ALGEBRA OF THE DIRAC-MATRICES

By Harish-Chandra

(*J. H. Bhabha Student, Cosmic Ray Research Unit, Indian Institute of Science, Bangalore*)

Received April 30, 1945

(Communicated by Prof. H. J. Bhabha, F.A.SC., F.R.S.)

§ 1. The Dirac matrices γ_a $(a=1, 2, 3, 4)$ are characterised by the commutation rules

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\,\delta_{ab}. \tag{1}$$

These four matrices give rise to a set of 16 quantities

$$\left.\begin{array}{l} 1 \\ \gamma_a \\ i\gamma_a\gamma_b \quad (a \neq b) \\ i\gamma_a\gamma_b\gamma_c \ (a, b, c \text{ all different}) \\ \gamma_1\gamma_2\gamma_3\gamma_4 \end{array}\right\} \tag{2}$$

which is closed under multiplication if we regard two quantities which differ by a numerical factor -1, i or $-i$ as essentially the same. As is well known these sixteen matrices are linearly independent and apart from equivalence possess only one four-dimensional representation which is irreducible. Several important identities concerning these matrices, which are independent of any particular representation have been established by Pauli (1936) by making use of the well-known results following from Schur's theorem. The object of the present paper is to point out that the commutation rules of the above 15 quantities (all excluding 1) can be expressed quite elegantly by one single formula and that the above-mentioned identities can therefore be derived directly from the commutation rules. It seems that perhaps the present method is more general and powerful than that of Pauli. The identities are obtained in a form such that the five elements of any pentad (see Eddington, 1936) can be regarded as basal elements. The use of the matrix B of Pauli is avoided so that the identities (34_1) and (34_6) of his paper can now be generalised to the case $\phi^+ \neq \psi^+$, $\phi \neq \psi$. This was not possible previously. Some new tensor identities are also obtained.

Further the present method yields quite easily the matrix determinant of a quantity composed linearly from the above sixteen matrices. As is

30

well known this determinant is independent of the representation used. Fddington (1936) has already calculated it by using the rather cumbersome method of employing a particular representation. As a physical application of the evaluation of this determinant we shall consider the case of a charged particle of spin $\frac{1}{2}$ having an *explicit* spin interaction with the electromagnetic field. It will be shown that apart from quantum effects and upto the first approximation the particle behaves as if it possessed a pure magnetic dipole moment, which points either along or opposite to the magnetic field in the rest system. This magnetic moment which arises from the explicit spin interaction is to be distinguished from the usual magnetic moment of the electron which is due purely to quantum effects, and which would therefore disappear if the non-commutability of the different operators were ignored.

§ 2. For the purpose of this paper it is more convenient to multiply the γ's by i and use

$$E_a \equiv i\gamma_a \tag{3}$$

instead of γ_a. Therefore

$$E_aE_b + E_bE_a = -2\,\delta_{ab} \tag{4}$$

The matrices E_1, E_2, E_3, E_4 anticommute and their squares are equal to -1. Put*

$$E_5 \equiv iE_1E_2E_3E_4 \tag{5}$$

so that

$$E_5{}^2 = -1,\ E_aE_5 = -E_5E_a\ (a = 1, 2, 3, 4). \tag{6}$$

Following Eddington we define

$$\begin{aligned} E_{\mu\nu} &\equiv E_\mu E_\nu \qquad (\mu, \nu = 1, 2, 3, 4, 5, \mu \neq \nu) \\ \left.\begin{aligned} E_{0\nu} &\equiv E_\nu \\ E_{\nu 0} &\equiv -E_{0\nu} = -E_\nu \end{aligned}\right\} & (\nu = 1, 2, \cdots, 5) \\ E_{\nu\nu} &\equiv 0 \qquad (\nu = 0, 1, \cdots 5). \end{aligned} \tag{7}$$

Then the following equations hold (*cf.* Eddington *l.c.*) for $\mu, \nu = 0, 1, \cdots 5$

$$E_{\mu\nu} = -E_{\nu\mu} \tag{8a}$$

$$E_{\mu\nu}{}^2 \equiv E_{\mu\nu}E_{\mu\nu} = -1\ (\mu \neq \nu) \tag{8b}$$

$$E_{\mu\nu}E_{\mu\rho} = E_{\nu\rho}\ (\mu, \nu, \rho \text{ all different}) \tag{8c}$$

$$E_{\mu\nu}E_{\sigma\rho} = E_{\sigma\rho}E_{\mu\nu} = \mp iE_{\lambda\tau}\ (\mu, \nu, \sigma, \rho, \lambda, \tau \text{ all different}) \tag{8d}$$

* The present definition of E_5 differs in sign from that of Eddington. The advantage is that now

$$E_5 = iE_1E_2E_3E_4 = i\gamma_1\gamma_2\gamma_3\gamma_4 = i\gamma_5$$

corresponding to (3). γ_5 is the same as in Pauli's paper.

A3

In (8*d*) the positive or the negative sign is to be chosen according as (μ, ν, σ, ρ, λ, τ) is an odd or even permutation of (0, 1, 2, 3, 4, 5). It is easy to see that (8) is equivalent to the single equation

$$E_{\lambda\mu} E_{\nu\rho} = -\delta_{\lambda\nu}\delta_{\mu\rho} + \delta_{\mu\nu}\delta_{\lambda\rho} + E_{\lambda\nu}\delta_{\mu\rho} - E_{\mu\nu}\delta_{\lambda\rho} - E_{\lambda\rho}\delta_{\mu\nu} + E_{\mu\rho}\delta_{\lambda\nu} - \frac{i}{2}\epsilon_{\lambda\mu\nu\rho\sigma\tau} E^{\sigma\tau} \tag{9}$$

Here $\delta_{\mu\nu}$ is the usual Kronecker's symbol

$$\delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

and $\epsilon_{\lambda\mu\nu\rho\sigma\tau}$ is antisymmetric in all 6 indices and $\epsilon_{012345} = 1$. It is convenient to make the convention that the same index appearing once below and once above in the same term implies a summation. Thus for example

$$E_\rho{}^\nu E_{\nu\sigma} = \sum_{\nu=0}^{5} E_{\rho\nu} E_{\nu\sigma}$$

while no summation is intended in the expression

$$E_{\rho\nu} E_{\nu\sigma}$$

In fact (9) can be looked upon as a tensor equation in a six-dimensional space whose metric tensor is $\delta_{\mu\nu}$. Equation (9) is invariant to any orthogonal transformation of this six-dimensional space, if we regard $E_{\lambda\mu}$ as an antisymmetric tensor. Also if $F_{\lambda\mu}$ is any set of fifteen E-numbers†, antisymmetric in λ, μ and satisfying the same commutation rules as (9), then it is not difficult to prove that

$$F_{\lambda\mu} = a_\lambda{}^\nu a_\mu{}^\rho E_{\nu\rho}$$

where $a_\lambda{}^\nu$ are the coefficients of an orthogonal transformation of the six-dimensional space, *i.e.*,

$$\sum_\nu a_\lambda{}^\nu a_\mu{}^\nu = \delta_{\lambda\mu}$$
$$| a_\lambda{}^\nu | = 1$$

where $| a_\lambda{}^\nu |$ denotes the six-dimensional determinant of the transformation. Thus every matrix transformation

$$F_{\lambda\mu} = \Lambda E_{\lambda\mu} \Lambda^{-1}$$

is equivalent to an orthogonal transformation of the six-dimensional space. The converse is also true due to the equivalence of all four-dimensional representations.

† Any linear combination of the E's and 1 is called an E-number (*cf.* Eddington, *l.c.*). Every matrix with 4 rows and 4 columns is an E-number due to the linear independence of the E's and 1.

For future use we note the following relations which follow directly from (9):—

$$E_{\lambda\mu} E_{\nu\rho} + E_{\lambda\nu} E_{\mu\rho} = -\delta_{\lambda\nu}\delta_{\mu\rho} - \delta_{\lambda\mu}\delta_{\nu\rho} + 2\delta_{\mu\nu}\delta_{\lambda\rho} + E_{\lambda\nu}\delta_{\mu\rho} + E_{\lambda\mu}\delta_{\nu\rho} - 2E_{\lambda\rho}\delta_{\mu\nu} + E_{\mu\rho}\delta_{\lambda\nu} + E_{\nu\rho}\delta_{\lambda\mu} \qquad (10a)$$

$$E_{\sigma}{}^{\nu} E_{\nu\rho} = 5\delta_{\sigma\rho} - 4E_{\sigma\rho} \qquad (10b)$$

Also if

$$S = s + s_{\lambda\mu} E^{\lambda\mu} \quad (s_{\lambda\mu} = -s_{\mu\lambda}) \qquad (11a)$$

$$T = t + t_{\lambda\mu} E^{\lambda\mu} \quad (t_{\lambda\mu} = -t_{\mu\lambda}) \qquad (11b)$$

where s, $s_{\lambda\mu}$, t, $t_{\lambda\mu}$ are ordinary numbers it follows from (9) that

$$ST = st - 2s_{\lambda\mu} t^{\lambda\mu} + \left(st_{\lambda\mu} + ts_{\lambda\mu} - 4s_{\lambda\nu} t^{\nu}{}_{\mu} - \frac{i}{2} s^{\alpha\beta} t^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta\lambda\mu}\right) E^{\lambda\mu} \qquad (12)$$

so that

$$ST - TS = -8 s_{\lambda\nu} t^{\nu}{}_{\mu} E^{\lambda\mu} \qquad (13a)$$

$$ST + TS = 2st - 4s_{\lambda\mu} t^{\lambda\mu} + 2\left(st_{\lambda\mu} + ts_{\lambda\mu} - \frac{i}{2} s^{\alpha\beta} t^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta\lambda\mu}\right) E^{\lambda\mu} \qquad (13b)$$

In particular on choosing $S = E_{\alpha\beta}$, (13) gives

$$TE_{\alpha\beta} - E_{\alpha\beta} T = 4\left(t_{\alpha\lambda} E^{\lambda}{}_{\beta} - t_{\beta\lambda} E^{\lambda}{}_{\alpha}\right) \qquad (14a)$$

$$TE_{\alpha\beta} + E_{\alpha\beta} T = -4t_{\alpha\beta} + 2tE_{\alpha\beta} - it^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta\lambda\rho} E^{\lambda\rho} \qquad (14b)$$

On contracting with $t^{\lambda\mu}$ (10a) yields

$$TE_{\nu\rho} + t^{\lambda\mu} E_{\lambda\nu} E_{\mu\rho} = -3t_{\nu\rho} + 3t_{\nu}{}^{\lambda} E_{\lambda\rho} - t_{\rho}{}^{\lambda} E_{\lambda\nu} + T\delta_{\nu\rho} + (E_{\nu\rho} - \delta_{\nu\rho})\, t \qquad (15)$$

Multiplying (14a) by $E^{\beta}{}_{\gamma}$ on the right and using (15) we obtain

$$E_{\alpha\beta} TE^{\beta}{}_{\gamma} = T\delta_{\alpha\gamma} + 4(\delta_{\alpha\gamma} - E_{\alpha\gamma})\, t + 4\left(t_{\alpha}{}^{\beta} E_{\beta\gamma} + t_{\gamma}{}^{\beta} E_{\beta\alpha}\right) - 8t_{\alpha\gamma} \qquad (16)$$

On contracting α, γ (16) gives the well-known result

$$2T - E^{\alpha\beta} TE_{\alpha\beta} = 32t.$$

Multiplying (14a) by $E_{\alpha\beta}$ on the right we get

$$E_{\alpha\beta} TE_{\alpha\beta} = -T + 4t_{\lambda\alpha} E^{\lambda}{}_{\alpha} + 4t_{\lambda\beta} E^{\lambda}{}_{\beta} - 8t_{\alpha\rho} E_{\alpha\beta} \quad (\alpha \neq \beta) \qquad (17)$$

As is well known (*cf.* Pauli, 1936) it follows from the commutation rules (9) that the spur of $E_{\lambda\mu}$ is zero. Therefore for any four-rowed representation

$$sp\,(T) = 4t$$

$$sp\,(E_{\lambda\mu} T) = -8t_{\lambda\mu}$$

Also since 1, $E_{\lambda\mu}$ are 16 linearly independent matrices T in (11b) can be any arbitrary matrix of 4 rows and 4 columns. Let ψ and ϕ be any two matrices with 4 rows and 1 column and ψ^{+} and ϕ^{+} with 1 row and 4

A3 a

34 Harish-Chandra

columns. Then $\phi\phi^+$, $\psi\psi^+$, $\phi\psi^+$ and $\psi\phi^+$ are square matrices with 4 rows and 4 columns and we can choose T equal to any of them. We notice that for $\mathrm{T}=\phi\phi^+$

$$t=\tfrac{1}{4}\, sp\,(\mathrm{T})=\tfrac{1}{4}\,\phi^+\phi \tag{18a}$$

$$t_{\lambda\mu}=-\tfrac{1}{8}\, sp\,(\mathrm{E}_{\lambda\mu}\,\mathrm{T})=-\tfrac{1}{8}\,\phi^+\mathrm{E}_{\lambda\mu}\,\phi. \tag{18b}$$

Substituting these values in (11*b*) and (14*b*) and multiplying by ψ^+ on the left and ψ on the right we obtain

$$\psi^+\phi\cdot\phi^+\psi=\tfrac{1}{4}\,\psi^+\psi\cdot\phi^+\phi-\tfrac{1}{8}\,\psi^+\mathrm{E}^{\lambda\mu}\,\psi\cdot\phi^+\mathrm{E}_{\lambda\mu}\phi \tag{19}$$

$$\begin{aligned}\psi^+\phi\cdot\phi^+\mathrm{E}_{\alpha\beta}\psi+\psi^+\mathrm{E}_{\alpha\beta}\phi\cdot\phi^+\psi&=\tfrac{1}{2}\,\psi^+\psi\cdot\phi^+\mathrm{E}_{\alpha\beta}\phi+\tfrac{1}{2}\,\psi^+\mathrm{E}_{\alpha\beta}\psi\cdot\phi^+\phi\\&\quad+\frac{i}{8}\,\phi^+\mathrm{E}^{\gamma\delta}\phi\cdot\psi^+\mathrm{E}^{\lambda\rho}\psi\cdot\epsilon_{\alpha\beta\gamma\delta\lambda\rho}\end{aligned} \tag{20}$$

On choosing $\mathrm{T}=\psi\phi^+$ (16) gives in the same way

$$\psi^+\mathrm{E}_{\alpha\beta}\psi\cdot\phi^+\mathrm{E}^{\beta}{}_{\alpha}\phi=\psi^+\psi\cdot\phi^+\phi+\psi^+\phi\cdot\phi^+\psi-\psi^+\mathrm{E}_{\alpha\beta}\phi\cdot\phi^+\mathrm{E}^{\beta}{}_{\alpha}\psi \tag{21}$$

$$\begin{aligned}\psi^+\mathrm{E}_{\alpha\beta}\psi\cdot\phi^+\mathrm{E}^{\beta}{}_{\gamma}\phi&=-\psi^+\mathrm{E}_{\alpha\gamma}\phi\cdot\phi^+\psi+\psi^+\phi\cdot\phi^+\mathrm{E}_{\alpha\gamma}\psi\\&\quad-\tfrac{1}{2}\,\psi^+\mathrm{E}_{\beta\gamma}\phi\cdot\phi^+\mathrm{E}_{\alpha}{}^{\beta}\psi+\tfrac{1}{2}\,\psi^+\mathrm{E}_{\beta\alpha}\phi\cdot\phi^+\mathrm{E}_{\gamma}{}^{\beta}\psi,\ (\alpha\neq\gamma).\end{aligned} \tag{22}$$

Similarly on putting $\mathrm{T}=\phi\phi^+$ (17) gives

$$\begin{aligned}\psi^+\mathrm{E}_{\alpha\beta}\,\phi\cdot\phi^+\mathrm{E}_{\alpha\beta}\,\psi&=-\psi^+\phi\cdot\phi^+\psi-\tfrac{1}{2}\,\phi^+\mathrm{E}_{\lambda\alpha}\,\phi\cdot\psi^+\mathrm{E}^{\lambda}{}_{\alpha}\,\psi\\&\quad-\tfrac{1}{2}\,\phi^+\mathrm{E}_{\lambda\beta}\phi\cdot\psi^+\mathrm{E}^{\lambda}{}_{\beta}\psi+\psi^+\mathrm{E}_{\alpha\beta}\psi\cdot\phi^+\mathrm{E}_{\alpha\beta}\phi,\ (\alpha\neq\beta).\end{aligned} \tag{23}$$

Now let *a*, *b*, *c*, *d* be indices which run from 1 to 4 only. Then following Pauli (1936) we put

$$\left.\begin{aligned}\psi^+\psi&\equiv i\Omega_0\\ \psi^+\mathrm{E}_{0a}\psi=\psi^+\mathrm{E}_a\psi=i\psi^+\gamma_a\psi&\equiv i\mathrm{S}_a\\ \psi^+\mathrm{E}_{a5}\psi=-\psi^+\gamma_a\gamma_5\psi=i\psi^+\hat{\gamma}_a\psi&\equiv i\hat{\mathrm{S}}_a\\ \psi^+\mathrm{E}_{ab}\psi=-\psi^+\gamma_a\gamma_b\psi&\equiv\mathrm{M}_{ab}\,(a\neq b)\\ \mathrm{M}_{aa}&=0\\ \psi^+\mathrm{E}_{05}\psi=i\psi^+\gamma_5\psi&\equiv i\Omega_5\\ \hat{\mathrm{M}}_{ab}&=\tfrac{1}{2}\,\epsilon_{abcd}\,\mathrm{M}^{cd}\end{aligned}\right\} \tag{24}$$

Here

$$\gamma_5\equiv\gamma_1\gamma_2\gamma_3\gamma_4=\mathrm{E}_1\mathrm{E}_2\mathrm{E}_3\mathrm{E}_4=-i\mathrm{E}_5$$

and

$$\hat{\gamma}_a\equiv i\gamma_a\gamma_5=\frac{i}{3!}\,\epsilon_{abcd}\,\gamma^b\gamma^c\gamma^d$$

where the tensor ϵ_{abcd} is antisymmetric in all the four indices and $\epsilon_{1234}=1$. If the corresponding quantities constructed from ϕ^+, ϕ be distinguished by a dash, (19) can be written in the following form:

$$\psi^+\phi\cdot\phi^+\psi = -\tfrac{1}{4}\,\Omega_0\Omega_0' + \tfrac{1}{4}\,S_aS'^a + \tfrac{1}{4}\,\hat{S}_a\hat{S}'^a - \tfrac{1}{8}\,M_{ab}M'^{ab} + \tfrac{1}{4}\,\Omega_5\Omega_5' \tag{25}$$

Choosing $\alpha = 0$, $\beta = 5$ in (20) we get

$$i\psi^+\phi\cdot\phi^+\gamma_5\psi + i\psi^+\gamma_5\phi\cdot\phi^+\psi = -\tfrac{1}{2}\,\Omega_0\Omega_5' - \tfrac{1}{2}\,\Omega_5\Omega_0' + \frac{i}{4}\,M^{ab}\hat{M}_{ab}' \tag{26}$$

Simlarly putting $\alpha = 0$ in (21) we obtain

$$S_aS'^a + \Omega_5\Omega_5' = -\,\Omega_0\Omega_0' + \psi^+\phi\cdot\phi^+\psi - \psi^+\gamma_a\phi\cdot\phi^+\gamma_a\psi - \psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi. \tag{27}$$

On the other hand we get on taking $\alpha = 5$

$$\hat{S}_a\hat{S}'^a + \Omega_5\Omega_5' = -\,\Omega_0\Omega_0' + \psi^+\phi\cdot\phi^+\psi - \psi^+\hat{\gamma}^a\phi\cdot\phi^+\hat{\gamma}_a\psi - \psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi. \tag{28}$$

On putting $\alpha = 0$, $\gamma = 5$ (22) gives

$$\begin{aligned} -\,S_a\hat{S}'^a = {} & -\,i\psi^+\gamma_5\phi\cdot\phi^+\psi + i\phi^+\gamma_5\psi\cdot\psi^+\phi \\ & + \tfrac{1}{2}\,\phi^+\gamma_a\psi\cdot\psi^+\hat{\gamma}^a\phi + \tfrac{1}{2}\,\psi^+\gamma_a\phi\cdot\phi^+\hat{\gamma}^a\psi \end{aligned} \tag{29}$$

Also if we put $\alpha = 0$, $\beta = 5$ in (23) we get

$$-\,\psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi = -\,\phi^+\psi\cdot\psi^+\phi + \tfrac{1}{2}\,S_aS'^a + \tfrac{1}{2}\,\hat{S}_a\hat{S}'^a \tag{30}$$

which is the same as equation (44 P).* From (30) and (25) we get

$$-\,\psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi - \psi^+\phi\cdot\phi^+\psi = \tfrac{1}{2}\,\Omega_0\Omega_0' + \tfrac{1}{8}\,M_{ab}M'^{ab} - \tfrac{1}{2}\,\Omega_5\Omega_5' \tag{31}$$

which corresponds to equation (43 P). Equations (26), (27), (28), (29) and (31) can be written in the following form:

$$\begin{aligned} \tfrac{1}{4}\,M^{ab}\hat{M}'_{ab} - \frac{i}{2}\,(\Omega_0\Omega_5' + \Omega_5\Omega_0') = {} & (\psi^+\phi\cdot\phi^+\gamma_5\psi - \phi^+\phi\cdot\psi^+\gamma_5\psi) \\ & + (\phi^+\psi\cdot\psi^+\gamma_5\phi - \psi^+\psi\cdot\phi^+\gamma_5\phi) \end{aligned} \tag{32}$$

$$\begin{aligned} 2\,S_aS^{a'} = {} & -2\,\Omega_0\Omega_0' - 2\,\Omega_5\Omega_5' - (\psi^+\gamma_a\phi\cdot\phi^+\gamma^a\psi - \psi^+\gamma_a\psi\cdot\phi^+\gamma^a\phi) \\ & - (\psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi - \psi^+\gamma_5\psi\cdot\phi^+\gamma_5\phi) + (\psi^+\phi\;\phi^+\psi - \psi^+\psi\phi^+\phi) \end{aligned} \tag{33}$$

$$\begin{aligned} 2\,\hat{S}_a\hat{S}'^a = {} & -2\,\Omega_0\Omega_0' - 2\,\Omega_5\Omega_5' - (\psi^+\hat{\gamma}_a\phi\cdot\phi^+\hat{\gamma}^a\psi - \psi^+\hat{\gamma}_a\psi\cdot\phi^+\hat{\gamma}^a\phi) \\ & - (\psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi - \psi^+\gamma_5\psi\cdot\phi^+\gamma_5\phi) + (\psi^+\phi\cdot\phi^+\psi - \psi^+\psi\cdot\phi^+\phi) \end{aligned} \tag{34}$$

$$\begin{aligned} -2\,S_a\hat{S}'^a = {} & i\,\{\phi^+\gamma_5\psi\cdot\psi^+\phi - \psi^+\gamma_5\phi\cdot\phi^+\psi\} \\ & + \tfrac{1}{2}\,\{\phi^+\gamma_a\psi\cdot\psi^+\hat{\gamma}^a\phi + \phi^+\hat{\gamma}_a\psi\cdot\psi^+\gamma^a\phi - 2\,\phi^+\hat{\gamma}_a\phi\cdot\psi^+\gamma^a\psi\} \end{aligned} \tag{35}$$

$$\begin{aligned} \tfrac{1}{2}\,M_{ab}M'^{ab} = {} & \Omega_0\Omega_0' - \Omega_5\Omega_5' - 2\,(\psi^+\phi\cdot\phi^+\psi - \psi^+\psi\cdot\phi^+\phi) \\ & - 2\,(\psi^+\gamma_5\phi\cdot\phi^+\gamma_5\psi - \phi^+\gamma_5\phi\psi^+\gamma_5\psi) \end{aligned} \tag{36}$$

* P refers to Pauli's paper.

36 Harish-Chandra

Equations (32) to (36) are the generalisation of equations (34_4 P), (34_1 P), (34_3 P), (34_5 P) and (34_2 P) respectively. Equations (32) and (36) have already been given by Pauli as equations (47 P) and (43 P). The others were not obtained by him. It is noteworthy that we have derived the *identities directly without employing the matrix B of Pauli. One more* interesting identity can be derived by interchanging ϕ^+ and ψ^+ in (23) and putting $\alpha = 0,\ \beta = 5$

$$\tfrac{1}{2}\,\phi^+\gamma_a\psi\cdot\psi^+\gamma^a\phi + \tfrac{1}{2}\,\phi^+\hat{\gamma}_a\psi\cdot\psi^+\hat{\gamma}^a\phi = -\,\Omega_5\Omega_5' - \Omega_0\Omega_0'.$$

However on putting $\phi^+ = \psi^+$ and $\phi = \psi$ it degenerates merely into the sum of (34_1 P) and (34_3 P).

It may be mentioned here that it is possible to derive *tensor* identities *in addition to the invariant identities* given above, by choosing other suitable sets of values for α, β and γ in the equations (20) to (22). On putting $\phi = \psi$ and $\phi^+ = \psi^+$ the following identities are obtained:

$$\hat{M}_{ab}\hat{S}^b - \Omega_0 S_a = 0 \quad [\alpha = 0,\ \beta = a \text{ in } (20)] \tag{37a}$$

$$\hat{M}_{ab}S^b + \Omega_0\hat{S}_a = 0 \quad [\alpha = a,\ \beta = 5 \text{ in } (20)] \tag{37b}$$

$$M_{ab}\Omega_5 + i\Omega_0\hat{M}_{ab} + i\,(S_a\hat{S}_b - S_b\hat{S}_a) = 0 \quad [\alpha = a,\ \beta = b \text{ in } (20)] \tag{37c}$$

$$M_{ab}S^b + i\Omega_5\hat{S}_a = 0 \quad [\alpha = 0,\ \gamma = a \text{ in } (22)] \tag{*37d}$$

$$M_{ab}\hat{S}^b - i\Omega_5 S_a = 0 \quad [\alpha = a,\ \gamma = 5 \text{ in } (22)] \tag{37e}$$

$$S_aS_b + \hat{S}_a\hat{S}_b + M_{ac}M^c{}_b = -\,\delta_{ab}\Omega_0{}^2 \quad [\alpha = a,\ \beta = b \text{ in (21) or (22)}] \tag{37f}$$

The generalised identities for the case $\phi \neq \psi$ and $\phi^+ \neq \psi^+$ can be obtained by similar substitutions and they need not be given here explicitly.

§ 3. Now we shall calculate the matrix determinant of T. As is well known this determinant, which we denote by det. T is the same for all four-dimensional representations of $E_{\lambda\mu}$. In fact it is equal to the independent term in the characteristic equation of T. It is therefore sufficient to determine the characteristic equation. For this purpose we make use of (12) and find that

$$T - t = t_{\lambda\mu}E^{\lambda\mu}$$

$$(T - t)^2 = -\,2\,t_{\mu\nu}t^{\mu\nu} - \frac{i}{2}\,t^{\alpha\beta}t^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho}$$

$$\{(T - t)^2 + 2\,t_{\mu\nu}t^{\mu\nu}\}^2 - \tfrac{1}{2}\,t^{\alpha\beta}t^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta\lambda\rho}\epsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'}t_{\alpha'\beta'}t_{\gamma'\delta'}$$
$$= \frac{i}{8}\,\epsilon^{\alpha\beta\mu\nu\sigma\tau}t_{\mu\nu}t_{\sigma\tau}\epsilon^{\gamma\delta\mu'\nu'\sigma'\tau'}t_{\mu'\nu'}t_{\sigma'\tau'}\epsilon_{\alpha\beta\gamma\delta\lambda\rho}E^{\lambda\rho}$$

* This identity was mentioned by Prof. Bhabha in a lecture

$$= \frac{i}{8}\,\epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau}\, 16\,(t_{\alpha\beta} t_{\lambda\rho} - t_{\lambda\beta} t_{\alpha\rho} - t_{\alpha\lambda} t_{\beta\rho})\, \mathrm{E}^{\lambda\rho}$$

$$= 2\, i \epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau}\,(t_{\alpha\beta} t_{\lambda\rho} \mathrm{E}^{\lambda\rho} - 2\, t_{\alpha\lambda} t_{\beta\rho} \mathrm{E}^{\lambda\rho})$$

$$= 2\, i \epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau}\,(\mathrm{T} - t) - 4\, i\; t_{\alpha\lambda} t_{\beta\rho} \epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} \mathrm{E}^{\lambda\rho} \tag{38}$$

Now

$$t_{\alpha\lambda} t_{\beta\rho} \epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\mu\nu} t_{\sigma\tau} = \tfrac{1}{6}\, t_{\lambda\rho} \epsilon^{\alpha\beta\mu\nu\sigma\tau} t_{\alpha\beta} t_{\mu\nu} t_{\sigma\tau} \tag{39}$$

(39) is easily verified for a 'tensor' $t_{\mu\nu}$ whose only non-vanishing components are t_{01}, t_{23}, t_{45}. Since by a suitable orthogonal transformation every antisymmetrical tensor $t_{\mu\nu}$ can be brought into this form, it follows that the invariant equation (39) holds for every $t_{\mu\nu}$. Also

$$\epsilon_{\alpha\beta\gamma\delta\lambda\rho} \epsilon^{\lambda\rho\alpha'\beta'\gamma'\delta'} t_{\alpha'\beta'} t_{\gamma'\delta'} = 16\,(t_{\alpha\beta} t_{\gamma\delta} - t_{\alpha\gamma} t_{\beta\delta} - t_{\alpha\delta} t_{\gamma\beta}). \tag{40}$$

Substituting (39) and (40) in (38) we get

$$\{(\mathrm{T} - t)^2 + 2\, t_{\mu\nu} t^{\mu\nu}\}^2 - 8\,\{(t^{\alpha\beta} t_{\alpha\beta})^2 - 2\, t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha}\}$$
$$- \tfrac{4}{3}\, i\, t_{\alpha\beta} t_{\gamma\delta} t_{\lambda\rho} \epsilon^{\alpha\beta\gamma\delta\lambda\rho}\,(\mathrm{T} - t) = 0. \tag{41}$$

Equation (41) is the characteristic equation of T. Putting $\mathrm{T} = 0$ on the left side of (41) we obtain the term independent of T so that

$$\det \mathrm{T} = (2\, t_{\mu\nu} t^{\mu\nu} + t^2)^2 - 8\,\{(t^{\alpha\beta} t_{\alpha\beta})^2 - 2\,(t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha})\}$$
$$+ \tfrac{4}{3} i\, t_{\alpha\beta} t_{\gamma\delta} t_{\lambda\rho} \epsilon^{\alpha\beta\gamma\delta\lambda\rho} t \tag{42}$$

(42) agrees with the result given by Eddington* (1936). For the purpose of the following discussion it is convenient to replace T by T′ where

$$\mathrm{T}' = t + \tfrac{1}{2}\, t_{\lambda\mu} \mathrm{E}^{\lambda\mu} \tag{43}$$

so that

$$\det \mathrm{T}' = (t^2 + \tfrac{1}{2}\, t_{\mu\nu} t^{\mu\nu})^2 - \tfrac{1}{2}\,\{(t^{\alpha\beta} t_{\alpha\beta})^2 - 2\, t^{\alpha\beta} t_{\beta\gamma} t^{\gamma\delta} t_{\delta\alpha}\}$$
$$+ \frac{i}{6}\, t_{\alpha\beta} t_{\gamma\delta} t_{\lambda\rho} \epsilon^{\alpha\beta\gamma\delta\lambda\rho} t \tag{44}$$

Let us now revert from $\mathrm{E}^{\lambda\mu}$ to the original matrices E_a and their products. Obviously T′ can be put in the form

$$\mathrm{T}' = t + t_a \mathrm{E}^a + \tfrac{1}{2}\, t_{ab} \mathrm{E}^{ab} + \frac{i}{6}\, \epsilon_{abcd} \mathrm{E}^a \mathrm{E}^b \mathrm{E}^c s^d$$
$$+ \frac{1}{24}\, s\, \epsilon_{abcd} \mathrm{E}^a \mathrm{E}^b \mathrm{E}^c \mathrm{E}^d \tag{45}$$

Here a, b, c, d run from 1 to 4 only and t_a and s_a are four-dimensional vectors while s is a scalar. From (5) we have

* Eddington has chosen $\mathrm{E}_5 = -\, i \mathrm{E}_1 \mathrm{E}_2 \mathrm{E}_3 \mathrm{E}_4$ and therefore in his case the sign of i in (42) is reversed.

$$\frac{1}{24}\ \epsilon_{abcd}E^aE^bE^cE^d = -\,iE_5 = -\,iE_{05} \tag{46a}$$

$$\tfrac{1}{6}\ \epsilon_{abcd}E^aE^bE^c = iE_5E_d = -\,iE_{d5}. \tag{46b}$$

On comparing (43) and (45) and using (46) we find

$$\left.\begin{aligned} t_{0a} &= t_a \\ t_{a5} &= s_a \\ t_{05} &= s. \end{aligned}\right\} \tag{47}$$

For brevity the following notation for any vector A_a or a tensor B_{ab} is introduced

$$|\,A_a\,|^2 \equiv A_aA^a \tag{48a}$$

$$|\,B_{ab}\,|^2 \equiv \tfrac{1}{2}\,B_{ab}B^{ab} \tag{48b}$$

Then

$$\begin{aligned}\tfrac{1}{2}\,t^{\mu\nu}t_{\mu\nu} &= t_{0a}t^{0a} + |\,t_{ab}\,|^2 + t_{05}t^{05} + t_{a5}t^{a5} \\ &= |\,t_a\,|^2 + |\,t_{ab}\,|^2 + |\,s_a\,|^2 + s^2 \end{aligned}\tag{49}$$

Also put

$$(t^2)_{\alpha\gamma} = (t^2)_{\gamma\alpha} \equiv t_\alpha{}^\beta t_{\beta\gamma} = t_\alpha{}^0t_{0\gamma} + t_\alpha{}^at_{a\gamma} + t_\alpha{}^5t_{5\gamma}$$

so that

$$(t^2)_{00} = -\,s^2 - |\,t_a\,|^2 \tag{50a}$$

$$(t^2)_{0b} = t^at_{ab} - ss_b \tag{50b}$$

$$(t^2)_{ab} = -\,t_at_b + t_a{}^ct_{cb} - s_as_b \tag{50c}$$

$$(t^2)_{a5} = -\,st_a + t_a{}^bs_b \tag{50d}$$

$$(t^2)_{05} = t^a\,s_a \tag{50e}$$

$$(t^2)_{55} = -\,s^2 - |\,s_a\,|^2 \tag{50f}$$

Thus we have

$$\begin{aligned} t_{\alpha\beta}t^{\beta\gamma}t_{\gamma\delta}t^{\delta\alpha} &= (t^2)_{\alpha\gamma}(t^2)^{\alpha\gamma} \\ &= (t^2)_{00}(t^2)^{00} + (t^2)_{55}(t^2)^{55} + 2\,(t^2)_{05}(t^2)^{05} + 2\,(t^2)_{0a}(t^2)^{0a} \\ &\quad + 2\,(t^2)_{a5}(t^2)^{a5} + (t^2)_{ab}(t^2)^{ab} \\ &= (s^2 + |\,t_a\,|^2)^2 + (s^2 + |\,s_a\,|^2)^2 + 2\,(t^as_a)^2 + 2\,|\,t^at_{ab} - ss_b\,|^2 \\ &\quad + 2\,|\,t_a{}^bs_b - st_a\,|^2 + 2\,|\,t_at_b + s_as_b - t_a{}^ct_{cb}\,|^2 \end{aligned}\tag{51}$$

From (44), (47), (49) and (51) we have

$$\begin{aligned}\text{det. } T' &= (t^2 + |\,t_a\,|^2 + |\,t_{ab}\,|^2 + |\,s_a\,|^2 + s^2)^2 - 2\,(|\,t_a\,|^2 + |\,t_{ab}\,|^2 + |\,s_a\,|^2 + s^2)^2 \\ &\quad + (s^2 + |\,t_a\,|^2)^2 + (s^2 + |\,s_a\,|^2)^2 + 2\,(t^as_a)^2 + 2\,|\,t^at_{ab} - ss_b\,|^2 \\ &\quad + 2\,|\,t_a{}^bs_b - st_a\,|^2 + 2\,|\,t_at_b + s_as_b - t_a{}^ct_{cb}\,|^2 \\ &\quad + i\,s\,t_{ab}t_{cd}\epsilon^{abcd}t + 4i\,t_at_{bc}s_d\epsilon^{abcd}t \end{aligned}\tag{52}$$

since

$$t_{\alpha\beta}t_{\gamma\delta}t_{\lambda\rho}\epsilon^{\alpha\beta\gamma\delta\lambda\rho} = 6\,t_{05}t_{ab}t_{cd}\epsilon^{05abcd} + 24\,t_{0a}t_{bc}t_{d5}\epsilon^{0abcd5}$$

Algebra of the Dirac-Matrices 39

Instead of E_a or γ_a which satisfy (4) or (1) respectively it is convenient to introduce the matrices a_μ ($\mu = 0, 1, 2, 3$) whose commutation rules are

$$a_\mu a_\nu + a_\nu a_\mu = 2\, g_{\mu\nu} \tag{53}$$

where $g_{\mu\nu}$ is the usual metric tensor of flat space-time ($g_{\mu\nu} = 0$, $\mu \neq \nu$, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$). For this purpose it is sufficient to choose $(a_1, a_2, a_3) = (E_1, E_2, E_3)$ and $a_0 = -iE_4 = \gamma_4$. From now onwards the greek indices run from 0 to 3. We make the convention that for any tensor $A_{ab\ldots}$

$$i^n A_{(0)_n} = A_{(4)_n} \tag{54}$$

where $A_{(4)_n}$ denotes that among $a, b, \cdots$ there are n indices equal to 4. The new quantity $A_{(0)_n}$ defined by (54) is obtained by replacing each index 4 by 0. Since for raising the greek indices we use the tensor $g^{\mu\nu}$ the following relation holds

$$A_{\mu\nu} \ldots B^{\mu\nu} \cdots = (-1)^m A_{ab} \ldots B^{ab} \cdots$$

where m is the total number of indices $a, b, \cdots$ in each of the tensors $A_{ab} \ldots$ and $B^{ab} \cdots$ which are contracted together. Also from (54)

$$\epsilon_{0123} = -\epsilon_{1230} = i\epsilon_{1234} = i.$$

Therefore we put

$$\epsilon_{\mu\nu\sigma\tau} = i\, \eta_{\mu\nu\sigma\tau}$$

where $\eta_{0123} = 1$. Thus (45) and (52) can be written as

$$T' = t - t_\mu a^\mu + \tfrac{1}{2} t_{\mu\nu} a^\mu a^\nu - \tfrac{1}{6} \eta_{\mu\nu\sigma\tau}\, a^\mu a^\nu a^\sigma s^\tau$$
$$- \frac{1}{24} s\, \eta_{\mu\nu\sigma\tau}\, a^\mu a^\nu a^\sigma a^\tau \tag{55}$$

$$\det T' = (t^2 - |t_\mu|^2 + |t_{\mu\nu}|^2 - |s_\mu|^2 + s^2)^2 - 2(-|t_\mu|^2 + |t_{\mu\nu}|^2$$
$$- |s_\mu|^2 + s^2)^2$$
$$+ (s^2 - |t_\mu|^2)^2 + (s^2 - |s_\mu|^2)^2 + 2(t^\mu s_\mu)^2 - 2|ss_\mu + t^\nu t_{\nu\mu}|^2$$
$$- 2|st_\mu + t_\mu{}^\nu s_\nu|^2 + 2|t_\mu t_\nu + s_\mu s_\nu + t_\mu{}^\sigma t_{\sigma\nu}|^2$$
$$- s\, t_{\mu\nu} t_{\sigma\tau} \eta^{\mu\nu\sigma\tau} t - 4\, t_\mu t_{\nu\sigma} s_\tau\, \eta^{\mu\nu\sigma\tau} t. \tag{56}$$

A notation similar to (48) has been employed in (56) for greek indices also.

The above result can immediately be applied to discuss the case of a particle of spin $\frac{1}{2}$ having a charge- and dipole-interaction with an electromagnetic (or meson) field. If p_μ be the energy-momentum vector of the particle, ϕ_μ the electromagnetic potentials and $F_{\mu\nu}$ the field-strengths the wave equation for the particle can be written as

$$\{a^\mu (p_\mu - g_1 \phi_\mu) + \frac{i}{2} g_2 a^\mu a^\nu F_{\mu\nu} + m\}\, \psi = 0. \tag{57}$$

Here g_1 is the charge and g_2 the dipole-strength and m the mass of the particle. Our object is to determine the classical analogue of this particle. Hence we treat p_μ, ϕ_μ, $F_{\mu\nu}$ as numerical quantities commuting with each other. This corresponds to the neglect of quantum effects. As is well known the condition for the existence of a solution of (57) is that

$$\det\{\alpha^\mu\pi_\mu + \frac{i}{2} g\alpha^\mu\alpha^\nu F_{\mu\nu} + m\} = 0. \tag{58}$$

Here $\pi_\mu = p_\mu - g_1\phi_\mu$ and we have written g instead of g_2 for simplicity. (58) corresponds to the classical equation of motion of the particle. Comparing (58) and (55) we find

$$\begin{aligned} t &= m \\ t_\mu &= -\pi_\mu \\ t_{\mu\nu} &= igF_{\mu\nu} \\ s_\mu &= 0,\ s = 0 \end{aligned}$$

so that on account of (56), (58) reduces to

$$(\pi_\mu\pi^\mu - m^2 + \tfrac{1}{2} g^2F_{\mu\nu}F^{\mu\nu})^2 - 4\,\pi_\mu\pi^\mu\,(\tfrac{1}{2} g^2F_{\nu\sigma}F^{\nu\sigma}) - 2\,(\tfrac{1}{2} g^2F_{\mu\nu}F^{\mu\nu})^2$$
$$- 4\,g^2\pi_\mu F^{\mu\nu}F_{\nu\sigma}\pi^\sigma + g^4F_\mu{}^\nu F_{\nu\sigma}F^{\sigma\tau}F_\tau{}^\mu = 0. \tag{59}$$

Now

$$F_{\mu\nu}F^{\nu\sigma}F_{\sigma\tau}F^{\tau\mu} = 2\,(\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu})^2 + (\tfrac{1}{2} F_{\mu\nu}F^{*\mu\nu})^2 \tag{60}$$

where $F^{*\mu\nu}$ is the tensor dual to $F^{\mu\nu}$ and is defined by

$$F^{*\mu\nu} = \tfrac{1}{2}\eta^{\mu\nu\sigma\tau}\,F_{\sigma\tau} \tag{61}$$

Equation (60) is easily verified for the particular Lorentz-frame in which only F_{01} and F_{23} are different from zero. Since it is a tensor equation it must hold for every other frame also. Thus we obtain from (59) and (60)

$$[\pi_\mu\pi^\mu - m^2 + \tfrac{1}{2} g^2F_{\mu\nu}F^{\mu\nu}]^2 + 4\,\{(\tfrac{1}{4} g^2F_{\mu\nu}F^{*\mu\nu})^2 - \pi_\mu\pi^\mu\,(\tfrac{1}{2} g^2F_{\nu\sigma}F^{\nu\sigma})$$
$$- g^2\pi_\mu F^{\mu\nu}F_{\nu\sigma}\pi^\sigma = 0. \tag{62}$$

When the explicit spin interaction is absent $g = 0$ and (62) reduces to the usual classical equations of a point-charge

$$\pi_\mu\pi^\mu - m^2 = 0. \tag{63}$$

In this case

$$\pi_\mu = m\,v_\mu \tag{64}$$

where v_μ is the classical four-velocity of the particle. Since we wish to retain only terms of the lowest order in g in (62) we can substitute (64) in the terms of (62) containing g, so that on ignoring terms of the order g^3, (62) becomes

$$(\pi_\mu\pi^\mu - m^2 + \tfrac{1}{2} g^2F_{\mu\nu}F^{\mu\nu})^2 - 4\,m^2g^2\,(\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} - F_\mu F^\mu) \tag{65}$$

where $F_\mu = F_{\mu\nu} v^\nu$. Therefore

$$\pi_\mu \pi^\mu - m^2 = \pm\, 2\, mg\, \sqrt{\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} - F_\mu F^\mu} \tag{66}$$

if the second and higher powers of g are neglected. Now

$$\tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} - F_\mu F^\mu = \mathbf{H}^2$$

where $\mathbf{H}$ is the magnetic field in the rest system of the particle. Therefore

$$\pi_\mu \pi^\mu - m^2 = \pm\, 2\, mg\, |\,\mathbf{H}\,| \tag{67}$$

For the non-relativistic case we find on putting $\pi_0 = m + W$ that

$$W = \frac{\pi^2}{2\,m} \pm g\,|\,\mathbf{H}\,| \tag{68}$$

where $\pi = (\pi_1, \pi_2, \pi_3)$. (68) shows that in the rest system and in a weak electromagnetic field the particle manifests only a *magnetic moment* g such that the direction of the moment is either along or opposite to the magnetic component of the field. This is precisely what is to be expected of a particle of spin $\tfrac{1}{2}$.

I am thankful to Prof. H. J. Bhabha for helpful comments and advice.

SUMMARY

The Dirac-matrices generate an algebra consisting of sixteen linearly independent elements. A formula is given for expressing the product of any two elements as a linear combination of these sixteen. This determines the structure of the algebra completely. It is shown that certain known identities concerning these matrices can be obtained comparatively easily by the present method. Some new identities are also deduced.

The characteristic equation of a general element of the algebra is derived and from it an expression is obtained for the determinant of any four-dimensional matrix representing the element. This expression is used to discuss the case of a particle of spin $\tfrac{1}{2}$ having an explicit spin interaction with the electromagnetic field. It is shown that in the classical limit $\hbar \to o$ and upto the first approximation in the interaction constant g the particle manifests only a *magnetic-moment* g in the rest system, the direction of the moment being either along or opposite to the magnetic field in the same system.

REFERENCES

1. Eddington .. *Relativity theory of electrons and protons*, Cambridge University Press, 1936.

2. Pauli .. *Ann. Inst. Poincaré*, 1936, **6** 109–36.