Paradoxical networks

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Study of networks is a discipline by itself. The subject is of great relevance in a wide variety of situations, from the neural network in the brain to the traffic network of big cities. Electrical networks dominated the scene till recently. By the middle of this century, network analysis had become a standard procedure, and the behaviour of networks could even be guessed. It therefore took everyone by storm when, in 1968, Braess[1] discovered a paradoxical behaviour in traffic networks, where addition of links or byways did not lead to either cheaper travel or reduction in traffic congestion. On the other hand, there appeared a new equilibrium at which the cost for all the individuals was higher than what existed before the link was added. It soon became clear that this is a general property of congested flows. In queueing networks also one meets an equivalent unusual behaviour. But what one was not prepared for is that such a puzzle could pop up in far simpler laboratory situations. In a recent paper in *Nature*, Cohen and Horowitz[2] described mechanical and electrical networks exhibiting an analogous phenomenon. In view of these discoveries it is of considerable importance and interest to know how this counterintuitive behaviour is possible.

Figure 1 shows a network[3,4] that behaves strangely. Consider a case where vehicles travel from point A to point C, via B_1 or B_2. The cost on a particular route depends on a variety of parameters. For a model calculation, take the cost (per person or vehicle) on AB_1 and B_2C to be the same, being equal to $10n$ where n is the number of vehicles on that route; similarly routes AB_2 and B_1C are identical, with a cost of $(50+n)$. In equilibrium, both routes will be equally congested. The traffic congestion can generally be relieved by adding links or byways. Suppose we connect B_1B_2 by a route through which traffic can flow. We work out the consequences of this added channel. On this new route B_1B_2 the cost is taken to be $(10+n)$ (Figure 1, *b*). With these initial conditions, the cost calculations given in the table are for the different configurations shown in Figure 1, *c*. Since, in configurations 2 and 3, different people will be paying differently, the system is not yet in equilibrium. In configuration 4, however, the traffic network is in equilibrium, with every driver paying 92. Earlier they were paying 83 each. Thus everyone is worse off. The new link has not solved the traffic problem.

Cohen and Horowitz discussed a system of springs and strings (see Figure 2). It behaves very interestingly. For a model calculation assume that the springs are massless, with a spring constant equal to unity so that the displacement is equal in magnitude to the force. The back-up strings are supposed to be inelastic. The central link string is also inelastic and has a length of, say 3/8 m. Under these conditions a weight of $\frac{1}{2}$ N results in a separation X of $1\frac{3}{8}$ m between the weight and the supports, and the safety strings (of length 1 metre) are limp. Now the interesting question is: what happens when the central linking string is cut? Intuition suggests that the weight would come down and the distance X would increase. But calculations yield a value of $X=1\frac{1}{4}$ m (see Figure 2, *b*). Hence the weight is at a higher level than before. If the length L of the linking string were to vary smoothly from zero to 1 m while all other elements remained the same, then X (before central-string dissection) varies with L as shown in Figure 2, *c*. For L between 0.25 m and 0.75 m the weight would be higher, i.e. X would decrease, if the link was cut. In this range the system has a paradoxical behaviour.

Cohen and Horowitz also discussed exact electrical and hydrodynamic analogues of the spring–string mechanical problem. In the electrical problem, current I behaves like the mechanical force and voltage V behaves like the displacement X. A resistor (with resistance R) obeying Ohm's law $V=IR$, is the electrical analogue of a massless spring obeying Hooke's law. An inextensible string can be represented electrically by a Zener diode whose characteristic voltage V_z (for $I>0$) is analogous to a string of constant length. Figure 3, *a* shows the electrical network. A Zener diode of characteristic voltage $V=\frac{3}{8}V$ connected across the bridge provides another path for current conduction (see fig. 3b). This is like the central linking string of the previous problem. Just as

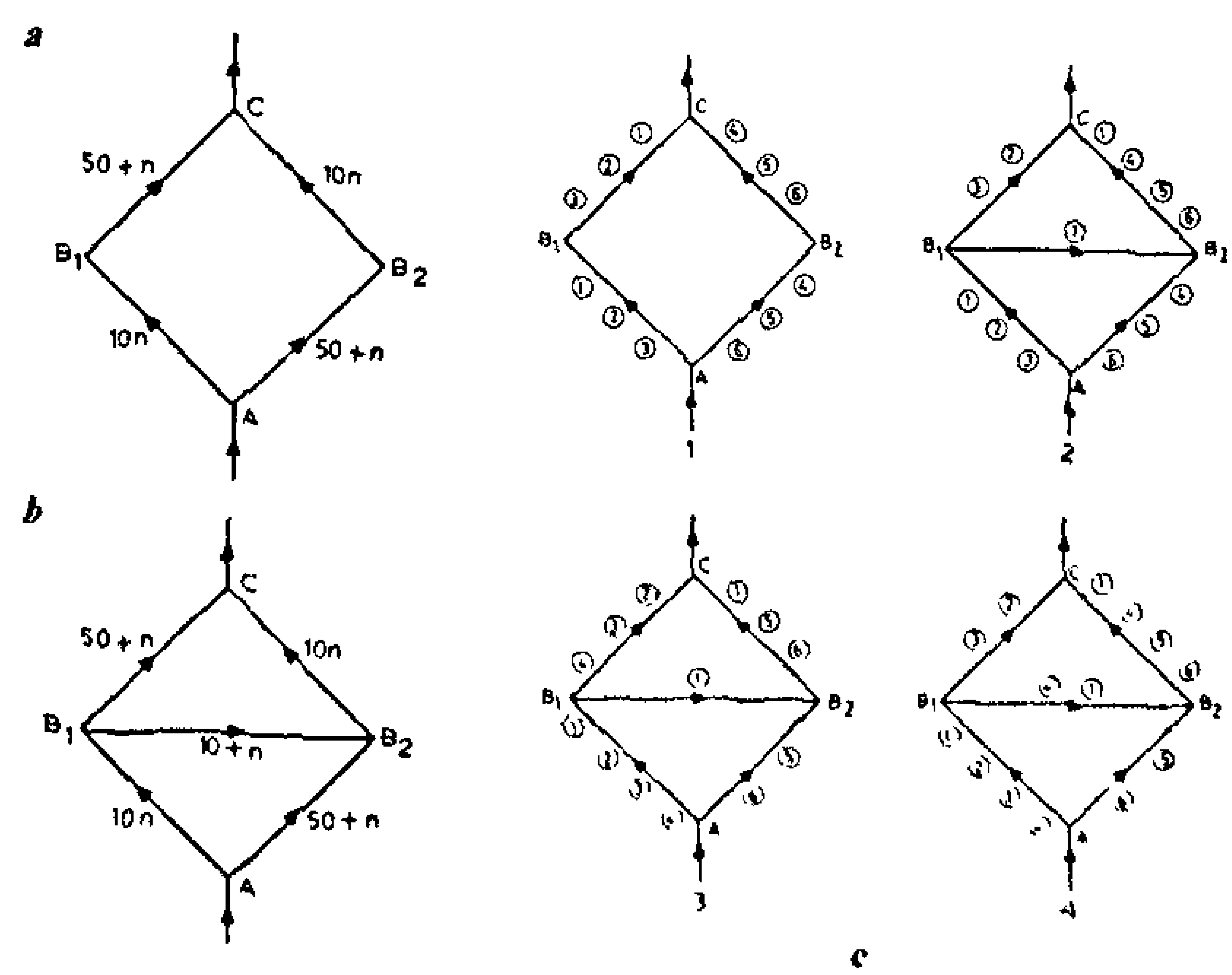

Figure 1. A traffic network (*a*) with the traffic flowing from A to C via B_1 or B_2, and (*b*) with a central link route (*c*) Four configurations for which the cost per individual to go from A to C, has been calculated. Encircled numbers indicate different vehicles travelling from A to C via the various possible routes.

Configuration	Cost per individual along			Remarks
	AB_1C	AB_2C	AB_1B_2C	
1	30+53=83 (for 1,2,3)	30+53=83 (for 4,5,6)	Nil	Equilibrium
2	30+52=82 (for 2,3)	53+40=93 (for 4,5,6)	30+11+40=81 (for 1)	
3	40+53=93 (for 2,3,4)	52+30=82 (for 5,6)	40+11+30=81 (for 1)	
4	40+52=92 (for 3,2)	52+40=92 (for 5,6)	40+12+40=92 (for 1,4)	New equilibrium

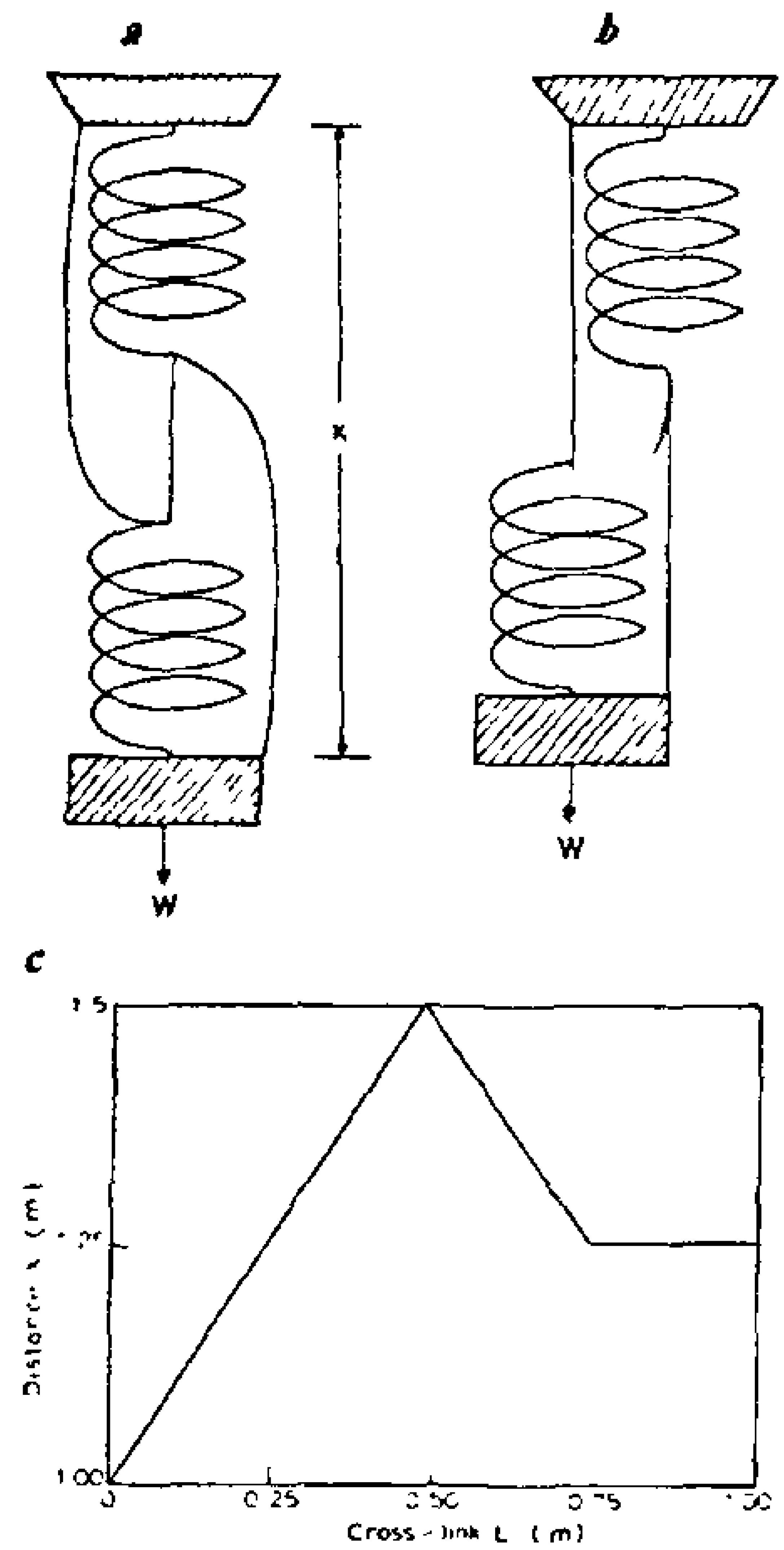

Figure 2. A spring–string system *a*, With the central link string intact, the support strings are limp *b*, Without the central link string, the weight gets lifted up for a certain range of lengths of the central string *c*, Variation of distance *X* from the support to the weight for different lengths *L* of the central link string

in the previous case here also one gets counterintuitive behaviour of increase in net resistance on connecting the central Zener diode. A paradoxical change in voltage drop occurs for $(1-3I/2) < V' < (1-I/2)$.

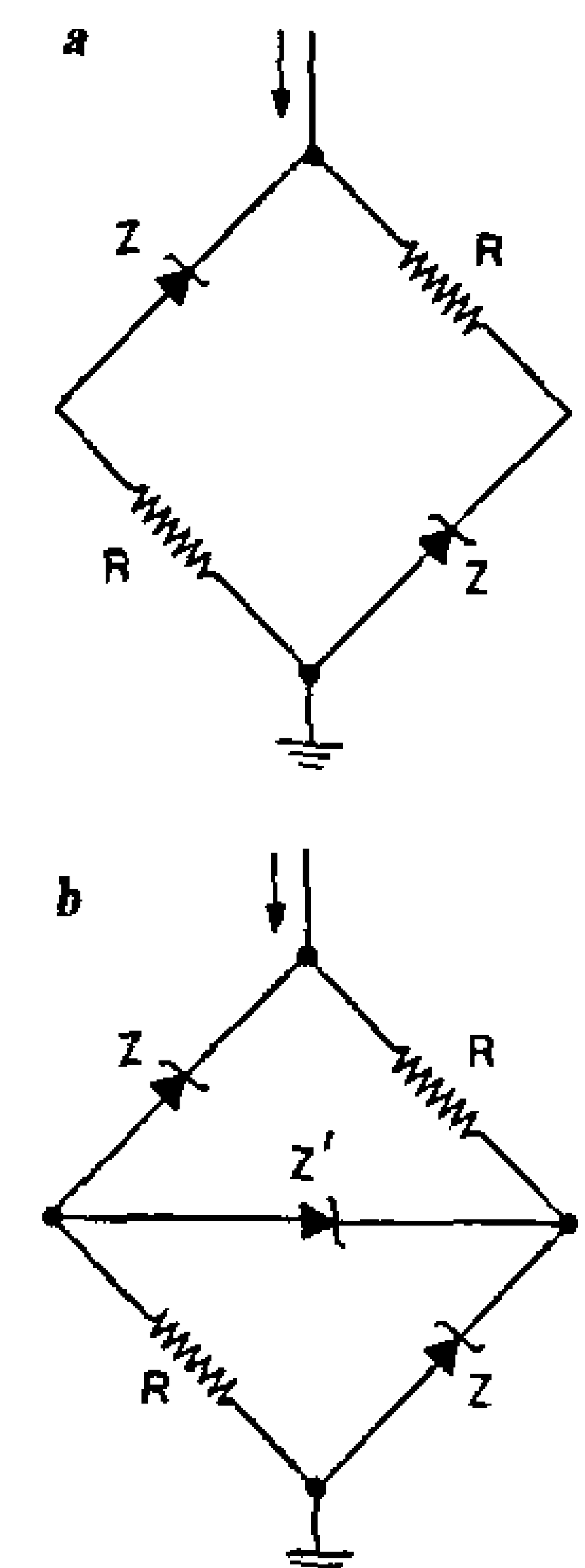

Figure 3. An electric network analogous to the spring-string system in Figure 2 (*a*) without the bridging Zener diode, (*b*) with bridging Zener diode.

In the hydrodynamic analogue, constant pressure difference valves act as Zener diodes (inextensible support strings) and Poiseuille flow in tubes mimics resistors (springs). Hence it should be possible to get unexpected flow behaviour in a proper combination of such elements.

Cohen and Horowitz also discuss an electrical network where addition of a

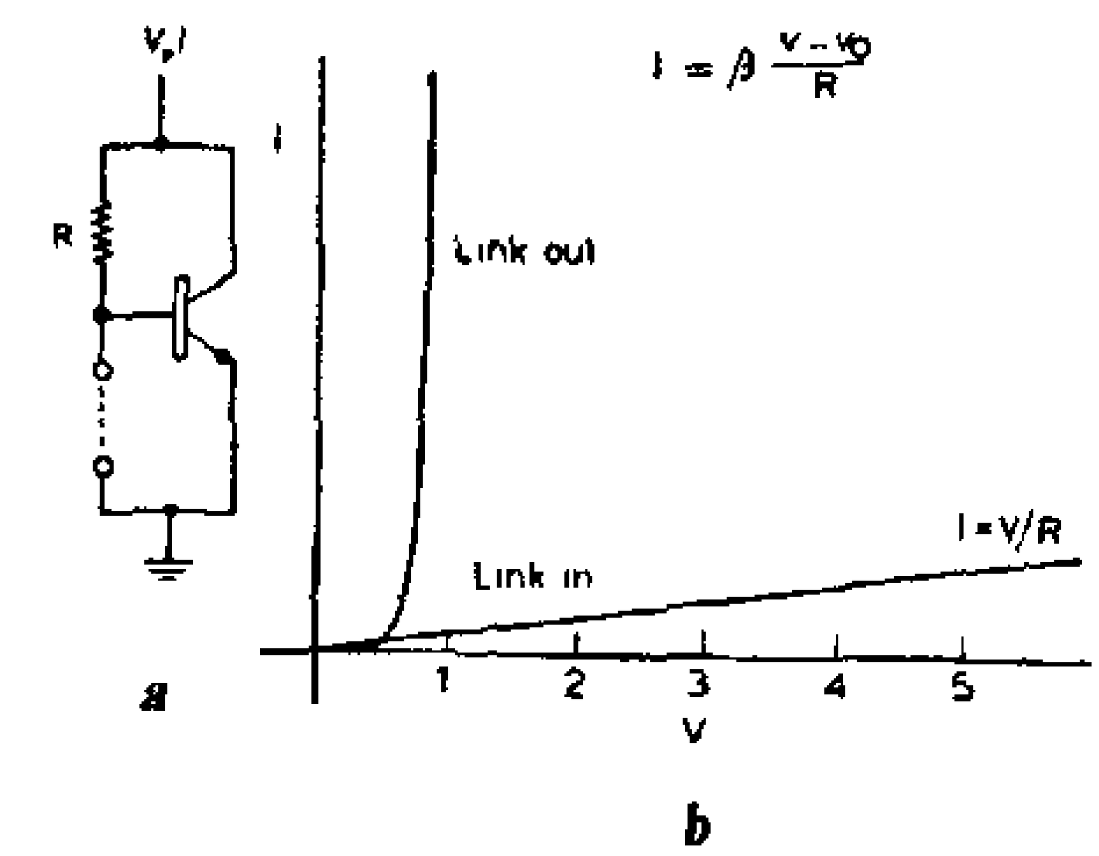

Figure 4. A transistor–resistor network (*a*) and its *I–V* characteristic ($\beta \approx 10^2$, $V_0 \simeq 0.6$ V) (*b*)

path can reduce current flow. But this behaviour is not counterintuitive. Figure 4, *a* shows a circuit consisting of a transistor and a resistor. When the path shown by a dashed line is open the circuit allows current to flow through the transistor for any applied voltage greater than 1 volt. If now the dashed path is connected, thus providing an additional current path, current no longer flows through the transistor and the circuit behaves like a single resistor *R* with a greatly reduced conduction across the circuit. This is not paradoxical because the circuit has a three-terminal device. Figure 4, *b* gives the *I-V* characteristics of this electrical sytem.

All these examples bring out an essential fact not hitherto appreciated in literature. Physical networks and systems need not behave as expected when additional paths are provided. As Cohen and Horowitz rightly point out, one of the important things to be solved is the specification of conditions under which general networks behave paradoxically.

1. Braess, D., *Unternehmensforschung*, 1968, **12**, 258.
2. Cohen, J. E. and Horowitz, P., *Nature*, 1991, **352**, 699.
3. Steinberg, R. and Zangwill, W. I., *Transportation Science*, 1983, **17**, 301.
4. Cohen, J. E., *Am. Sci.*, 1988, **76**, 576.

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