

The Problem of Plateau.

A CLASSICAL problem in the calculus of variations is to prove the existence of a surface of least area bounded by a given contour Γ . The Euler conditions of this variational problem form a system of non-linear partial differential equations, expressing that the required surface has mean curvature zero, or, is a "minimal surface". This problem of finding a minimal surface to pass through an assigned boundary—called Plateau's Problem, derives an exceptional interest from the circumstance that the surface can always be exhibited to the eye by a simple physical experiment. Dip a wire having the form of the given boundary-curve in soap-solution, and the film adhering to the wire when withdrawn from the solution is the surface required. Simple examples are furnished by the catenoid and the helicoid which correspond to the case when the bounding curve is a pair of equal pair circles, and a helix respectively. It is known from the theory of surface-tension that a very thin film must assume that form for which the surface-area is a minimum and that for such a surface the mean curvature vanishes at every point. Various forms of such surfaces of revolution were produced experimentally by J. A. F. Plateau who made an elaborate study of the phenomena of surface-tension and greatly facilitated the study of liquid films. (*However, while investigating these beautiful phenomena, he himself never saw them, having lost his sight in about 1846.*) Although the existence of a minimal surface passing through a given curve can be demonstrated simply by such experimental methods, it is remarkable that no theoretical proof of existence in the general case was known till very recently, when in about 1931-32, Rado and Douglas succeeded independently in giving a satisfactorily general existence proof. This appears all the more remarkable if we reflect that the names of such great mathematicians as Riemann, Weierstrass, H. A. Schwarz, Darboux and others are associated with the problem. Riemann, and after him Weierstrass and others succeeded in linearising the differential equations of the problem, showing that Plateau's problem is equivalent to finding a potential vector $X(u, v)$ in a domain B of the u, v -plane (boundary C) which gives a conformal mapping of B on a surface bounded by Γ (the given boundary-curve of the minimal surface), and on the basis of this definition obtained the solution for many interesting particular cases. Douglas has indeed considered the much more general and difficult problem of proving the existence of a minimal surface in an m -dimensional space, which is bounded by k given contours (Jordan curves) $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ and which has a prescribed topological structure, i.e., is required to be one-sided or two-sided and to have a prescribed genus. He has so far published a solution for two-sided minimal surfaces of the genus zero for $k=1$ and $k=2$, and also one-sided surfaces (the type of a Möbius band) with $k=1$, and he has announced a solution of the general problem in which he will make essential use of the theory of Abelian functions

on Riemann surfaces of arbitrary genus. In the meanwhile, R. Courant has developed (*Annals of Math.*, 1937, 38, No. 3, 679) an independent method on the line of Dirichlet's Principle which not only solves the original Plateau problem and the most general problem formulated by Douglas but is also capable of yielding solutions for the problem in cases apparently not accessible to Douglas' original method, in which parts of the boundaries are free on prescribed manifolds of any dimension less than m .

The problem may be formulated analytically as follows:— Suppose that the surface S under consideration to be represented by functions $x_\mu(u, v)$ of two parameters (u, v) , (or by a vector $\bar{X}(u, v)$ with the x_μ as components) in a given domain B of the u, v -plane with boundary C .

These functions shall be continuous in $B + C$, have piecewise continuous second derivatives in B , and map Γ on C . Then the problem is to minimise by one of these admissible functions

the integral $A(\bar{X}) = \iint_B \sqrt{EG - F^2} \, du \, dv$, where

with the usual notation

$$E = \sum_{\mu} \left(\frac{\partial x_{\mu}}{\partial u} \right)^2, \quad G = \sum_{\mu} \left(\frac{\partial x_{\mu}}{\partial v} \right)^2, \quad F = \sum_{\mu} \frac{\partial x_{\mu}}{\partial u} \frac{\partial x_{\mu}}{\partial v}.$$

Since the integral A is invariant under arbitrary transformations of the parameters (u, v) and their domain B , the latter, if Γ is a simple Jordan curve, may be chosen as the unit circle $u^2 + v^2 < 1$. It is then shown easily that the problem is equivalent to the problem of minimising the classical Dirichlet integral $D(X) =$

$$\frac{1}{2} \iint_B (\bar{X}_u^2 + \bar{X}_v^2) \, du \, dv.$$

The proof of the existence of a solution is carried out on the basis of the known solution of the boundary-value problem of the potential equation $\nabla^2 \rho(u, v) = 0$ for a domain in the u, v -plane bounded by k circles C_1, C_2, \dots, C_k , and a series of Lemmas of which the most important are the following:—

(1) Suppose that in a domain B of the u, v -plane bounded by k circles C_1, \dots, C_k , the vector $X(u, v)$ is continuous and has piecewise continuous derivatives in B and $D(X) \leq M$, and maps the circles C_1, \dots, C_k in a continuous way on the prescribed Jordan curves $\Gamma_1, \dots, \Gamma_k$ in the m -dimensional X -space. Let O be a point in B or outside B , C_r the part of the circle with radius r round O lying in B . Then there exists for every sufficiently small δ a value r_0 with $\delta \leq r_0 \leq \delta_0$, so that on every connected arc of C_{r_0} the oscillation of the vectors X does not exceed the quantity $\epsilon(\delta) = [4\pi M / |\log \delta|]^{\frac{1}{2}}$. In particular, for O on C and δ so small that C_r consists of a single arc, there exist two points A_1, A_2 on C at the same distance r_0 from O with $|X(A_1) - X(A_2)| \leq \epsilon(\delta)$.

(2) Let R on C_r be a point of non-equicontinuity for a sequence of vectors $\{X_n\}$ which satisfy the assumptions of the above lemma and map

C_p on a Jordan curve Γ_p in a monotone continuous way. Let b be a fixed arc on C_p with end-points A, B containing the point R but otherwise arbitrarily small and let b' the complementary arc of C_p . Then at least for a subsequence X_n the image β of b defined by X_n will cover all Γ_p , except for an arc β' whose diameter tends to zero as n increases. In other words: the mapping of C_p on Γ_p by X_n tends to a degeneration in such a way that any small neighbourhood of a point R is mapped on nearly the whole closed curve Γ_p . Since the Dirichlet integral $D(\bar{X})$ is invariant under conformal mapping, we may by a linear transformation, transform the unit-circle into itself so that 3 given points on C are co-ordinated to three fixed points on Γ . Assuming that this 3-point condition is satisfied and that the Dirichlet integral is capable of finite values, the solution of Problem I for a single contour Γ ($k=1$) is obtained as follows: There exists a minimising sequence $X_1 \dots X_n \dots$ of admissible vectors for which $D(X_n) \rightarrow d$ for $n \rightarrow \infty$ while always $D(\bar{X}_n) \geq d$ [d = lower bound of $D(\bar{X})$] and so these Dirichlet integrals are bounded. The boundary values of the \bar{X}_n are equi-continuous, since otherwise by the choice of a suitable subsequence we would have a point R of non-equicontinuity on C and this by lemma (2) above contradicts the 3-point condition which excludes the possibility of an arbitrarily small arc b of C being mapped on nearly the whole curve Γ . On account of the equi-continuity we can choose a subsequence of the \bar{X}_n —again called \bar{X}_n —which converges uniformly on C . With these boundary values we solve the boundary-value problem of $\nabla^2 X = 0$ for B and thus obtain a sequence of potential vectors having the same boundary values as the \bar{X}_n and having Dirichlet integrals not exceeding $D(\bar{X}_n)$, according to a known theorem which asserts that the minimal value of the Dirichlet integral is attained for and only for the function which solves the corresponding boundary-value problem. Since the new potential vectors are also admissible vectors in Problem I, they form a minimal sequence $\bar{X}_1 \dots \bar{X}_n \dots$ and the uniform convergence of their boundary-values implies their uniform convergence in $B + C$ to a potential vector $\bar{X} = \lim_{n \rightarrow \infty} \bar{X}_n$

satisfying the conditions of Problem I and $\nabla^2 \bar{X} = 0$ in B . In each concentric circle the derivatives of \bar{X}_n converge uniformly towards the derivatives of \bar{X} . Denoting by D_r the Dirichlet integral for a concentric circle of radius $r < 1$, we have therefore $D_r(\bar{X}) = \lim_{n \rightarrow \infty} D_r(\bar{X}_n)$

$$\leq \lim_{n \rightarrow \infty} D(\bar{X}_n) = d. \quad \text{Letting } r \rightarrow 1 \text{ we}$$

obtain $D(\bar{X}) \leq d$ and since the inequality sign would contradict the assumption that d is the

lower bound, we have $D(\bar{X}) = d$, i.e., \bar{X} solves Problem I. To prove that the solution defines a minimal surface, use is made of the remark that in using the minimising character of \bar{X} we need not observe the 3-point condition nor the potential character of \bar{X} , to replace \bar{X} by $z(r, \theta) = X(r, \phi)$ with $\phi = \theta + \epsilon \lambda(r, \theta)$, λ arbitrary with continuous first and second derivatives in $B + C$. Since z satisfies the conditions of Problem I, $D(z) \geq d$. Passing to the limit $\epsilon \rightarrow 0$, it is shown

$$\text{that this leads to the condition } \lim_{r \rightarrow 1} \int_0^{2\pi} \lambda(r, \theta) r \bar{X}_r \bar{X}_\theta d\theta = 0.$$

$$\text{Now have } w^2 \phi(w) = \sum_{\mu} \left(\frac{df_{\mu}}{d \log w} \right)^2$$

$$= \sum_{\mu} \left(r \frac{\partial x_{\mu}}{\partial r} - i \frac{\partial x_{\mu}}{\partial \theta} \right)^2 \text{ and so } -2r \bar{X}_r \bar{X}_\theta =$$

$I\{w^2 \phi(w)\}$, [$I(z)$ = imaginary part of z] is a potential function in B and from the above

$$\text{condition } \int_0^{2\pi} \lambda(r, \theta) r \bar{X}_r \bar{X}_\theta d\theta \rightarrow 0 \text{ as } r \rightarrow 1,$$

it is proved that $r \bar{X}_r \bar{X}_\theta$ vanishes identically in B . Since the imaginary part of $w^2 \phi(w)$ vanishes this function must be real and constant in B , i.e., $w^2 \phi(w) = C$, $\phi(w) = C/w^2$; but $\phi(w)$ is regular at $w = 0$ and so $C = 0$, i.e., $\phi(w) = 0$ which expresses the character of S as a minimal surface. Therewith is solved Plateau's problem for $k = 1$. It must be noticed that incidentally for $m = 2$ when Γ is a Jordan curve in the x, y -plane, we obtain Riemann's mapping theorem (a remark due to Douglas.)

For $k > 1$ certain additional conditions are required to prevent the degeneration of the domain B . On the basis of some fresh lemmas on the lower limits of $D(\bar{X})$, the solution of Plateau's problem for $k = 2$ is first given and then the solution for the general case $k > 1$ is constructed, by assuming that the problem has been solved for all lower values of k . As before general theorems on conformal mapping of multiply-connected domains result as by-products of the reasoning. In the latter part of the paper, Courant also gives an alternative and considerably simplified solution by making use of fundamental facts concerning conformal mapping of domains in a plane, and proves that his solution of Plateau's problem furnishes at the same time the surface of least area bounded by Γ .

We await the further papers of Prof. Courant with great interest.

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