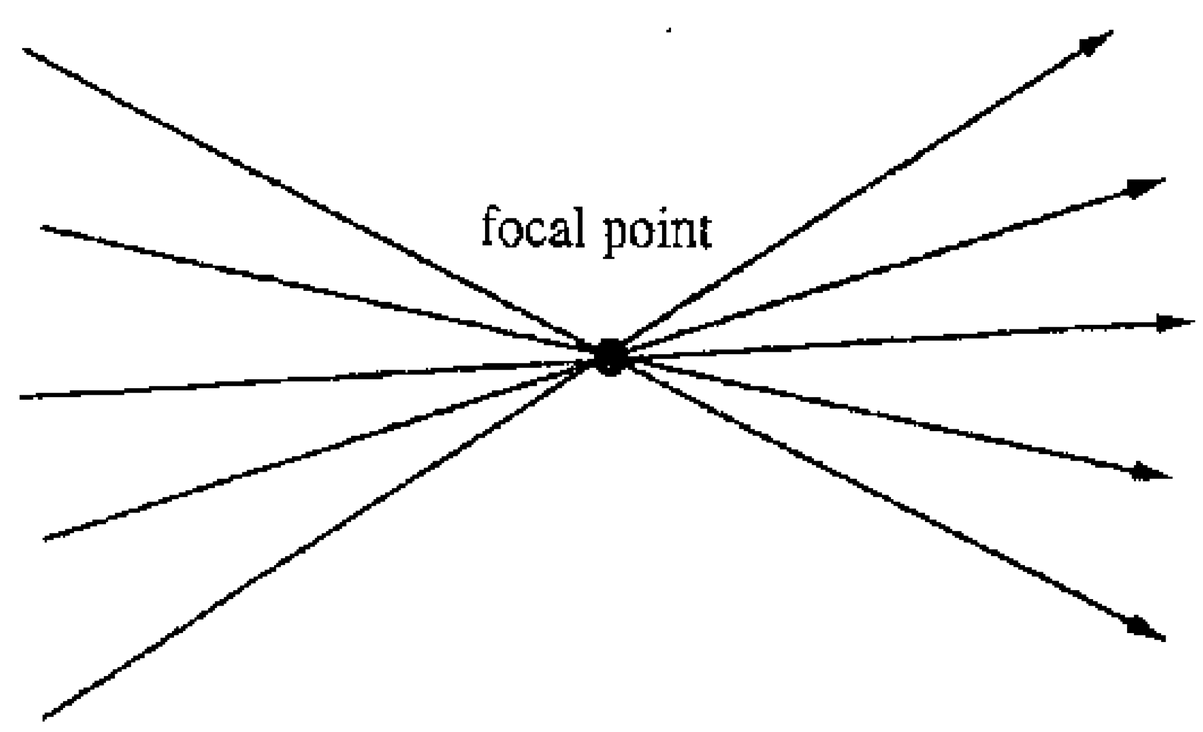

Figure 5.

Across a caustic, both the light intensity and the number of rays change discontinuously. The simplest caustic is shown in figure 4. Because energy travels along rays, it is clear that the light intensity is largest in caustics: caustics are bright places, dominating optical images. Caustics are mathematical catastrophes. (The dramatic terminology is more appropriate for another application of the mathematics, to the discontinuous collapse of elastic structures – e.g. bridges – when the loads are continuously increased.)

What catastrophe theory provides is a classification – a catalogue – of *stable* caustics. The innocent word 'stable' denotes an important concept. It means that the caustics to be considered are those whose form changes smoothly under a small change of circumstances. The point focus through which all rays pass (figure 5), although familiar from elementary optics, is unstable – all the artistry of the lensmaker is needed to produce it, and it explodes into 'aberrations' upon the slightest alteration of object position, lens orientation, etc.

3. THE CATALOGUE OF CAUSTICS

Caustics are classified by their *codimension,* that is the number of dimensions that must be explored to find them. Thus a point on a line, a curve in the plane, and a surface in space (figure 6) all have codimension one. This simplest catastrophe – called the *fold* – has an unimpressively simple geometry, but describes an impressive natural phenomenon, namely the rainbow.

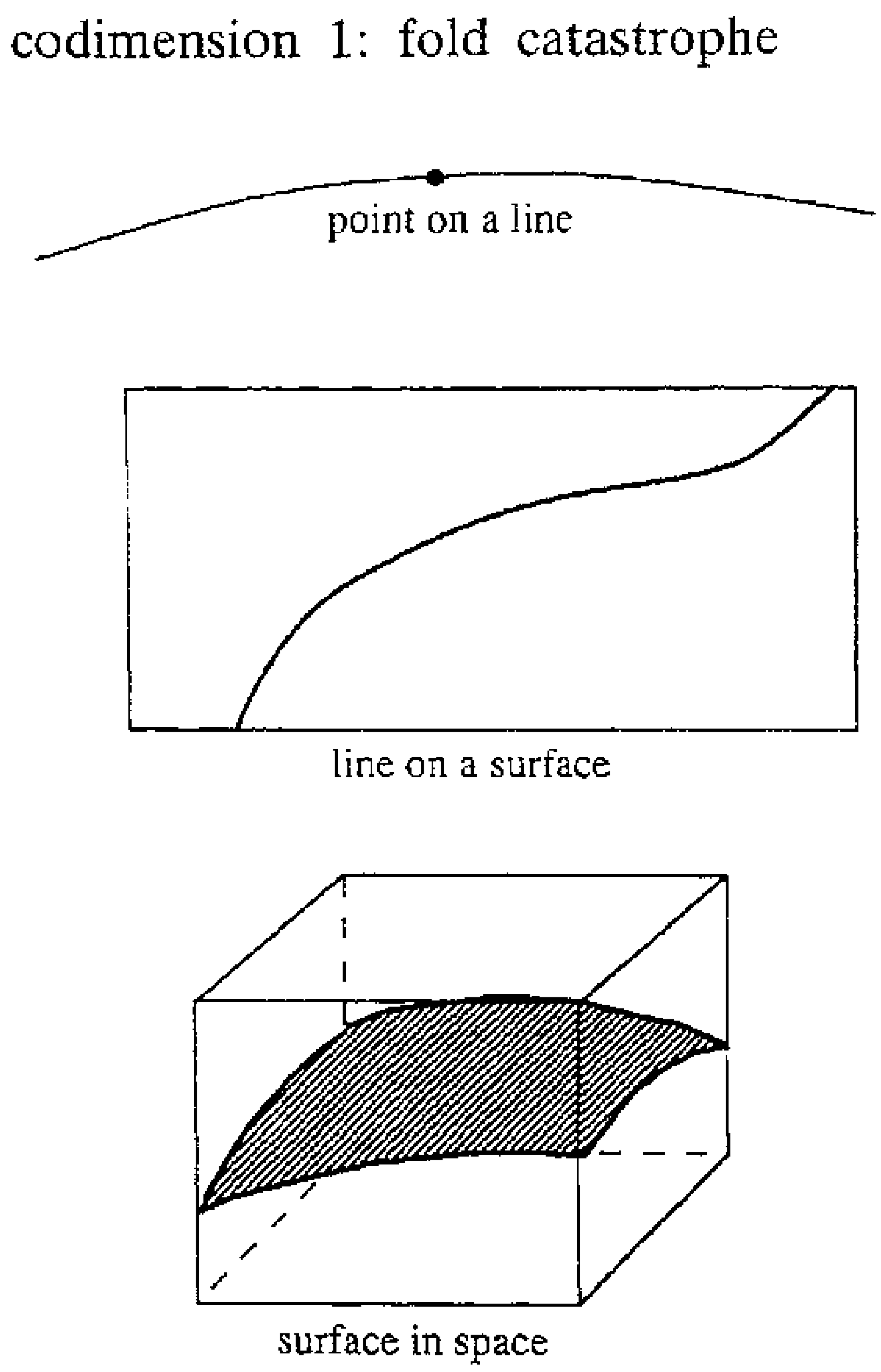

Figure 6.

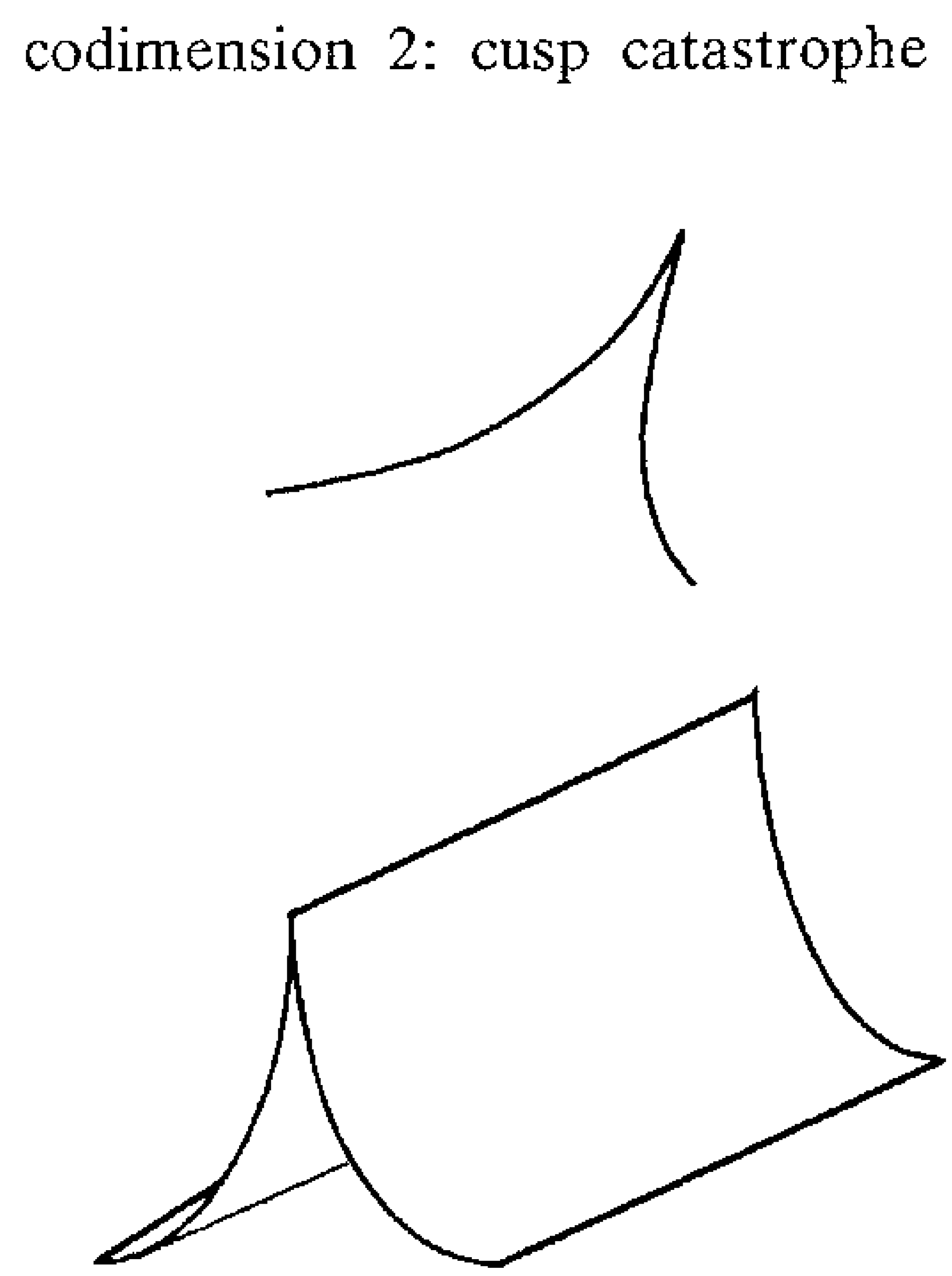

Figure 7.

As Francis Bacon wrote, "The rainbow is made in the sky out of a dripping cloud". Sunrays strike a raindrop and emerge after two refractions and one (internal) reflection. A graph of the direction of emergence against the latitude of incidence (first plotted by Descartes in 1638) reveals a curve with a minimum – a folded curve, hence the name. Thus in some directions two rays emerge, and in others none. The boundary is an angular caustic (at about 138° to the forward direction): the light emerging from each drop is particularly bright on a cone, and we see, brightly lit, all those drops on whose cones our eyes lie. These drops themselves lie on a cone with our eyes at the vertex, so that we see it as a circular caustic in 'skyspace'.

Next on the list is a catastrophe with codimension two: the *cusp.* A cusp (figure 7) is a point in the plane or a line in space, at which two folds (curves or surfaces) meet and touch. The fact that this is the only such catastrophe is a powerful result of the mathematics, telling us for example that certain other caustics are unstable – in the plane, for example, a fold line coming to an end, or an isolated focal point, or two fold lines meeting at a finite-angled corner.

Cusped caustics are easy to see, especially for those who wear glasses. Raindrops form 'lenses on the lenses', and because the glass is usually not perfectly clean the droplet lenses are irregular. Their foci (figure 8), forming stable caustic curves (rather than points) on the retina, can be seen by looking through such rained-on glasses at point sources (e.g. street lights) at night. In the laboratory the cusps can be produced very easily (figure 9) by a broadened laser beam refracted through a water droplet sprinkled on a dusty glass plate and then onto a distant screen.

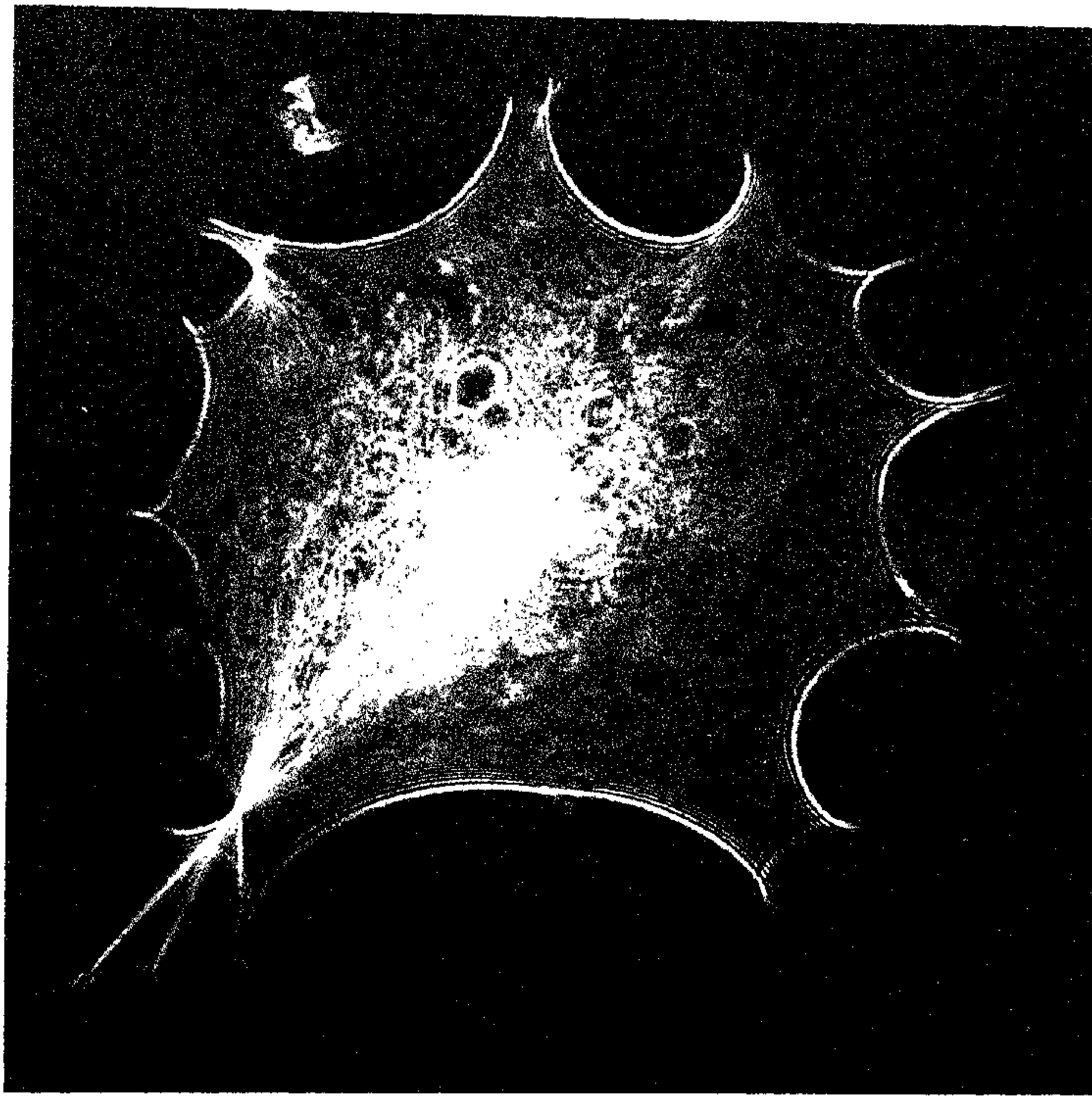

Figure 8.

Codimension-three catastrophes are points in space; there are three of them. In the *swallowtail* (figure 10), two cusped curves meet at the singular point. On a screen the swallowtail can be identified by its characteristic section, which is a self-crossing curve with two cusps. Swallowtails are often seen in water-droplet caustics (figure 11). It is hard to see the full three-dimensional swallowtail. One (not very effective) technique is to blow smoke into the focus and view the caustic surface by diffuse scattering. Another (better) is to synthesize the surface by moving the screen.

In the *elliptic umbilic* (figure 12) three cusp lines touch. The characteristic section of this catastrophe is a three-cusped 'triangle'. This can be seen (figure 13) in a laser beam refracted through irregular bathroom-window glass (the best glass has smooth irregularities about a millimetre across). A screen through the singular point would show an isolated focal point, which as already explained is not a codimension-two catastrophe and so is unstable. Now we can demonstrate the instability simply by moving the screen: instantly the focus explodes into a 'triangle' containing cusps, which *are* stable.

In the *hyperbolic umbilic* (figure 14) a smooth outer surface intersects a cusped inner surface. The characteristic section is a cusped curve within a smooth curve. Three of these can be seen on the bathroom-window caustic of figure 13, at each of the 'triangle' corners. A screen through the singular point would show a curve with a corner, which like an isolated focus is unstable. Again the instability can be demonstrated by moving the screen: the corner explodes into the stable 'cusped-curve-within-a-smooth-curve'.

The classification of catastrophes continues far beyond codimension three. Ever more complicated

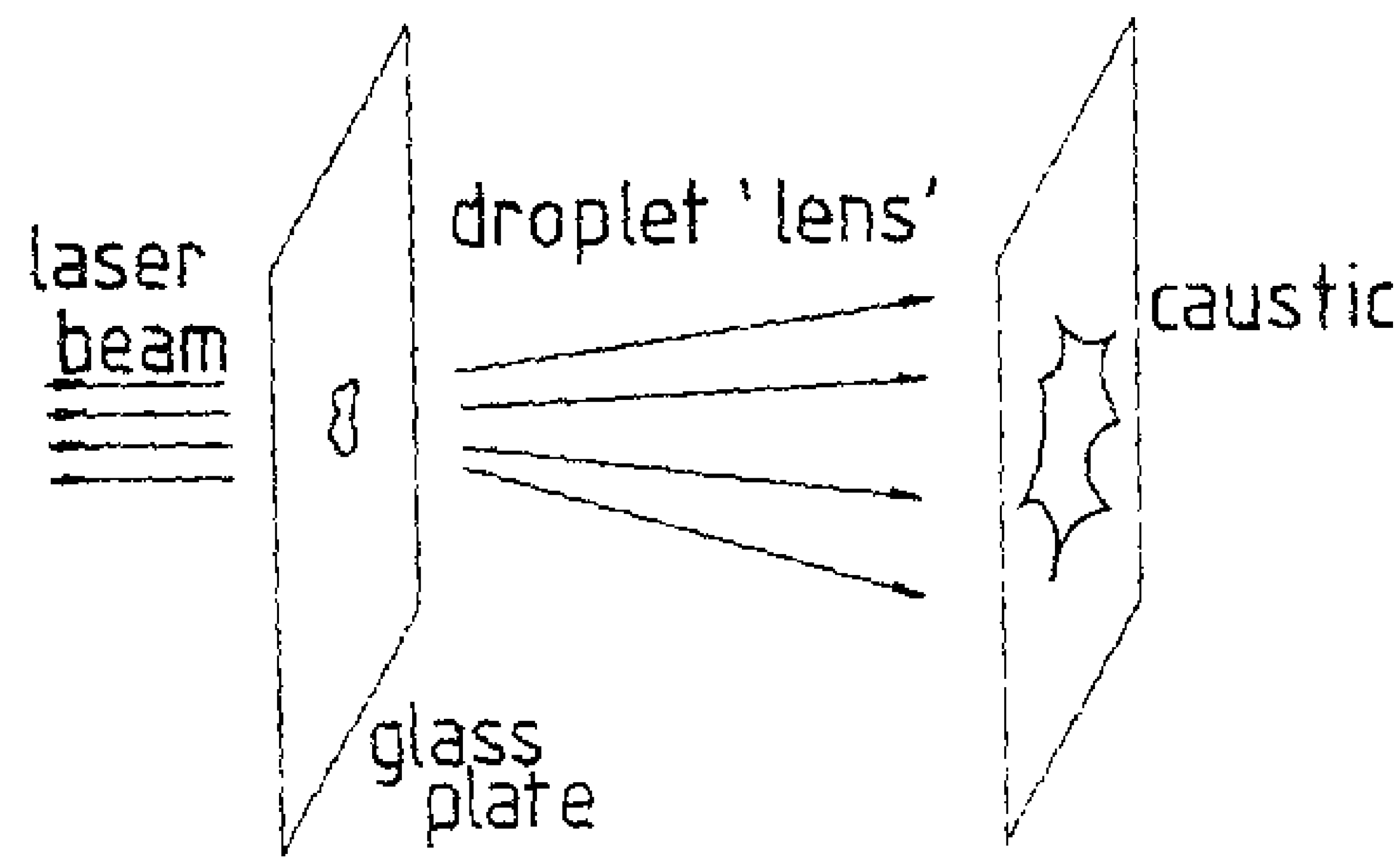

Figure 9.

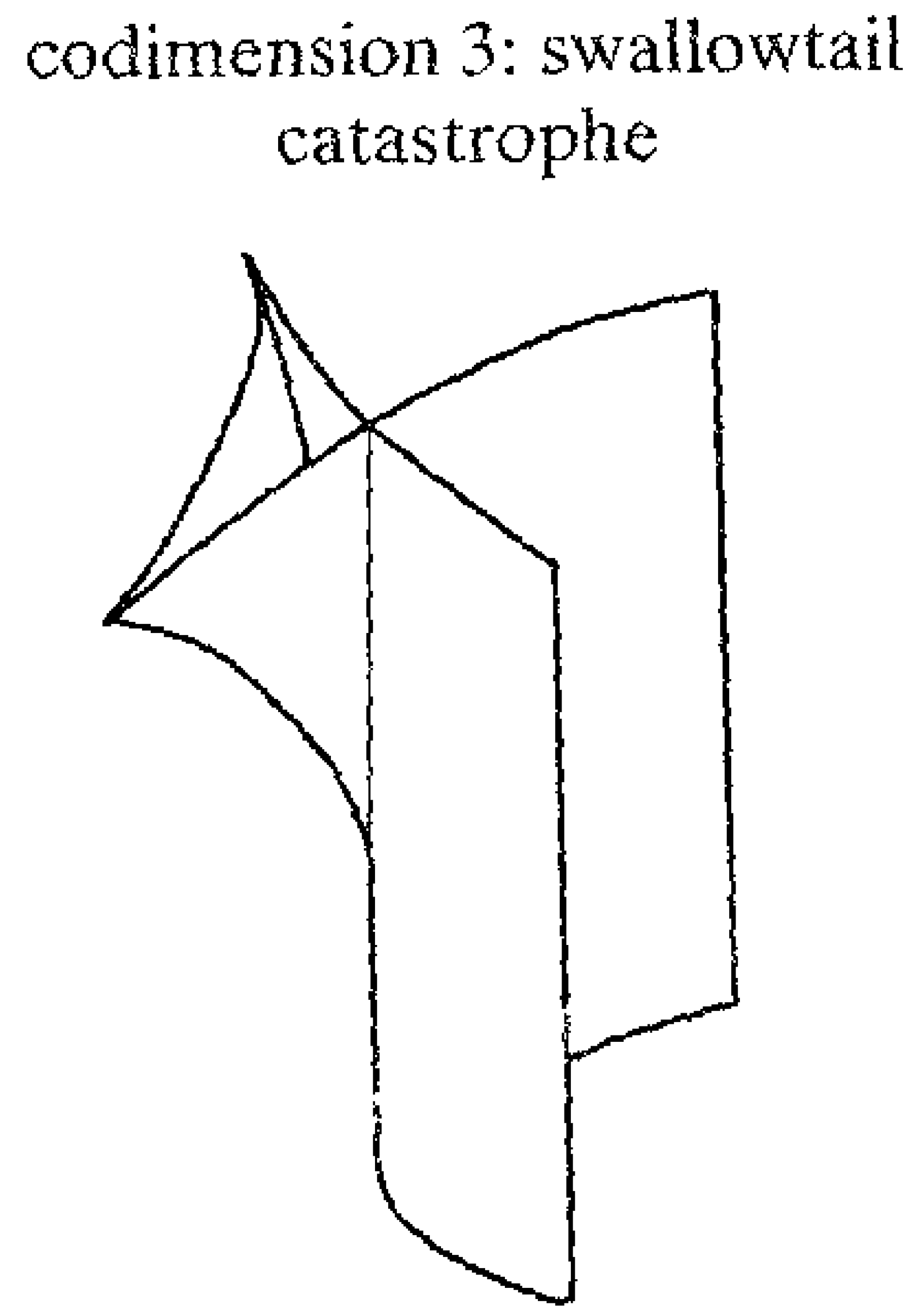

Figure 10.

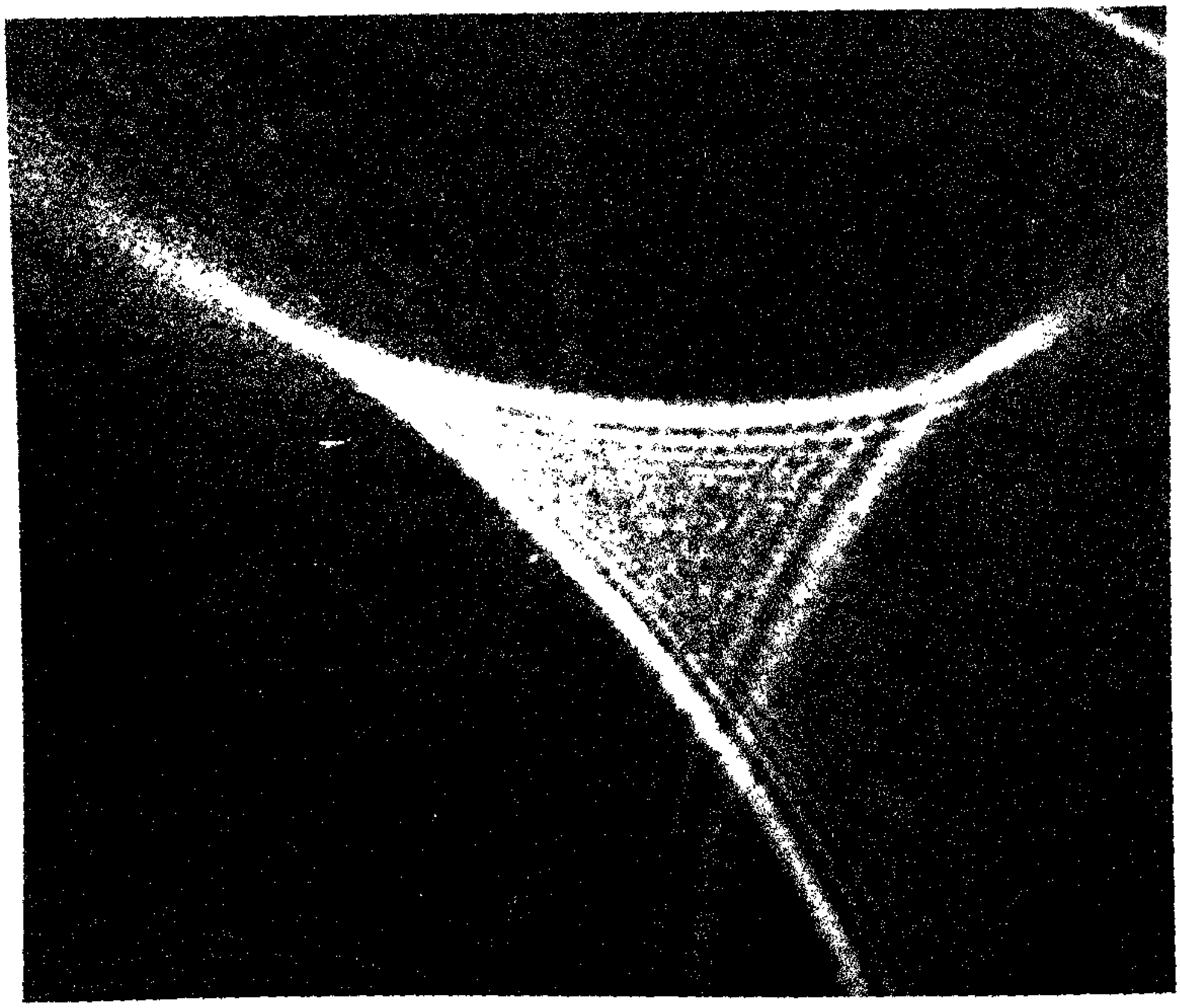

Figure 11.

codimension 3: elliptic umbilic

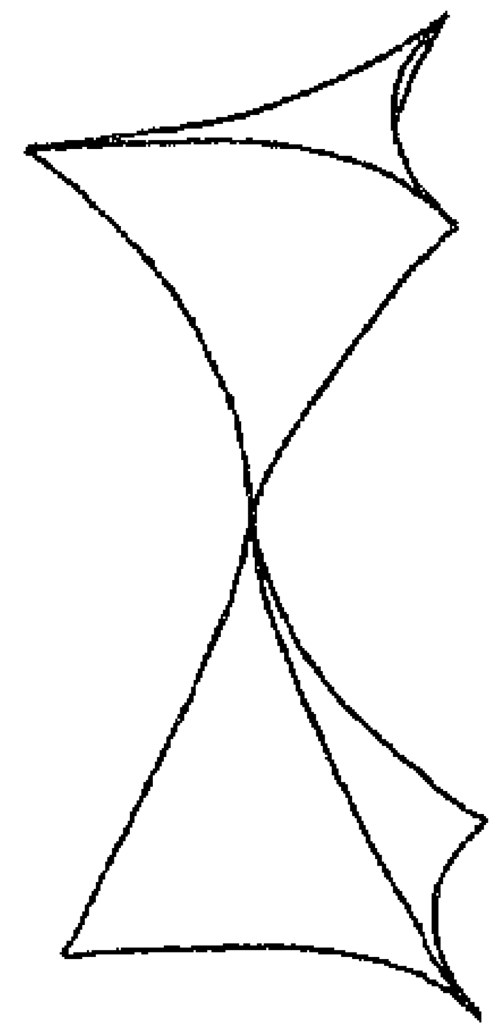

Figure 12.

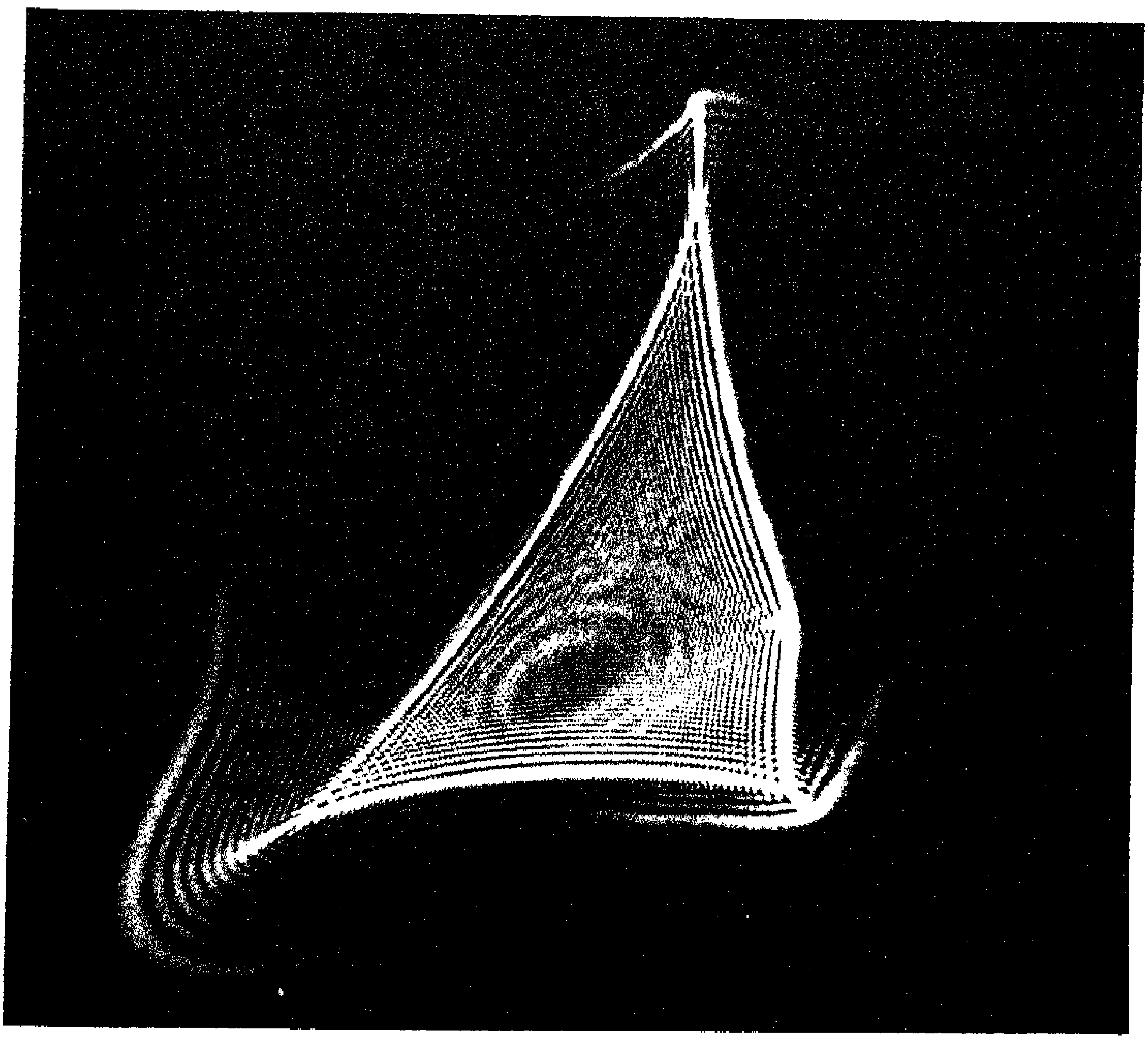

Figure 13.

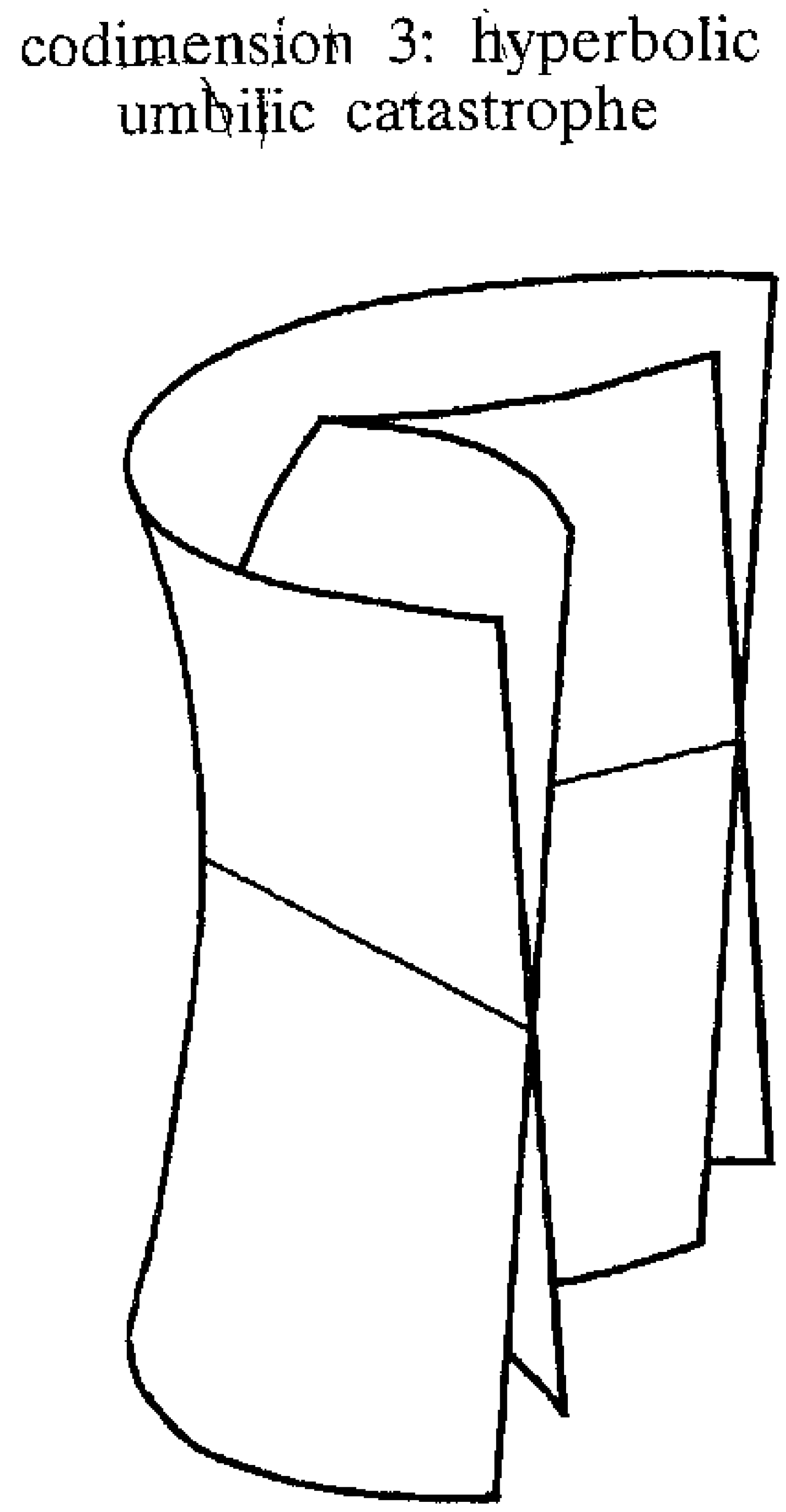

Figure 14.

forms proliferate, each containing lower catastrophes as structural elements. The catalogue has not yet been completed.

This sort of optics has the flavour of botany rather than physics. Mathematics tells us that stable caustics can have only certain geometries, and we search the jungle of images trying to find these 'exotic species'. And what different 'specimens' of a given catastrophe (such as the several hyperbolic umbilic caustics in figure 14) have in common is not the geometric identity of crystals or Ford cars but the topological similarity of two roses.

It is not fanciful to regard the optical catastrophes as 'atoms of form', playing a role in the physics of light that is closely analogous to that of real atoms in the physics of matter. The atoms occupy an intermediate regime – a mesoscale – between optics on the micro- and macro-scales, as summarized in the following table.

OPTICS		MATTER
Many connected catastrophes – caustic networks	MACROSCALE	Many atoms linked together – molecules, polymers, solids
Catastrophes	MESOSCALE	atoms
Wave (interference) patterns, decorating caustics with fine detail	MICROSCALE	subatomic structure – electrons, nuclei, quarks

4. Into the microscale: interference catastrophes

Although catastrophe theory is modern geometry, the physics to which I have been applying it is long superseded. For nearly two centuries we have known that the description of light in terms of rays ('geometrical optics') is an approximation to a much better theory, in which light is regarded as waves. The ray approximation is good on large scales (for example in the design of some optical instruments) but fails to reproduce the wave interference phenomena that occur when waves cross. Such interference – clearly visible on some of the images already presented – constitutes the optical microworld, the 'subatomic' level.

It might seem that on the deeper level of wave physics, catastrophe theory might be superseded like the geometrical optics it describes. But this is not the case. It turns out, remarkably, that each variety of stable caustic – each catastrophe – is decorated on fine scales with its own pattern of wave interference. Moreover, the catastrophe mathematics gives precise and detailed descriptions of the intricate intensity variations across the patterns. These descriptions are themselves short-wave approximations, but very refined ones.

First in the hierarchy is interference associated with a fold catastrophe. This pattern, together with the theoretical intensity graph, is shown in figure 15. This mathematics was discovered in 1838 – long before catastrophe theory – by Airy, who of course was not aware that he was describing the first pattern in a hierarchy.

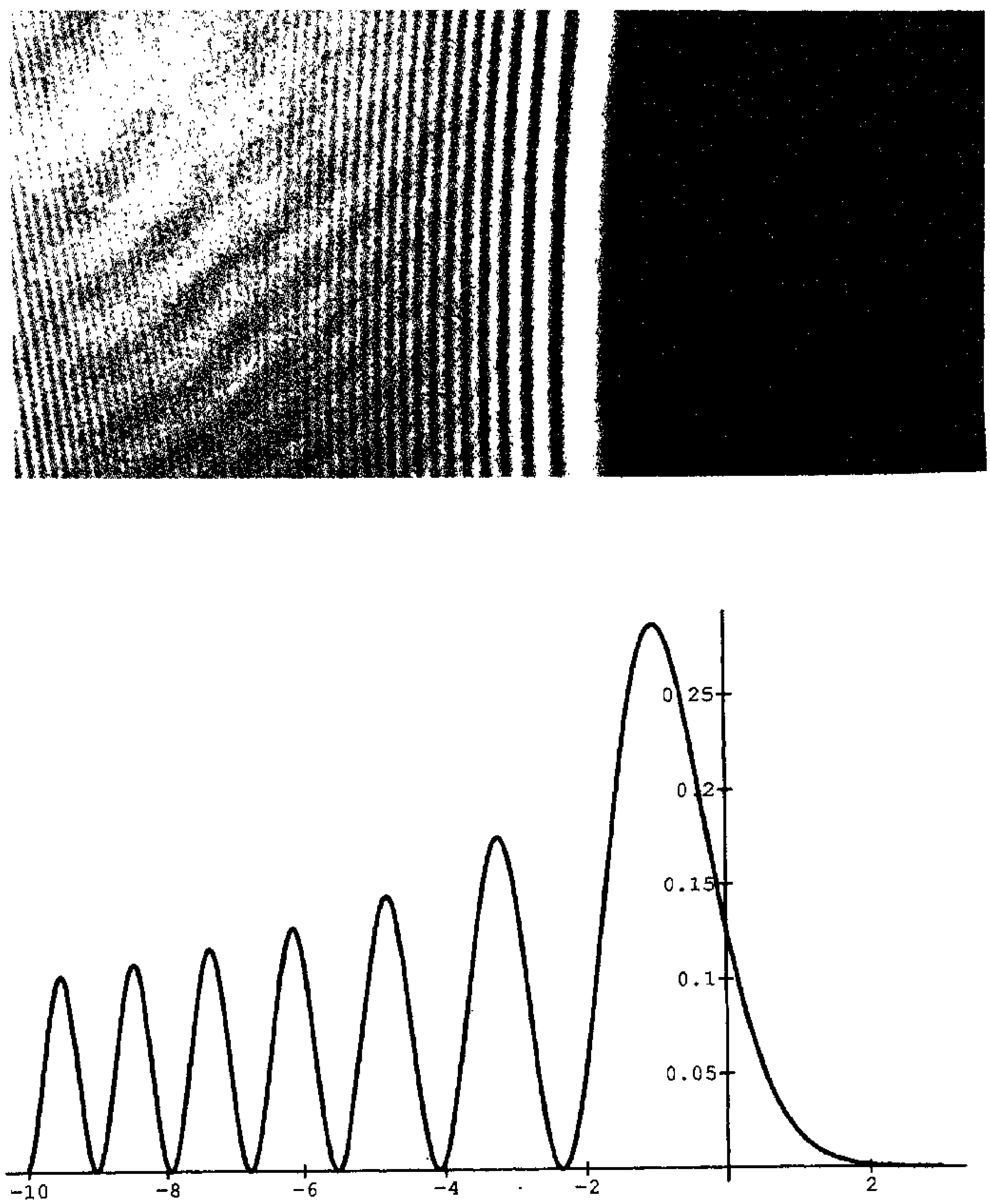

Figure 15.

Next is interference associated with a cusp. This caustic requires at least two dimensions for its display, and so its wave pattern is best displayed as an intensity map in the plane (figure 16a). Magnification (figure 16b) reveals a mass of minute detail in which the original geometrical cusp can barely be discerned, yet even the finest scales are reproduced in the theoretical contour map (figure 16c). The mathematics of this pattern was discovered in 1946 – again before catastrophe theory – by Pearcey, who also was not aware that his pattern was the second member of a hierarchy.

The three interference catastrophes of codimension three are patterns in space, which can be illustrated section by section on a screen. I show here one section from each of the elliptic (figure 17) and hyperbolic (figure 18) umbilics (generated by water-droplet lenses) together with computer simulations of the theoretical predictions. Obviously the agreement is very good.

Catastrophe theory's penetration into wave physics goes deeper than providing a series of intricate patterns. It also gives precise quantitative descriptions of two phenomena associated with the short-wave limit: as the wavelength gets smaller (in relation to the size of refracting and reflecting objects) the intensity on a caustic gets larger and the scale of interference detail gets smaller.

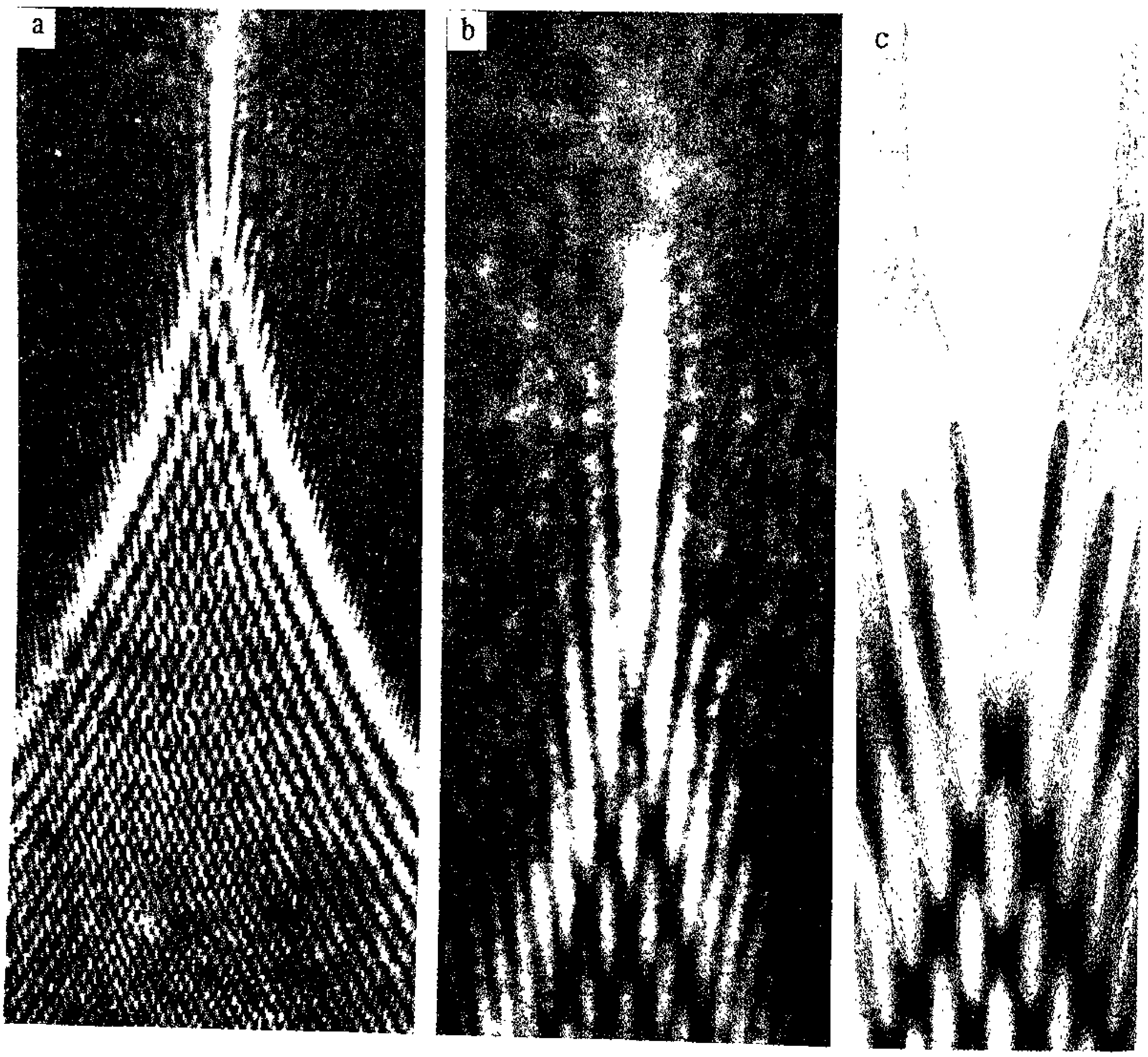

Figure 16.

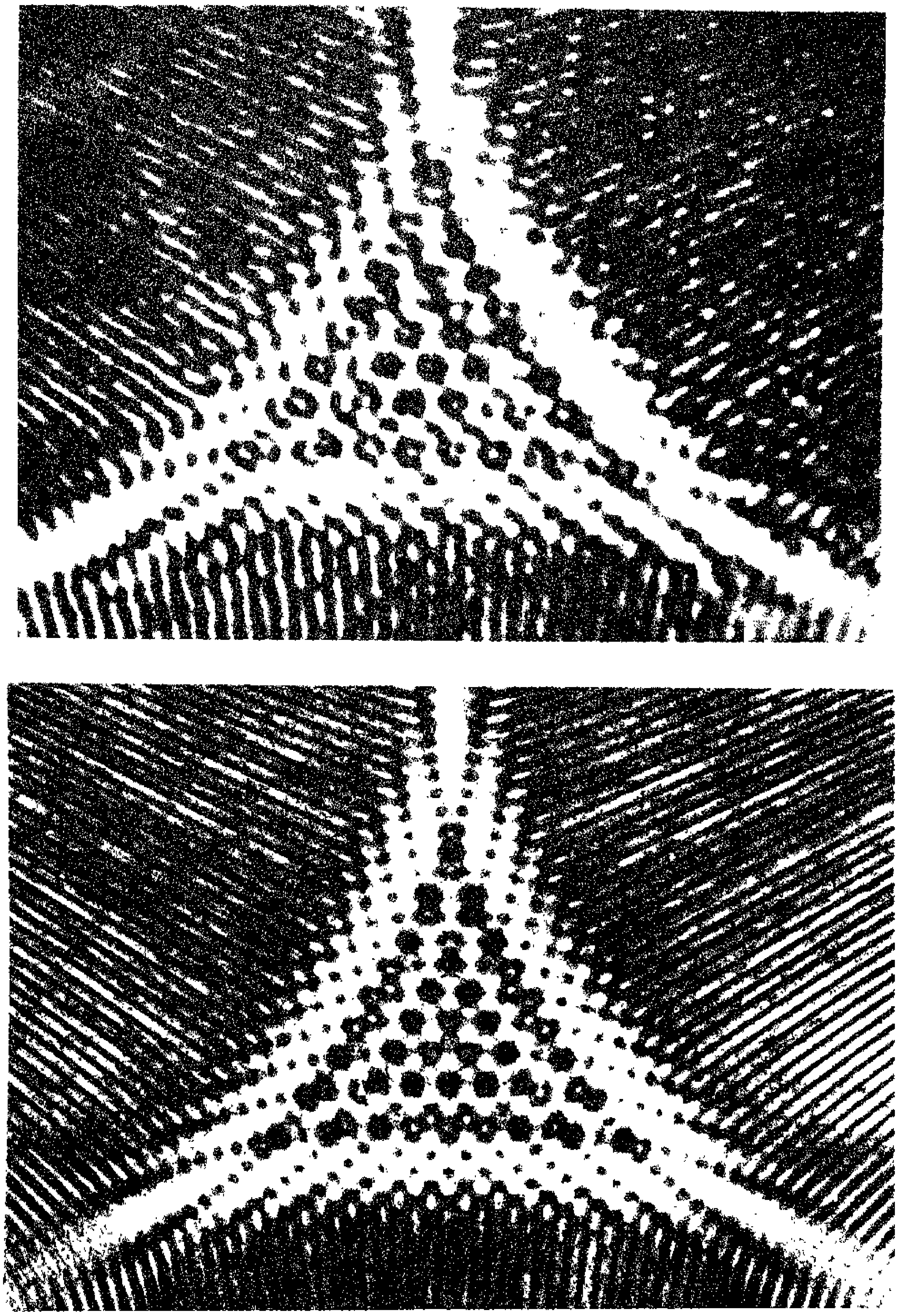

Figure 17.

5. UP TO THE MACROSCALE: CAUSTIC NETWORKS

The most familiar examples of caustics with many connected catastrophes are the networks formed by sunlight focused by the wavy water surface on the bottom of a swimming pool (figure 19a and b, page 1191). Usually these networks are very complicated and it is hard to discern even the 'atomic' catastrophes – let alone the 'subatomic' interference detail. The reason is that the caustics are blurred by the finite size of the sun's disc and the rapid motion of the water. Indeed, if you ask a child to draw one of these networks, the pattern that often results (figure 20) can be shown to be geometrically impossible if taken literally as a caustic. What has happened is that poor resolution makes several caustic curves appear as one.

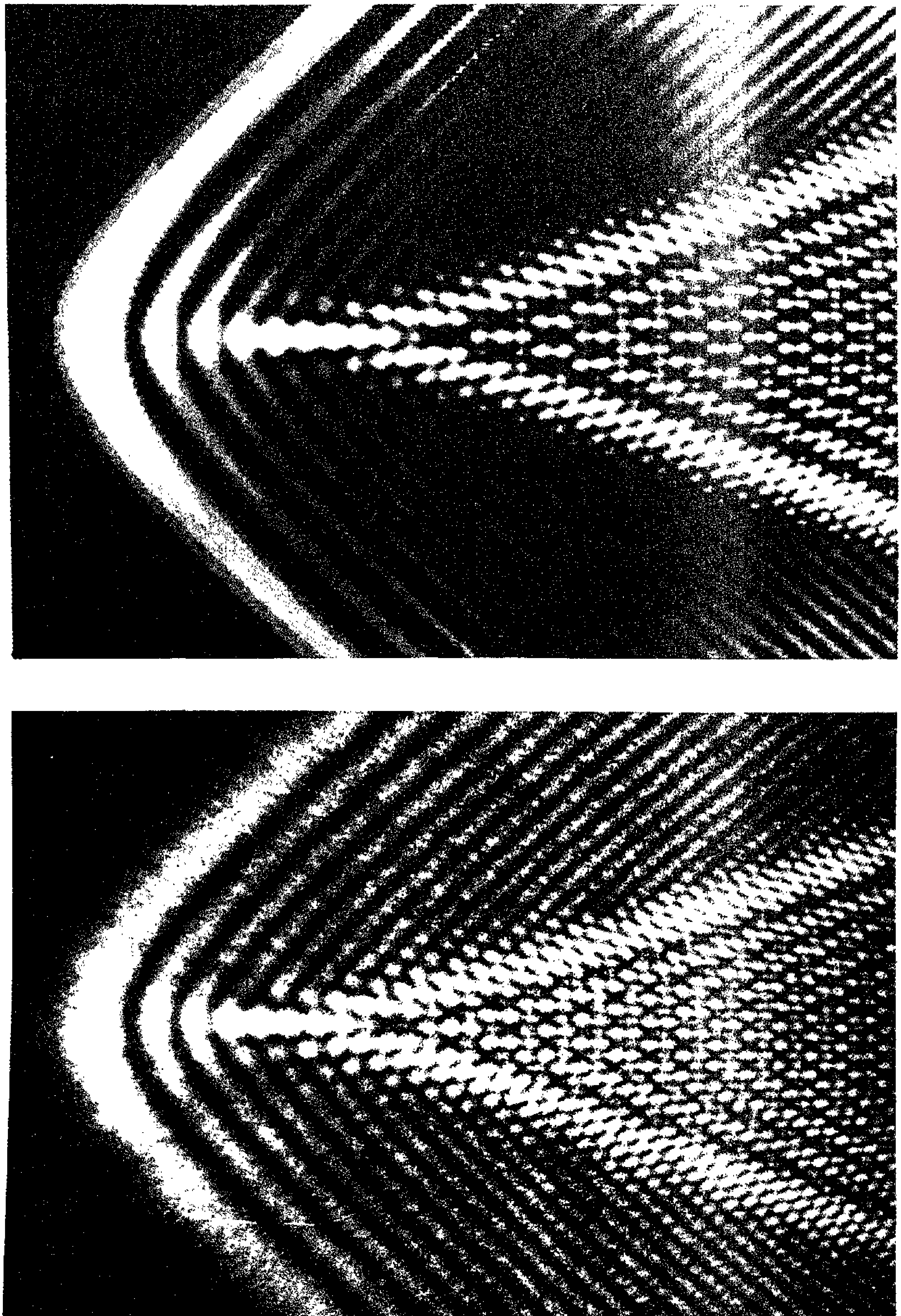

Figure 18.

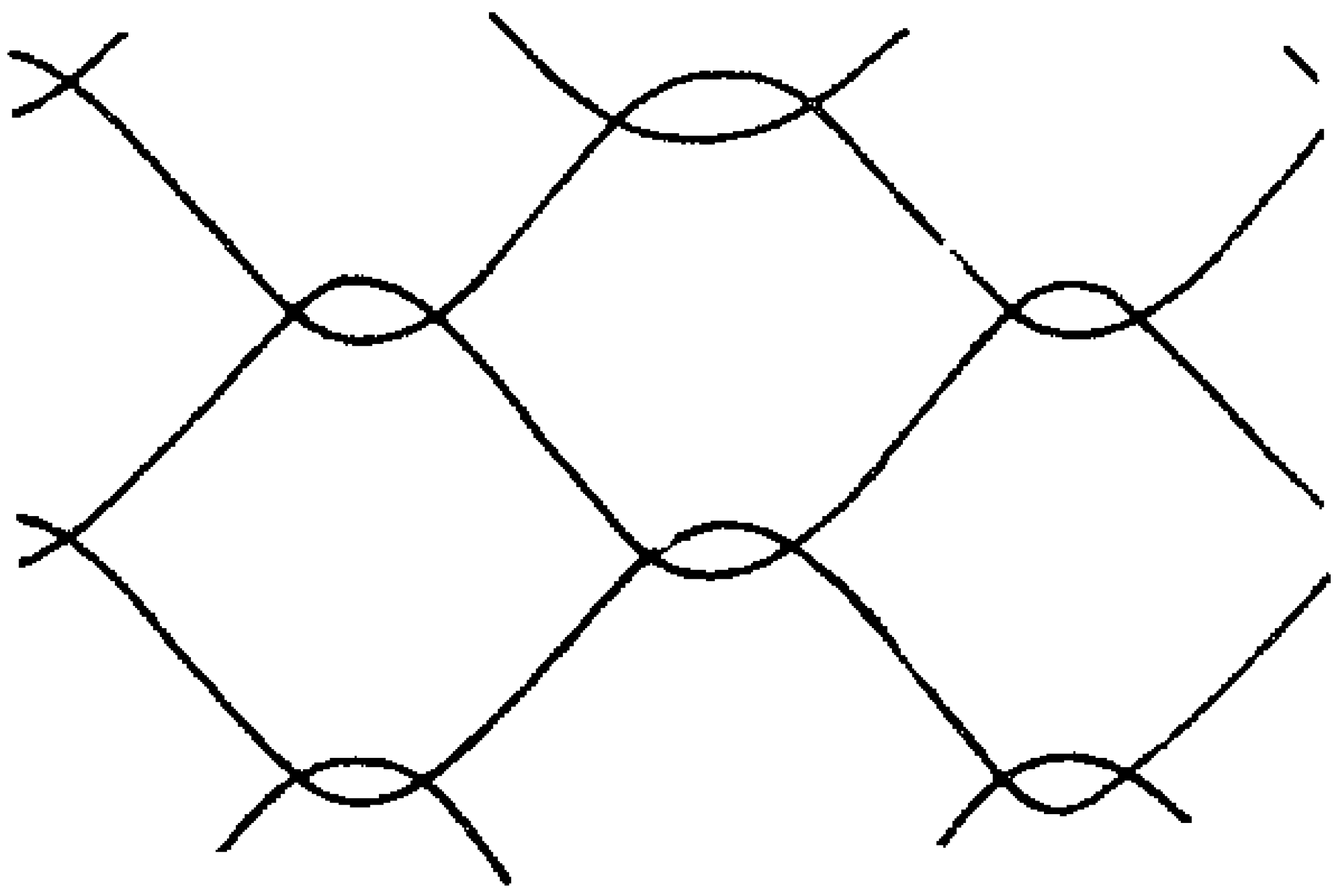

Figure 20.

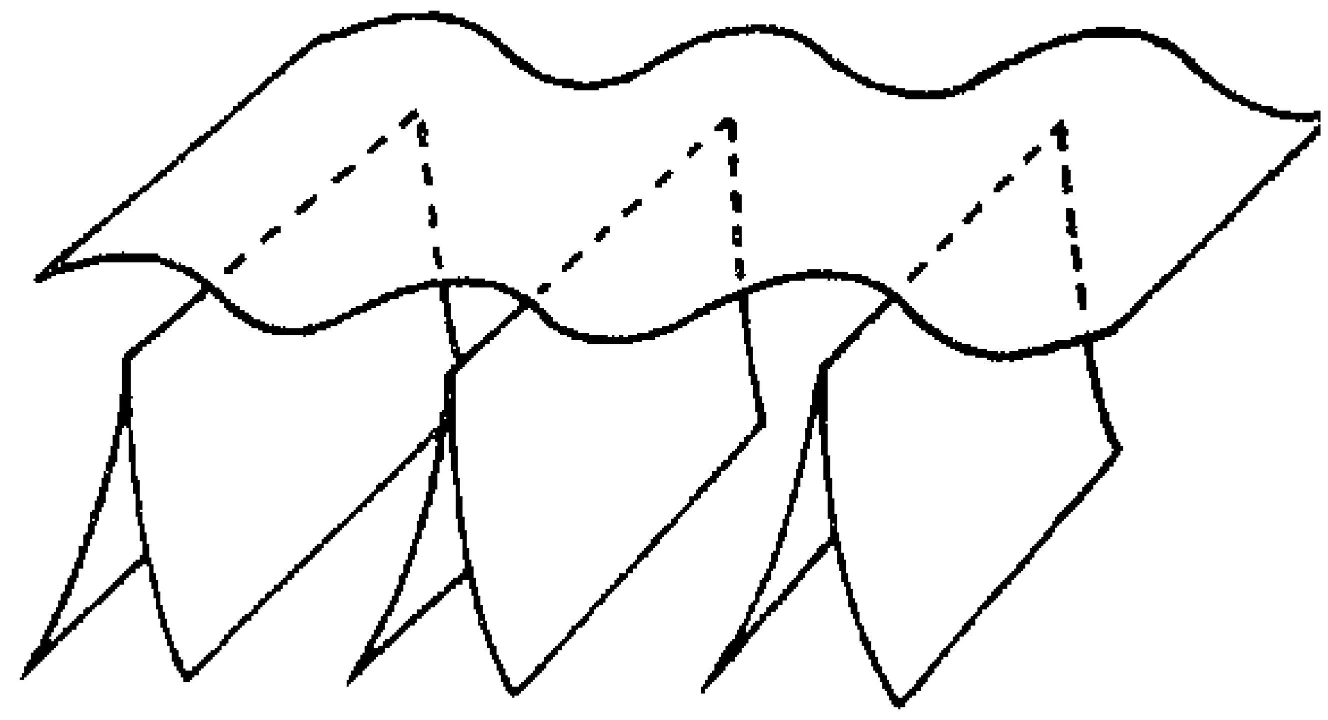

Figure 21.

The full analysis of these networks requires very high-order catastrophes and is more complicated than I wish to discuss here. But a good feeling for how catastrophe theory helps is gained by considering the caustics from a single train of water waves (figure 21). This consists of pairs of sheets joined at horizontal cusp lines. If the bottom of the pool – the screen – is below the level of the cusp, the observed caustic takes the form of a pair of parallel lines. This tells us that under poor resolution the lines we see might really be double, and indeed such arrays of line pairs can sometimes be seen (figure 22).

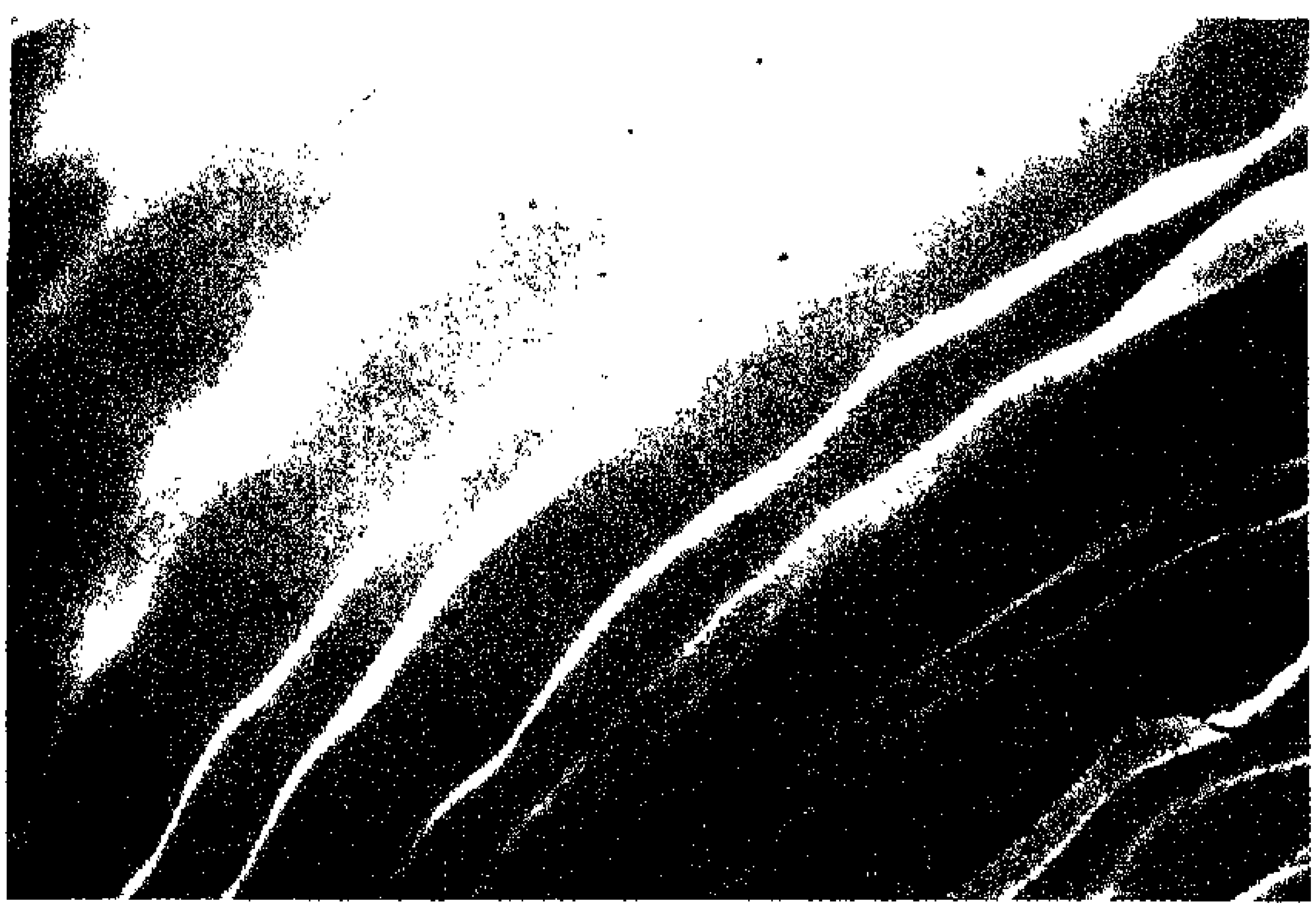

Figure 22.

Now imagine that the water wave crest is perturbed, for example by the addition of a weak wave travelling in another direction. This does not alter the fact that the caustic is cusp-edged, because catastrophe theory assures us that the cusp is stable. But the cusp line need no longer be straight and horizontal. Often it is curved so that only a part of the caustic surface intersects the screen, and this gives rise to a distinctive lips-shaped caustic (figure 23), with two cusps. Such lips can sometimes be seen (figure 24).

If the additional water wave is powerful enough, the cusp-edged caustic surface can be broken up. One of the many patterns that can then be seen is shown in figure 25 for the case where

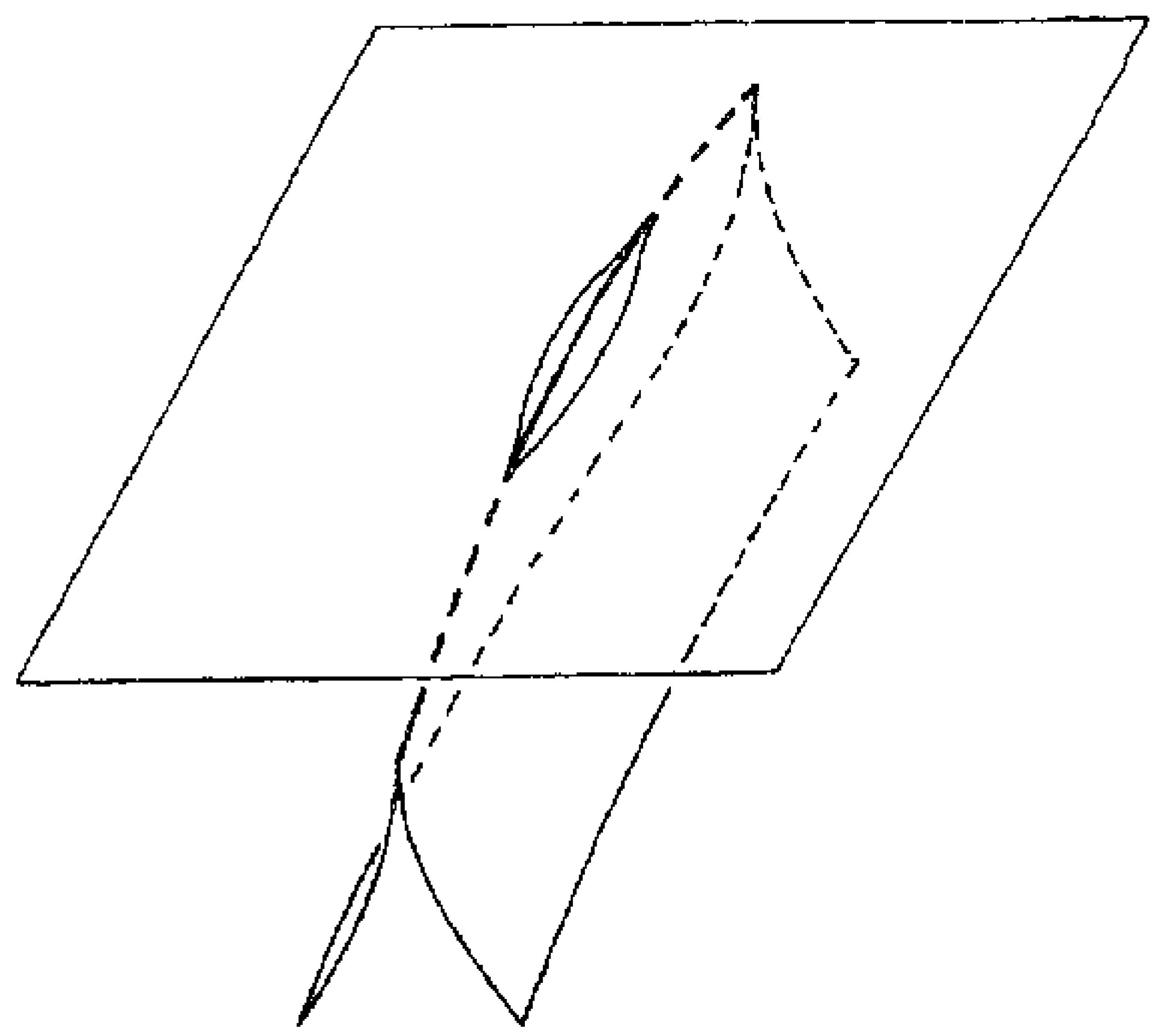

Figure 23.

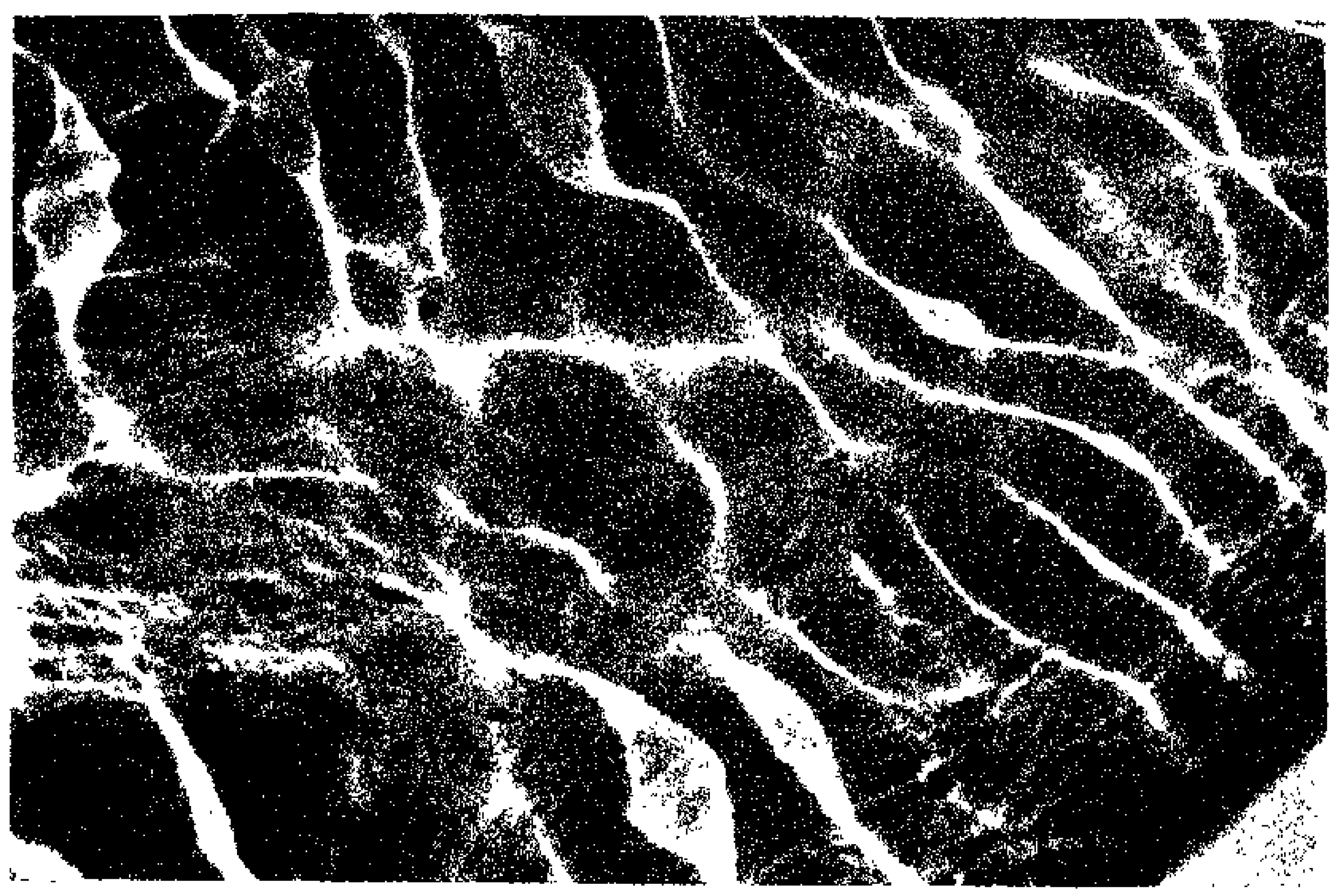

Figure 24.

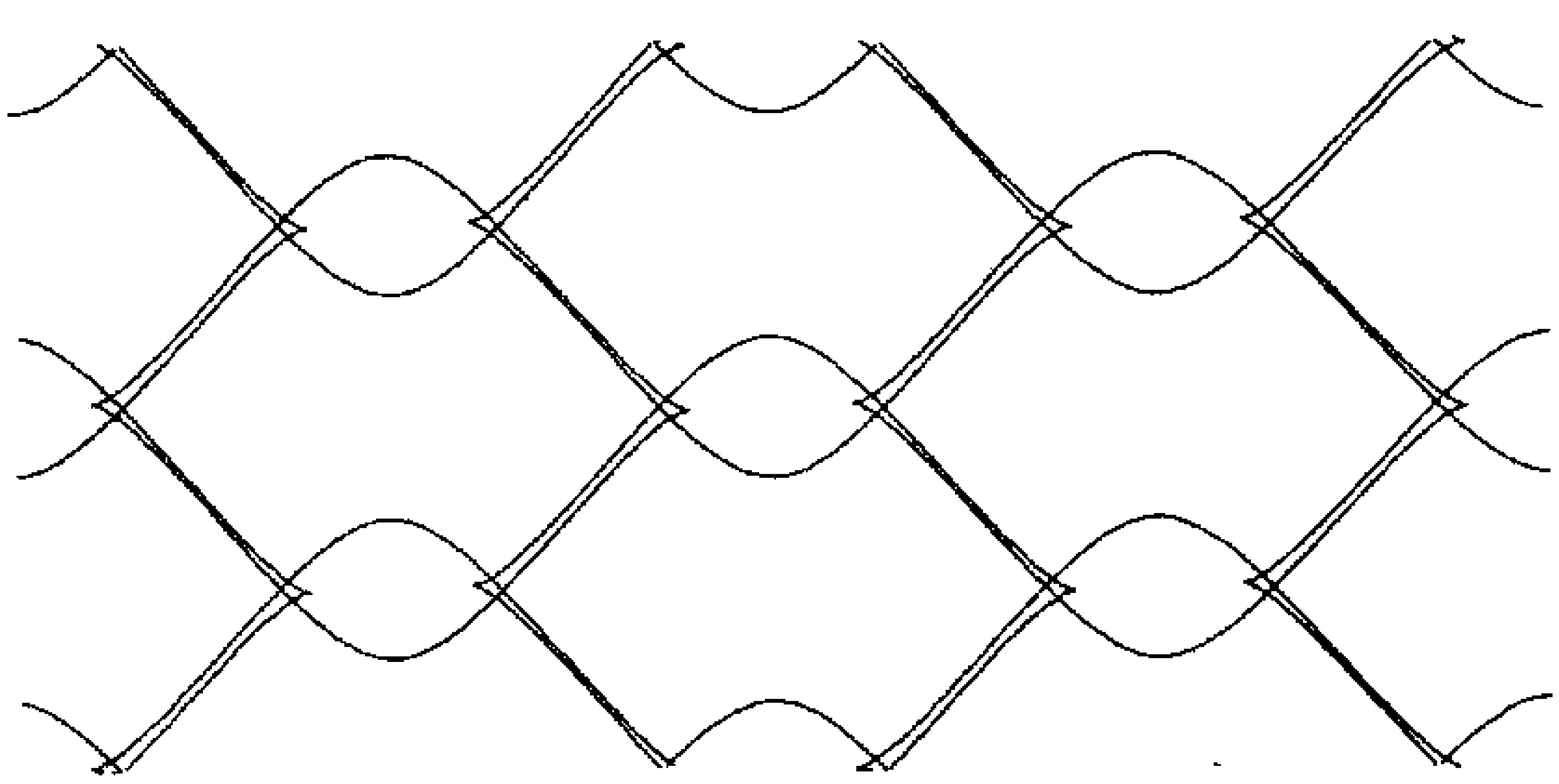

Figure 25.

there are not two single waves but two trains of waves. This pattern is one of the resolutions of the child's drawing (figure 20), in which many of the lines are indeed line pairs and their junctions are the result of giant overlapping lips shapes. Again these caustics can be observed (figure 26).

Figure 26.

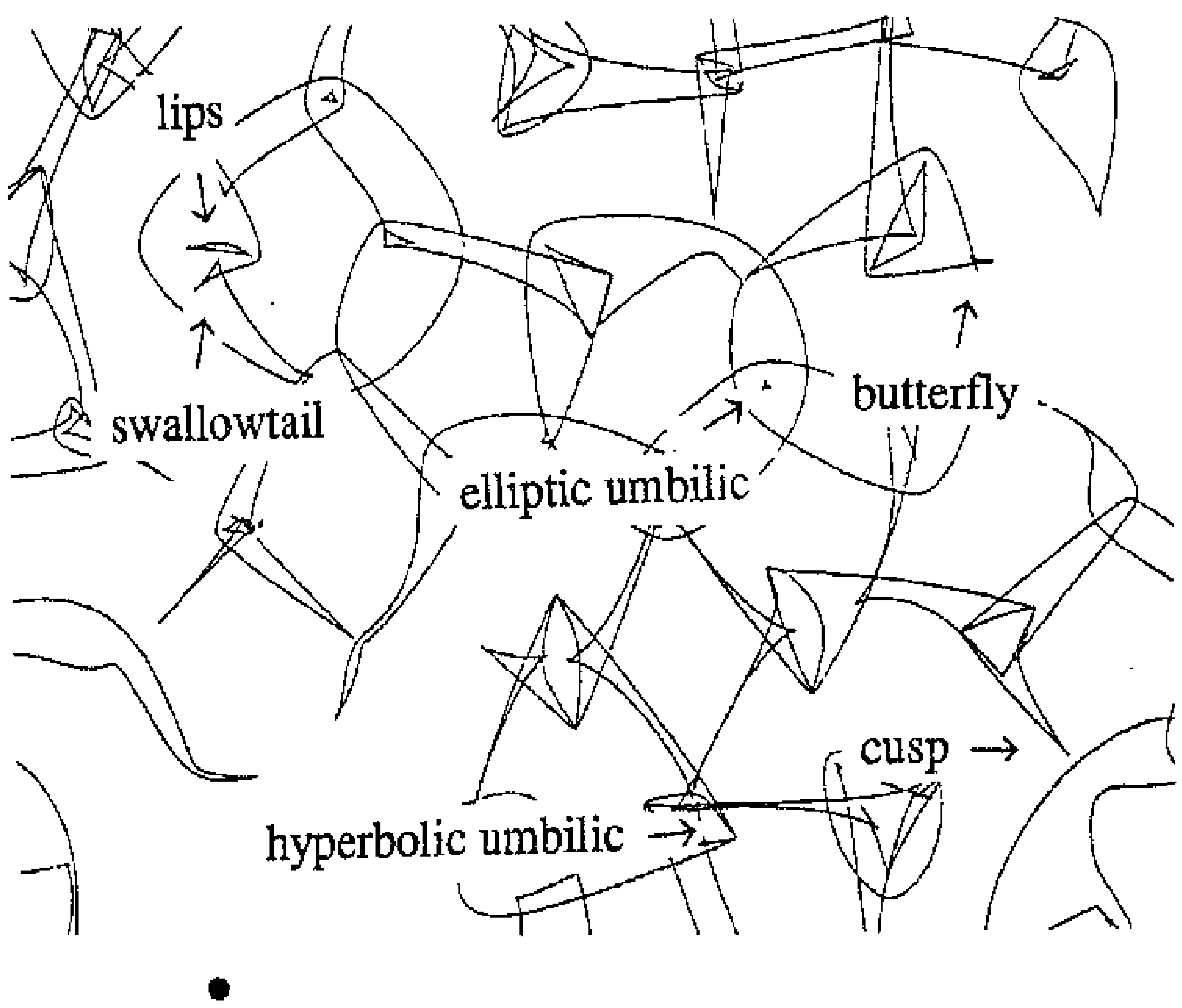

Figure 27.

Computer simulation of caustics avoids the blurring effect of poor resolution. The patterns (figure 27) produced on a screen by the superposition of several trains of water waves are then revealed as being intricate assemblies of all the catastrophic phenomena I have been telling you about. The fold lines are joined at cusp points, and there are lips, swallowtails and elliptic and hyperbolic umbilics, as well as sections of higher catastrophes such as the butterfly.

Turbulence in the atmosphere is sometimes strong enough to refract starlight into caustic networks dancing across the ground (especially for stars low in the sky, whose light has passed through more air). The networks are too faint to see, even at night, but when we look up at a star the repeated passage across our eyes of caustic surfaces in the networks give rise to the intensity fluctuations that we call 'twinkling'. One of the more surprising quantitative applications of catastrophe theory is to the statistics of the fluctuating intensity of twinkling starlight. And as Christopher Zeeman has pointed out, the fact that the surfaces tend to arrive in pairs gives new meaning to the nursery rhyme: "*Twinkle, twinkle,* little star...".

6. LAST WORDS

The message I have tried to convey is that caustics are naturally classified as mathematical catastrophes, and that many optical phenomena have caustics as their essential feature. Here is a partial list, which includes several topics I have not had time to discuss here:

> *rainbows ★ twinkling starlight ★ sparkling sunlight on the sea ★ nocturnal images through rained-on glasses ★ swimming-pool networks ★ bright-edged shadows of floating leaves and insects ★* occultations of stars by planetary atmospheres ★ mirages . . .*

It is always satisfying when abstract constructions match features of the external world – indeed, such matching is the essence of physics. It is especially pleasing when, as here, the phenomena are not confined to the laboratory but occur in nature where they can be seen by everybody.

FURTHER READING

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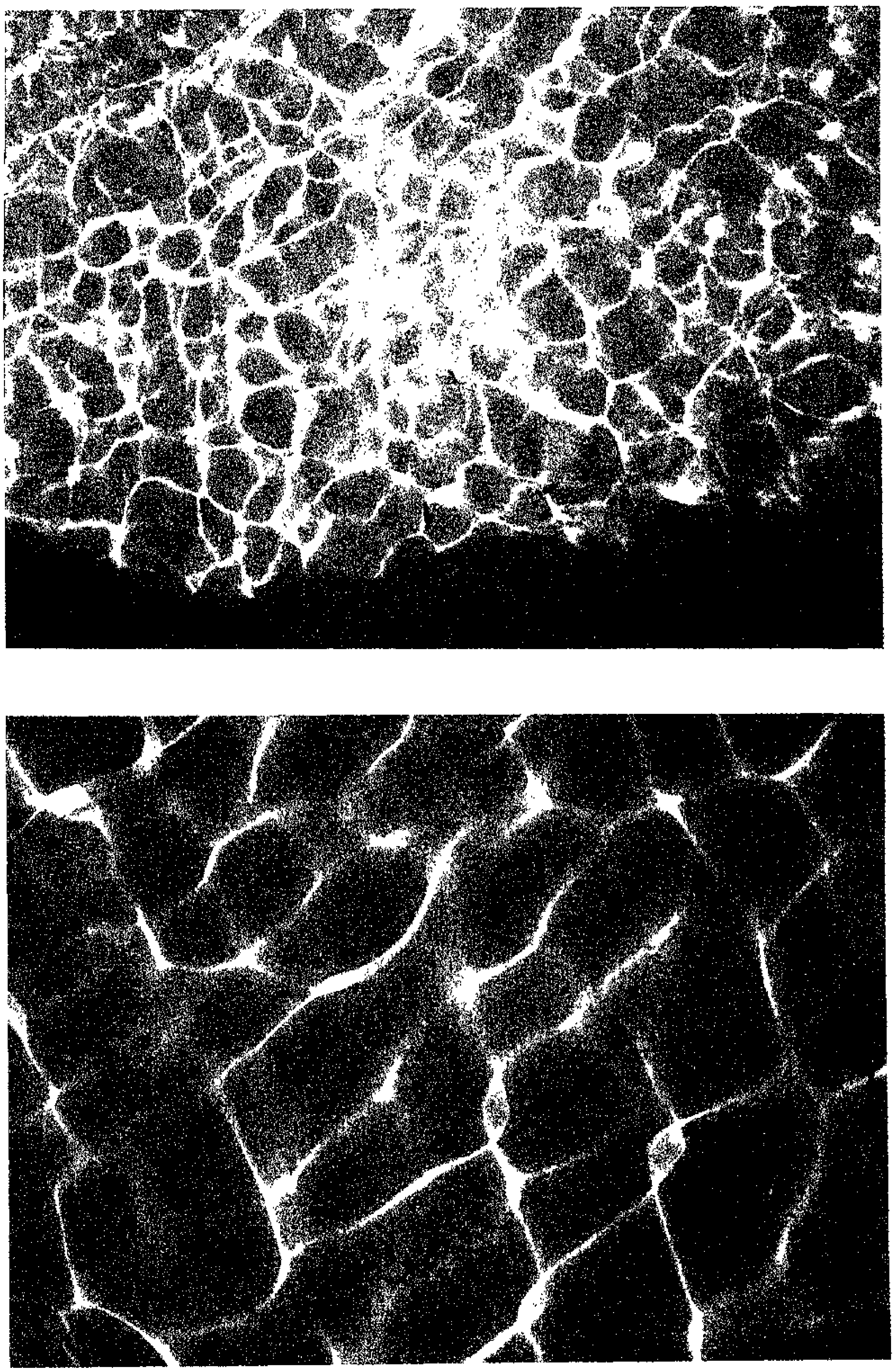

Figure 19.