single function. So to prove that a general point $P \in X$ is not a complete intersection, one needs only to show that $CH_0(X) \neq 0$. This in general requires techniques from projective geometry. The fact is that the curve X is an *elliptic curve* and they have very large Chow groups.

A similar definition can be made for any variety of any dimension. If X is an arbitrary variety, define $CH_0(X)$ to be the quotient of $F(X)$ by the subgroup generated by elements $\Sigma_{P \in Y} v_P(f)$, where f runs over all non-zero functions on Y and Y runs over all curves on X. It will follow by definition, that if a point is a complete intersection then its Chern class is zero. The converse is true for affine varieties over **C**, a theorem due to Murthy[10]. The proof requires deep analysis of what are known as *vector bundles* (or *projective modules*) over these varieties. The point I want to stress is that it is such unifying general theories which ultimately pay rich dividends.

Example 3.1: We will conclude this article by showing that the Chow group of Y, the cubic surface we defined earlier is zero.

First let us prove that the Chow group of a plane curve of degree atmost two is zero. So let $X = \{f(x,y) = 0\}$ be a plane curve and assume that $\deg f \leq 2$. If $\deg f = 1$, then $f = 0$ is a line and hence any point on X is defined by the vanishing of one function. Thus by definition the Chow group is zero. So let us assume that $\deg f = 2$. If f is a product of two non-constant polynomials, then both must be of degree 1 so that X is a union of two lines and since any point on X must lie on one of these lines we are done. So let us assume that f is irreducible. Then by a change of variables, we can assume that $f = x^2 - g(y)$, where $\deg g \leq 2$. Now again by a change of variables, we can assume that $f = x^2 - y$ if $\deg g = 1$ and if $\deg g = 2$ then after a further change of variables we can assume that $f = xy - 1$. In either case if $P = (a,b) \in X$, then it is the set of zeroes of the single polynomial $x - a$. So the Chow group of X is zero.

Now let us go back to the cubic surface $Y = \{f = x^3 + y^3 + z^3 - 1 = 0\}$. Let $P = (a,b,c)$ be any point on Y. We will only treat the case when $a + c \neq 0$ and $b \neq 1$, the rest of the cases being similar. Let d be chosen so that $a + d(b-1) + c = 0$. Consider the intersection, denoted by X, of the plane $x + d(y-1) + z = 0$ with Y. By choice, $P \in X$. Substituting the expression for z from the linear equation, we get the equation g defining X in the (x,y)-plane. Since f is cubic so is g. It is clear that $g(x,y) = (y-1)h(x,y)$, where h is of degree 2. Since $b \neq 1$, $P \in C = \{h = 0\} \subset Y$. But the Chow group of C is zero by the previous paragraph and hence the class of P in the Chow group of Y is zero. Since P was an arbitrary point we get that $CH_0(Y) = 0$.

For a more detailed account with proofs of many of the above discussions, the reader may see ref. 11.

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Projective algebraic varieties

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1. Projective geometry

In their famous book *Geometry and the Imagination*, Hilbert and Cohn–Vossen[1] describe projective geometry as a study of 'geometrical facts that can be formulated and proved without any measurement or comparison of distances or angles'. They give the following example:

> ... if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure.

To give a physical analogy, imagine the point of projection as a light source, and the first planar figure as the outline of some object; then the projected figure is its shadow on a wall. It is clear that problems of *perspective* in drawing or painting contain the elements of projective geometry.

A simple example of a result from projective geometry is *Desargues' theorem.* Two triangles (in space) are said to be *in perspective* if they 'cast the same shadow on a wall', i.e. have the same image in a plane, on projecting from a certain point P (called the *centre of perspective*). Equivalently, triangles $\triangle ABC$ and $\triangle A'B'C'$ are in perspective if the lines $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ have a point P in common (i.e. are *concurrent*).

Desargues' theorem states that if two triangles are in perspective, then the three points of intersection of corresponding pairs of sides, namely $\overline{AB} \cap \overline{A'B'}$, $\overline{BC} \cap \overline{B'C'}$ and $\overline{AC} \cap \overline{A'C'}$, are *collinear* (i.e. lie on a line). If the two triangles are not in the same plane, a bit of thought will convince the reader that in fact the 3 pairs of corresponding sides meet at points which must lie on *the line of intersection of the two planes*, and in particular, the 3 points are collinear. The case when the triangles are in a plane may be proved by a limiting argument, or by showing that they are obtained by projection from a pair of triangles in space which are not in the same plane, but are in perspective.

Another old result is *Pappus' theorem*, that if A, B, C and A', B', C' are two sets of 3 points lying on two lines L, L' respectively, and L, L' meet at a point, then the 3 points of intersection $\overline{AB'} \cap \overline{A'B}$, $\overline{BC'} \cap \overline{B'C}$ and $\overline{CA'} \cap \overline{C'A}$ are collinear. *Brianςon's theorem* states that if a hexagon $ABCDEF$ is circumscribed about a conic, then the 3 'diagonals' $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ meet at a point.

2. The projective plane

In thinking about properties of plane figures preserved under projection from a point outside the plane, one realizes quickly that the plane seems to have 'missing points' which 'reappear' after the projection, while certain other points 'disappear'. To illustrate this, if $\mathbf{P}$, $\mathbf{P}'$ are distinct planes in space, and O is a point outside them, then the projection from $\mathbf{P}$ to $\mathbf{P}'$ is the following transformation: given a point Q of $\mathbf{P}$, we associate to it the point $Q' = \overline{OQ} \cap \mathbf{P}'$, the point where the line through O and Q meets the plane $\mathbf{P}'$. But this 'association' does not always work: a point Q of $\mathbf{P}$ such that the line $\overline{OQ}$ is parallel to the plane $\mathbf{P}'$ does not have any point of $\mathbf{P}'$ associated to it while a point Q' of $\mathbf{P}'$ such that $\overline{OQ'}$ is parallel to $\mathbf{P}$ does not come from any point of $\mathbf{P}$.

The idea needed to remedy this situation is suggested by considering the above situations as limiting cases. Thus, if $\overline{OQ}$ is parallel to $\mathbf{P}'$, then Q is a limit of a sequence Q_n such that the lines $\overline{OQ_n}$ meet $\mathbf{P}'$; a little thought might convince the reader that the points $Q'_n = \overline{OQ_n} \cap \mathbf{P}'$ 'go off to infinity' in $\mathbf{P}'$. The situation of 'reappearing' points is explained similarly.

In the last century, Poncelet defined the *projective plane* by adding certain 'ideal points' which lie 'at infinity', according to the following rules.

1. Any affine ('finite') line is augmented by one point at infinity; such an augmented line is called a *projective line.*
2. Two affine lines are assigned the same point at infinity precisely when they are parallel.
3. The points at infinity all lie on a projective line, the 'line at infinity'.

Thus two affine lines (in the original 'finite' plane) are parallel precisely if the corresponding augmented projective lines meet at infinity. In the projective plane, any two distinct lines meet at exactly 1 point.

One introduces 'coordinates' in the projective plane as follows: a point P in the projective plane is represented by *three* numbers (or rather an ordered triplet), say (x, y, z), such that

(i) $(x, y, z) \neq (0, 0, 0)$
(ii) (x, y, z) and (x', y', z') represent the same point precisely when there is a non-zero number λ such that $(x', y', z') = (\lambda x, \lambda y, \lambda z)$.

These are called *homogeneous coordinates.* The 'finite' affine plane consists of the subset where $z \neq 0$; any point P in this subset has exactly one representative set of homogeneous coordinates of the form $(x, y, 1)$; now we regard the ordered pair (x, y) as the 'usual' (Cartesian) coordinates of the 'finite' point P. The points with $z = 0$ are the points at infinity.

A line in the Cartesian plane has an equation $ax + by + c = 0$ for some numbers a, b, c, where at least one of a, b is non-zero; this may be regarded as obtained by setting $z = 1$ in the homogeneous equation $ax + by + cz = 0$. More generally, let $f(x, y, z) = 0$ be a homogeneous polynomial equation (of positive degree). Then we can consider the *projective curve* of solutions of this equation, defined as the set of points P in the projective plane such that for any set of representative homogeneous coordinates (x_0, y_0, z_0) for P, we have $f(x_0, y_0, z_0) = 0$. This vanishing condition is unchanged if we substitute a different (proportional) set of homogeneous coordinates, since f is a *homogeneous* polynomial, so that $f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x, y, z)$, where d is the degree of f. If we consider the curve given by a linear equation $ax + by + cz = 0$ where $(a, b) \neq (0, 0)$, then there is exactly one point on this with $z = 0$, which has homogeneous coordinates $(b, -a, 0)$. Thus, such a projective line is obtained from the 'finite' (affine) line by adjoining 1 point at infinity. The line at infinity is defined by the linear equation $z=0$, which has degree 1, and so we are justified in calling it a line.

A line in $\mathbf{P}^2$ is determined by an ordered triplet of coefficients (a,b,c), not all zero, such that proportional triplets determine the same line. Thus, the lines in $\mathbf{P}^2$ are naturally parametrized by points of a projective plane $\check{\mathbf{P}}^2$, called the *dual projective plane*. Thus, there is a 'duality' between the projective plane and the 'dual' plane of lines, so that any assertion involving incidence of points and lines in one plane is equivalent to a 'dual' assertion in the other plane. For example, the 'dual' of a point $P=(x_0,y_0,z_0)$ is the set of lines passing through P, *which forms a line in the dual projective space*: the line (a,b,c) passes through P precisely when $ax_0+by_0+cz_0=0$ which is a homogeneous linear relation between a,b,c. The dual of Desargues theorem is its converse, while the dual of Brianҫon's theorem is *Pascal's theorem*: if a hexagon is inscribed in a conic, the 3 points of intersection of opposite pairs of sides are collinear. The relation of duality is explored in depth in the book of Hilbert and Cohn–Vossen.

3. Projective varieties

More generally, we may define projective n-space, denoted $\mathbf{P}^n$, using homogeneous coordinates, as the set of $(n+1)$-tuples $(x_0,x_1,\ldots,x_n)$, where at least one x_i is non-zero, and where proportional $(n+1)$-tuples represent the same point. The usual affine (or Euclidean) n-space is the subset consisting of points where $x_n\neq 0$, and a point with usual (Cartesian) coordinates $(a_1,\ldots,a_n)$ corresponds to the point with homogeneous coordinates $(a_1,\ldots,a_n,1)$. More generally, projective n-space is the union of $n+1$ subsets $U_i=\{x_i\neq 0\}$, $i=0,\ldots,n$, each of which is isomorphic to affine n-space. As in the case of the plane, the hyperplanes in $\mathbf{P}^n$ are again naturally parametrized by a projective space $\check{\mathbf{P}}^n$, the *dual projective space*.

Again, given a finite collection of homogeneous polynomials

$$f_1(x_0,\ldots,x_n),\ldots,f_r(x_0,\ldots,x_n),$$

we can consider the subset $V(f_1,\ldots,f_r)$ of projective space of points whose homogeneous coordinates satisfy the system of equations

$$f_1(x_0,\ldots,x_r)=\cdots=f_r(x_0,\ldots,x_r)=0.$$

These vanishing conditions remain unchanged if we replace the homogeneous coordinates by a proportional set. A set which can be described in the form $V(f_1,\ldots,f_r)$ is called a *projective algebraic variety* (or just projective variety). These are basic objects of study in algebraic geometry. Note that if $X\subset\mathbf{P}^n$ is a projective variety, then it is the union of subsets $X_i=X\cap U_i$, where $X_i\subset U_i\cong\mathbf{A}^n$ is an affine variety. This reduces 'local' properties of a projective variety (properties of the variety in a neighbourhood of a point) to similar properties of affine varieties, considered in Mohan Kumar's article (page 218, this issue); an example of a local property is *smoothness*. Thus in a sense, the importance of projective varieties stems from their good 'global' properties.

From the description of the projective n-space in terms of homogeneous coordinates, we see that it may be also described as *the set of lines through the origin* in an $(n+1)$-dimensional vector space. This is because any such line consists of all vectors proportional to a non-zero vector $(x_0,\ldots,x_n)$, which we call a generator of the line; another non-zero vector is a generator of the same line precisely when it has a proportional set of coordinates.

The vector space which is underlying the above discussion may be over any given field k; the corresponding projective space is denoted $\mathbf{P}^n_k$, if we want to call attention to the field under consideration. Examples for k are the real numbers $\mathbf{R}$ (when the vector space is the usual Euclidean space), the complex numbers $\mathbf{C}$, the field of rational numbers $\mathbf{Q}$, or the finite field $\mathbf{F}_q$ with q elements, where q is a power of a prime number (any finite field must have a prime power number of elements). A projective variety in $\mathbf{P}^n_k$ is said to be *defined over* k. The study of varieties defined over $\mathbf{Q}$ or a finite field $\mathbf{F}_q$ is equivalent to interesting problems on Diophantine equations (in number theory); see ref. 2 for an introduction. Notions from projective geometry in fact often help shed new light on such number theoretic problems. For example, one may consider the *Fermat curve* of degree n, defined in $\mathbf{P}^2$ by $x^n+y^n-z^n=0$. The still unproven assertion known as *Fermat's last theorem* describes the set of points of $\mathbf{P}^2_{\mathbf{Q}}$ on this curve (it says that if $n\geq 3$, there are no such points all of whose coordinates are non-zero). Considered as a curve over the complex number field $\mathbf{C}$, one can show that its *genus* (we will discuss this notion below) is $(n-1)(n-2)/2$, which is at least 2 if n is at least 4; a deep, general theorem of Faltings then implies that there are only a finite number of points of $\mathbf{P}^2_{\mathbf{Q}}$ on the Fermat curve.

Classically, one concentrated on the projective spaces over the field $\mathbf{C}$ of complex numbers. In this case, each complex line in $\mathbf{C}^{n+1}$ meets the unit sphere S^{2n+1} of $\mathbf{C}^{n+1}$ in a circle. In particular, there is a surjective mapping $f: S^{2n+1}\to\mathbf{P}^n_{\mathbf{C}}$ whose fibres are circles, which is called the *Hopf fibration*. Since $\mathbf{P}^n_{\mathbf{C}}$ is the continuous image of the compact space S^{2n+1}, $\mathbf{P}^n_{\mathbf{C}}$ is compact. Thus a projective variety over $\mathbf{C}$ is a closed subset of a compact space; hence any complex projective variety is a compact topological space.

Since $\mathbf{P}^n_{\mathbf{C}}$ is the union of $n+1$ open subsets each isomorphic to $\mathbf{C}^n$, $\mathbf{P}^n_{\mathbf{C}}$ is an n-dimensional complex manifold (hence has dimension $2n$ as a topological manifold). In particular, it makes sense to ask if a complex projective variety $X \subset \mathbf{P}^n_{\mathbf{C}}$ is a complex manifold in a neighbourhood of some point P; if this is the case, we call P *a smooth point* (or *non-singular point*) of X. If all points of X are smooth, i.e. X is a submanifold of $\mathbf{P}^n_{\mathbf{C}}$, then X is called a *non-singular projective variety*. These varieties are the principal objects of study in projective geometry, since it is often for these that one is able to obtain the deepest and most beautiful results. The study of properties of a more general variety X is often reduced to those of a collection of auxilliary non-singular projective varieties Y_i associated to X; thus a very important problem on singular (i.e. non-smooth) varieties is that of *resolution of singularities*, discussed in more detail in Abhyankar's article (see page 229, this issue).

The problem of classifying smooth projective varieties over $\mathbf{C}$ up to isomorphism is one of the basic problems on projective varieties, which has been the subject of much old and recent research, including that of the 1990 Fields medallist Mori; see ref. 3 for an introduction to these ideas.

The projective space $\mathbf{P}^n_{\mathbf{R}}$ over the real numbers is also a compact topological manifold of dimension n. Instead of the Hopf fibration, one has a fibration $S^n \to \mathbf{P}^n_{\mathbf{R}}$, which collapses (i.e. identifies) all pairs of antipodal points of S^n. The real projective spaces are important in topology; in particular, $\mathbf{P}^2_{\mathbf{R}}$ is perhaps the simplest compact, non-orientable surface without boundary. A theorem in topology states that $\mathbf{P}^n_{\mathbf{R}}$ is orientable precisely when n is odd. The space $\mathbf{P}^3_{\mathbf{R}}$ is the underlying manifold of the Lie group $SO_3(\mathbf{R})$ of rotations in Euclidean space $\mathbf{R}^3$.

4. Some examples

Example 4.1: Let X_λ be the subvariety of $\mathbf{P}^2_{\mathbf{C}}$ defined by the equation

$$f_\lambda(x,y,z) = x^3 + y^3 + z^3 + \lambda xyz = 0,$$

where λ is a constant, and x, y, z are the homogeneous coordinates on $\mathbf{P}^2_{\mathbf{C}}$. Thus for each complex number λ, X_λ is a plane cubic curve. One computes that if $\lambda^3 \neq -27$, then for each point $P \in X_\lambda$, at least one of the partial derivatives

$$\frac{\partial f_\lambda}{\partial x}, \frac{\partial f_\lambda}{\partial y}, \frac{\partial f_\lambda}{\partial z}$$

is non-zero at P. This implies that X_λ is non-singular. If $\lambda^3 = -27$, then X_λ has 3 singular points $(1, 1, -3/\lambda)$, $(1, \omega, -3\omega^2/\lambda)$, $(1, \omega^2, -3\omega/\lambda)$, where ω is a primitive cube root of 1.

Note that for each λ, a 'general' line intersects X_λ in 3 distinct points. This is a special case of Bezout's theorem, discussed below.

Example 4.2: Consider the mapping $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^3_{\mathbf{C}}$ given by $(s,t) \mapsto (s^3, s^2t, st^2, t^3)$, where s, t are homogeneous coordinates on $\mathbf{P}^1$. This is well defined, because $\lambda(s,t) = (\lambda s, \lambda t) \mapsto (\lambda^3 s^3, \lambda^3 s^2 t, \lambda^3 st^2, \lambda^3 t^3) = \lambda^3(s^3, s^2t, st^2, t^3)$. Let $X \subset \mathbf{P}^3$ be the image,

$$X = \{(s^3, s^2t, st^2, t^3) \mid s, t \in \mathbf{C}, \text{ and } s \neq 0 \text{ or } t \neq 0\}.$$

If x, y, z, w are the homogeneous coordinates in $\mathbf{P}^3$, we see that X is defined by the equations

$$f_1 = xw - yz = 0, f_2 = xz - y^2 = 0, f_3 = yw - z^2 = 0,$$

and further that any homogeneous polynomial vanishing on X is a linear combination $f_1g_1 + f_2g_2 + f_3g_3$, where the g_i are homogeneous of the same degree. However, X is also defined by the two equations

$$f_2 = xz - y^2 = 0,\ wf_1 - zf_3 = z^3 - 2yzw + xw^2 = 0,$$

as one sees by considering the cases $x \neq 0$, $x = 0$ separately. This is analogous to the situation for affine varieties described in Mohan Kumar's article: X is a *set theoretic complete intersection* in $\mathbf{P}^3_{\mathbf{C}}$. Kronecker's problem described there, for curves in $\mathbf{A}^3$, has an analogue in $\mathbf{P}^3_{\mathbf{C}}$: is every projective space curve a set theoretic complete intersection? However, unlike in the affine case, there is a topological restriction: we must consider only connected curves in $\mathbf{P}^3_{\mathbf{C}}$, since one can prove that any set theoretic complete intersection curve in $\mathbf{P}^3_{\mathbf{C}}$ is connected. This is a special case of general theorems relating the topology of a projective variety to that of its hypersurface sections; such results are known as *Lefschetz theorems* after S. Lefschetz, who first obtained interesting results of this type.

From the parametric representation of X, or from the above equations, one sees that a general plane in $\mathbf{P}^3$ intersects X in 3 distinct points. If X is any space curve, we define its *degree* to be the number of points of intersection with a general plane. It is not so easy to guess the degree of a space curve from its set of defining equations, as we see in the above example.

Example 4.3: The mapping $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^3_{\mathbf{C}}$ in the above example can be generalized as follows: for any positive integers n, d let $M_0, \ldots, M_N$ be the distinct monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$. Then there is a mapping $\mathbf{P}^n \to \mathbf{P}^N$, the *Veronese embedding*, given by $(x_0, \ldots, x_n) \mapsto (M_0, \ldots, M_N)$. One can show that this mapping is one-one, and its image is a smooth, projective variety in $\mathbf{P}^N$, isomorphic to $\mathbf{P}^n$. The simplest non-trivial example is when $n = 1$, $d = 2$, which embeds $\mathbf{P}^1$ as a conic in $\mathbf{P}^2$. Under the Veronese embedding $\mathbf{P}^n \to \mathbf{P}^N$ given by monomials of degree d, the inverse image of a hyperplane in $\mathbf{P}^N$ is a hypersurface in $\mathbf{P}^n$ of degree d, since it is the zero set of a linear homogeneous polynomial in $M_0, \ldots, M_N$, which is just a homogeneous polynomial in $x_0, \ldots, x_n$ of degree d. Conversely, every homogeneous polynomial of degree d is obtained this way. Hence the collection of hypersurfaces of degree d in $\mathbf{P}^n$ is naturally isomorphic to the collection of hyperplanes in $\mathbf{P}^N$, which is just the dual projective space $\mathbf{P}^{N}$. Thus, the collection of hypersurfaces of a fixed degree in $\mathbf{P}^n$ is itself a projective space.

We also see that if $X \subset \mathbf{P}^n$ is a hypersurface of degree d, then $\mathbf{P}^n - X$ is an affine variety, since it is isomorphic via the Veronese embedding to a closed subset of $\mathbf{P}^N - H \cong \mathbf{A}^N$ for some hyperplane H.

Example 4.4: Let $\mathbf{G}(k,n)$ denote the set of k-dimensional subspaces of $\mathbf{C}^n$. If $k=1$, $\mathbf{G}(1,n)=\mathbf{P}^{n-1}$. In general, $\mathbf{G}(k,n)$ has a natural structure as a smooth projective variety, the *Grassmann variety* (or *Grassmannian*). This is defined as follows.

If $W \subset \mathbf{C}^n$ is a k-dimensional subspace, then it has a basis $v_1,\ldots,v_k$, where $v_i=(a_{i1},\ldots,a_{in})$ are vectors in $\mathbf{C}^n$ which form a linearly independent set. The linear independence is equivalent to the statement that the $k\times n$ matrix with rows v_i, that is $A=[a_{ij}]$, has rank k, i.e. has a non-zero (maximal) $k\times k$ minor. If we choose another basis $w_1,\ldots,w_k$ of W, then $w_i=\Sigma_j b_{ij} v_j$ for an invertible $k\times k$ matrix $B=[b_{ij}]$, and the new matrix A' so obtained, with rows w_i, is given by $A'=BA$. Hence if M is any maximal minor of A, and M' the corresponding minor of A', then $M'=\det(B)M$. The number of distinct maximal minors is the binomial coefficient $\binom{n}{k}$, so that if $N=\binom{n}{k}-1$, then the formula

$$F(W)=(M_0,\ldots,M_N)\in\mathbf{P}^N,$$

assigning to each subspace W the collection of the maximal minors of an associated $k\times n$ matrix, gives a well-defined mapping

$$F:\mathbf{G}(k,n)\to\mathbf{P}^N.$$

One can show that

(i) this mapping is one-one, i.e. if $F(W_1)=F(W_2)$, then $W_1=W_2$
(ii) the image of F is a non-singular projective variety in $\mathbf{P}^N$.

Hence F may be used to give $\mathbf{G}(k,n)$ the structure of a projective variety. The mapping F is called the Plücker embedding, the above homogeneous coordinates on $\mathbf{G}(k,n)$ are called *Plücker coordinates*. There is an explicit description of the equations satisfied by $\mathbf{G}(k,n)$ in $\mathbf{P}^N$. These are called—you guessed it—the Plücker equations! For details, see ref. 4 chapter 1.

5. Intersection theory

Next, we touch on *intersection theory*. Some typical problems solved using intersection theory are the following.

1. How many lines (i.e. linearly embedded $\mathbf{P}^1$'s) in $\mathbf{P}^3$ meet 4 given lines $L_1,\ldots,L_4$ which are in general position? (Answer: 2)
2. How many lines are contained in a general smooth cubic surface in $\mathbf{P}^3$? (Answer: 27)
3. How many conics in $\mathbf{P}^2$ are simultaneously tangent to five given conics $C_1,\ldots,C_5$ which are in general position? (Answer: 3264)
4. How many lines are contained in a general hypersurface of degree 5 in $\mathbf{P}^4$? (Answer: an exercise for the ambitious reader!)

The first non-trivial result in intersection theory is *Bezout's theorem*, which states that if X, Y are distinct irreducible plane curves of degrees m,n respectively, then X and Y meet at exactly mn points, provided the number of intersections are counted 'properly'. For example, a 'simple' tangential intersection of two curves counts twice, while in general, a point of intersection is counted r times if there is an r-fold order of contact of the curves. To make sense of this, one needs to define the order of contact (or *intersection multiplicity*) of 2 curves at a point P; if x,y are local affine coordinates, so that P is the origin, consider the smallest number of monomials $M_1,\ldots,M_r$ in x,y such that if $f(x,y)=0$, $g(x,y)=0$ are the affine equations of the curves, then any power series $h(x,y)$ can be written as a linear combination

$$h(x,y)=a(x,y)f(x,y)+b(x,y)g(x,y)+\sum_j c_j M_j$$

for some power series a, b and constants c_j (one can prove that such a finite set of monomials exists). Then r is defined to be the intersection multiplicity of X and Y at P. For example, the intersection multiplicity at the origin of the curves $y^2-x^3=0$ and $x^2y+y^3+x^5=0$ is 7, since we can choose $\{1,x,x^2,x^3,x^4,y,xy\}$ as the set of monomials M_j. The somewhat subtle definition of intersection multiplicity is needed, as the reader can see from this example: there seems no 'obvious' reason why 7 is the correct value to assign to this intersection.

A simple application of Bezout's theorem is to the following problem: if $X\subset\mathbf{P}^2$ is a nonsingular plane curve of degree d, how many tangents to X pass through a general point $P\in\mathbf{P}^2$? If X is defined by the homogeneous polynomial equation $F(x,y,z)=0$, the tangent to X at (a,b,c) is $F_X(a,b,c)X+F_Y(a,b,c)Y+F_Z(a,b,c)Z=0$, where F_X, F_Y, F_Z are the partial derivatives. Let $P=(x_0,y_0,z_0)$ and let Y be the curve defined by the homogeneous polynomial equation $F_X(X,Y,Z)x_0+F_Y(X,Y,Z)y_0+F_Z(X,Y,Z)z_0=0$. Then the points of intersection of X and Y are precisely the points of X whose tangent lines pass through P. By Bezout's theorem, there are $d(d-1)$ such points. Hence $d(d-1)$ tangents to X pass through a general point P in the plane.

In higher dimensions, if X is a variety in $\mathbf{P}^n$ of dimension r, then a 'general' projective linear subspace $L \cong \mathbf{P}^{n-r}$ intersects X in a certain fixed, finite number of points, say d; then d is called the *degree* of X. The generalization of *Bezout*'s theorem to $\mathbf{P}^n$ states that if $X_1,\ldots,X_r$ are irreducible varieties in $\mathbf{P}^n$ of dimensions adding up to n, and of degrees $d_1,\ldots,d_r$ respectively, such that $X_1 \cap \cdots \cap X_r$ consists of finitely many points, then the number of points of intersection (counted properly) is the product $d_1 d_2 \cdots d_r$. Again, one needs to define local intersection multiplicities suitably, but this is more subtle than in the case of plane curves.

More generally, there is a theory of intersection multiplicities for defining intersection numbers of appropriate subvarieties of any smooth, projective variety. An interesting example is given by the *Schubert calculus*, which is a set of rules for computing intersection numbers of certain special subvarieties of Grassmannians, now called Schubert cycles. In the last century, Schubert had developed rules for computing these intersection numbers, which seemed to give the right answers in cases where the computations could be done rigorously another way. One of the famous *Hilbert problems* was to give a rigorous justification for the Schubert calculus. All of the questions posed at the beginning of this section can be reduced to Schubert calculus; the last two problems would perhaps be difficult to solve any other way.

There is a vast literature, both classical and modern, on intersection theory; a comprehensive modern source is Fulton's book[5].

6. Families of projective varieties

One example of a family of projective varieties is given by the family of all projective linear subspaces $\mathbf{P}^k$ in $\mathbf{P}^n$, which is just the Grassmannian $G(k+1, n+1)$. We have also seen in example 4.3 that the collection of all hypersurfaces of a fixed degree in $\mathbf{P}^n$ are naturally parametrized by a projective space.

More generally, we may consider all subvarieties of a fixed dimension r and degree d in $\mathbf{P}^n$. If X is such a variety, then for any $r+1$ 'general' hyperplanes $H_1,\ldots,H_{r+1}$, the intersection $X \cap H_1 \cap \cdots \cap H_{r+1}$ is empty, since intersecting with each hyperplane ought to reduce the dimension by 1. All $(r+1)$-tuples of hyperplanes are parametrized by the points of $(\mathbf{P}^n)^{r+1} = \mathbf{P}^n \times \cdots \times \mathbf{P}^n$. Let $x_{i0},\cdots,x_{in}$ be homogeneous co-ordinates on the ith factor $\mathbf{P}^n$. One can show that the subset of $(r+1)$-tuples such that the intersection is non-empty is the zero set in $(\mathbf{P}^n)^{r+1}$ of a polynomial $F(x_{1,0},\cdots,x_{r+1,n})$ which is separately homogeneous of degree d in each of the $r+1$ sets of $n+1$ variables; the polynomial is determined by X up to a constant factor. Thus the collection of all such subvarieties X is parametrized by the points in a certain subset of the projective space constructed using the vector space of such polynomials F. This subset is in fact a projective variety, called the *Chow variety*.

The construction of the Chow variety is the forerunner to the construction of other such families of varieties, and ultimately, the *moduli spaces* parametrizing isomorphism classes of varieties of a given type, like the moduli space of curves of genus g mentioned earlier. Moduli theory, particularly the moduli of *vector bundles*, has been the object of study of several Indian mathematicians, like M. S. Narasimhan, C. S. Seshadri, S. Ramanan, and several of their students and co-workers. Seshadri's article[6] gives a nice overview of moduli theory.

The Chow variety and other similar varieties, notably the *Hilbert scheme* defined by Grothendieck, lead to the construction of variety structures on other sets. For example, if X, Y are projective varieties, then the set Hom (X, Y) of all morphisms from X to Y has a natural structure as a variety; the idea is that if $X \times Y \subset \mathbf{P}^n$, the graph of a morphism from X to Y is a subvariety of $\mathbf{P}^n$ of a certain dimension and degree, hence determines a point in a suitable Chow variety. If we fix the degree of the graph, then we obtain a corresponding variety $\mathrm{Hom}^d(X,Y)$; then $\mathrm{Hom}(X,Y)$ is the disjoint union of all the $\mathrm{Hom}^d(X,Y)$. In particular, $\mathrm{Hom}(X,Y)$ is *finite dimensional* in a neighbourhood of any morphism, which is in contrast to the situation for affine varieties. A particular case is the variety Aut (X) of automorphisms of a projective variety; it is a disjoint union of (isomorphic) connected components, and the connected component containing the identity morphism is an *algebraic group* (a Lie group which is an algebraic variety). For example, Aut $(\mathbf{P}^n_{\mathbf{C}})$ is the algebraic group $PGL_{n+1}(\mathbf{C}) = GL_{n+1}(\mathbf{C})$/(scalar matrices), called the *projective linear group*.

In contrast, as observed in Mohan Kumar's article, the structure of the automorphism group of even the affine plane is very complicated; in particular, it is 'infinite dimensional'.

7. Complex projective varieties

Projective varieties over the field $\mathbf{C}$ of complex numbers have certain remarkable topological and analytic properties, which we briefly discuss here. The book[4] of Griffiths and Harris develops algebraic geometry from this perspective, and the reader can find detailed explanations there.

First, we mention a result of Chow, which generalizes a fact known earlier for Riemann surfaces. If $X \subset \mathbf{P}^n_{\mathbf{C}}$ is a smooth projective variety, so that it is a complex manifold in a natural way, then one can define meromorphic functions on X in the sense of complex analysis; these are functions which can be locally expressed as ratios of analytic functions. Chow's theorem states that any meromorphic function on X is in fact a rational function, i.e. a ratio of homogeneous polynomials of the same degree. This has been generalized by Serre[7] into a principle, usually referred to as GAGA (from the title of Serre's paper), that (loosely speaking) states that any 'global analytic object' defined on a complex projective variety X is in fact algebraic. For example, any complex submanifold of a smooth projective variety X is an algebraic subvariety; any analytic mapping between complex projective varieties is a morphism of algebraic varieties; any holomorphic differential form on X is an algebraic differential form (a polynomial differential form on each affine open subset of X, in the sense considered in Mohan Kumar's article); any analytic vector bundle on X is algebraic.

The second general property relates to the so-called *cohomology groups* of a smooth projective variety; again this generalizes results which go back to the work of Riemann related to the Dirichlet principle. Recall that on a manifold M of dimension n, a (complex) differential r-form ω is an object (actually, a skew-symmetric tensor) locally defined by an expression

$$\omega = \sum_{1 \le i_1 < i_2 \cdots < i_r \le n} f_{i_1 i_2 \ldots i_r} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r},$$

where $x_1, \ldots, x_n$ are local coordinates, and the coefficients $f_{i_1 \ldots i_r}$ are complex valued functions (differentiable to any order); the 'wedge product' $\wedge$ is skew-symmetric, and the dx_i transform under a change of coordinates by the rule $\mathrm{d}y_i = \sum_j \frac{\partial y_i}{\partial x_j} \mathrm{d}x_j$. A 0-form is just a function. If $\Omega^r(M)$ is the C-vector space of r-forms, there is an operation $\mathrm{d}: \Omega^r(M) \to \Omega^{r+1}(M)$, called *exterior differentiation*, given by the rules

$$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2, \ \mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i \text{ if } f \text{ is a function}$$

and

$$\mathrm{d}(\omega_1 \wedge \omega_2) = (\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^r \omega_1 \wedge \mathrm{d}\omega_2$$

if ω_1 is an r-form. An r-form ω is called *closed* if $\mathrm{d}\omega = 0$; it is called *exact* if $\omega = \mathrm{d}\eta$ for an $(r-1)$-form η. One computes from the definitions that $\mathrm{d}(\mathrm{d}(\eta)) = 0$ for any form η, so that exact forms are closed. The rth *cohomology group* of a manifold M is defined as the quotient space

$$H^r(M, \mathbf{C}) = \frac{\text{closed } r\text{-forms}}{\text{exact } r\text{-forms}}$$

If M is compact, then one knows that $H^r(M, \mathbf{C})$ is a finite dimensional C-vector space, whose dimension is called the rth *Betti number* of X, denoted by $b_r(X)$. Thus if $b_r(X) = s$, we can find s closed r-forms $\omega_1, \ldots, \omega_s$ such that for any closed r-form ω, there are unique complex constants c_i such that $\omega - \Sigma_i c_i \omega_i$ is exact.

Riemann showed that if X is (in present terminology) a compact Riemann surface of genus g (i.e. topologically, X is a g-holed torus), then X has g linearly independent holomorphic 1-forms, and these forms and their complex conjugates give a basis for the first cohomology group $H^1(X, \mathbf{C})$. Of course, Riemann did not use the language of cohomology, which was invented later.

This was generalized by Hodge, as follows. If X is a complex projective variety, the cohomology groups $H^r(X, \mathbf{C})$ have a decomposition into C-subspaces

$$H^r(X, \mathbf{C}) = H^{r,0} \oplus H^{r-1,1} \oplus \cdots \oplus H^{0,r},$$

where $H^{r,0}$ is the vector space of holomorphic r-forms, and the remaining $H^{i,j}$ have a similar description in terms of 'harmonic' forms. There is a natural complex conjugation on $H^r(X, \mathbf{C})$ (since the exterior derivative is *a real* differential operator), and the decomposition has the property that $\overline{H^{t,r-t}} = H^{r-t,t}$. In particular, $H^{t,r-t}$ and $H^{r-t,t}$ have the same dimension. For example, this means that the odd Betti numbers of a complex projective variety are even. The decomposition is called the *Hodge decomposition*, and the study of related properties of varieties is called *Hodge theory*. Originally, this was done for smooth projective varieties, but an extension to arbitrary varieties (called 'mixed' Hodge theory) has been given by Deligne, using resolution of singularities.

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