Pancharatnam's route to the geometric phase

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This article focuses on the route by which Pancharatnam was led to the geometric phase. Barring a near-miraculous result from spherical trigonometry which he pulls out of the air, it is seen to be a systematic attack on problems of analysis and synthesis of polarized light, entirely following from just one basic principle and presented as a series of propositions. It is also noted that the phase was quite central to the interpretation of his own experiments and to his other theoretical work.

An 'unexpected geometrical result'

It is now quite well known (see articles by Berry and Bhandari in this issue and the references therein) that Pancharatnam, in his studies of the interference of polarized light, derived a phase angle which is an early example of the phase now recognized to occur more generally when a quantum system traverses a path in its space of states. In this article, the focus is on the precise chain of reasoning which led to the original conclusion[1]. The starting point is of course the Poincaré sphere representation of the state of polarization (see the accompanying box for a brief introduction). Pancharatnam uses only one property of the sphere to start with and deduces everything else that he needs. The major steps go as follows:

1) In his own words, (placed in quotes from now on with my added comments in square brackets) the fundamental property is that 'When a vibration of intensity I in the state of polarization C is decomposed into two vibrations in the opposite [orthogonal] states of polarization A and A', the intensities of the 'A-component' and the 'A′-component' are $I\cos^2\frac{1}{2}CA$ and $I\sin^2\frac{1}{2}CA$ [or $I\cos^2\frac{1}{2}CA'$] respectively'. The next remark is that since the sum of these two intensities is a constant, one obtains no constructive or destructive interference (in the sense of intensity variations sensitive to phase differences) when two such orthogonal beams are superposed.

2) The first problem tackled is the interference of two beams 1 and 2 with intensities I_1 and I_2 in non-orthogonal polarization states which are represented by points A and B on the Poincaré sphere. These are separated by an angle c (see Figure 1). The method used is to decompose 2 into (i) a component with intensity $I_2\sin^2(c/2)$ with polarization state A', orthogonal, i.e. diametrically opposite to A and (ii) a component of intensity $I_2\cos^2(c/2)$ with the same polarization state as beam 1, namely A. The interference of the two beams in the *same* polarization state follows the usual rule, viz. sum of the individual intensities and an interference term with twice the geometric mean of the two intensities modulated by the cosine of the phase difference. $I_A = I_1 + I_2\cos^2(c/2) + 2\sqrt{I_1}(\sqrt{I_2}\cos(c/2))\cos\delta$ The angle δ is the phase difference between the beam 1 and the component of beam 2 resolved along the polarization state A of 1. One then simply adds the intensity of the A′ component to this since, by step 1 above, there is no interference between orthogonal beams. The resulting formula for the intensity is

$$I_3 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(c/2)\cos\delta. \qquad (1)$$

3) The next step is deceptively simple. The angle δ was introduced as the phase difference between the beam 1 and the A′ component of 2. But '... we will be guilty of no internal inconsistency if we make the following statement by way of a definition: *the phase advance of one polarized beam over another* (not necessarily in the same state of polarization) *is the amount by which its phase must be retarded relative to the second, in order that the intensity resulting from their mutual interference may be a maximum*'. The consistency referred to is that δ so defined does not change if the intensities of the beams vary and changes in the expected way if either of the two beams is changed in phase. A truly mathematical spirit at work here! Thus is established a convention for comparing the phases of any two non-orthogonal states.

4) Pancharatnam now attacks the question which equation (1) has left unanswered – what is the polariz-

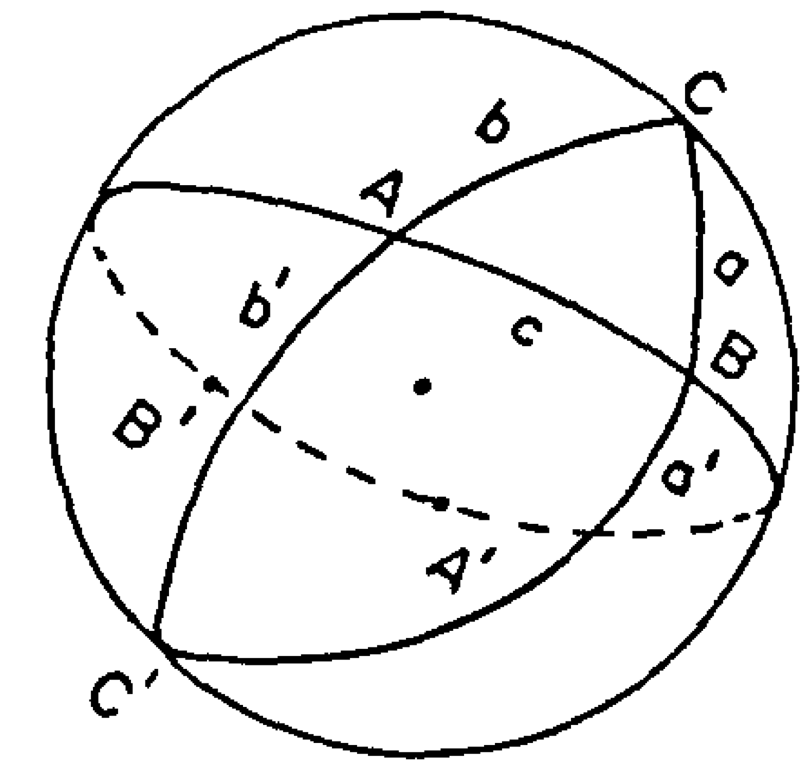


Figure 1.

ation state of the beam 3 made by combining beams 1 and 2 in step 2 above? On the Poincaré sphere in the figure, it is denoted by C, and the sides of the triangle ABC are labelled according to the usual convention, after their opposite angles. One can modify the question and ask for the intensities and phases of beams in polarization states A and B which combine to give intensity I_3 in the polarization state C. The beautiful device used at this point is to resolve both beams 1 and 2 along the state C′ which is orthogonal to C. Although both beams 1 and 2 have C′ components, these components must be equal in intensity and opposite in phase so that their resultant, the C′ component of the beam 3, which is in the state of polarization C, vanishes. *Using the intensity condition, we have*

$$I_1 \cos^2(b'/2) = I_2 \cos^2(a'/2). \quad (2)$$

This fixes the ratio of the intensities I_1 and I_2. A similar argument which fixes the ratio of I_1 and I_3 goes as *follows. Adding beam 1, with phase reversed, to 3 will* produce beam 2 in polarization state B. Hence, the B′ components of 1 and 3 *must have equal intensities. This leads to the condition*

$$I_3 \sin^2(a/2) = I_1 \sin^2(c/2). \quad (3)$$

Equations (2) and (3) fix the ratio of the three intensities to be the same as that of the squared sines of the opposite sides (arcs) to the points A, B, C representing the three polarization states, i.e.

$$\frac{I_1}{\sin^2(a/2)} = \frac{I_2}{\sin^2(b/2)} = \frac{I_3}{\sin^2(c/2)}. \quad (4)$$

5) Now that we know the intensities of beams 1 and 2, what about their relative phase? This information is already contained in equation (1) above which can now be used to determine δ knowing all the other quantities. Pancharatnam determines the phase as any experimenter would, by using intensity measurements! Using the *intensities from (4) in (1), the cosine of the phase* difference is given by

Box 1. *A recurring theme emerging from many of* the articles in this issue is that the Poincaré sphere and the related Stokes parameters became, in Pancharatnam's hands, powerful geometric tools for the understanding of polarized light. This box has the limited goal of supplying, *in one place for convenient reference*, the algebraic preliminaries, definitions and conventions which connect these two approaches with each other, and with the standard description of polarized light in terms of two simple harmonic motions of the same frequency. Considering a monochromatic plane wave travelling in the z direction, we have

$$E_x = a_1 \cos(\omega t + \phi_1) = \mathbf{Re}[a_1\, e^{-i(\omega t + \phi_1)}], \quad (1)$$

and a *similar equation with amplitude a_2 and* phase ϕ_2 for the y component. It is convenient to define two complex amplitudes, $z_1 = a_1 e^{-i\phi_1}$ and $z_2 = a_2 e^{-i\phi_2}$, and combine the pair into a column vector $\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ *(warning to electrical engineers, +i* produces a phase lag). Stokes, in the mid-nineteenth century, emphasized that the measurable *quantities* in optics (of his time, no femtosecond laser pulses!) were intensities, averaged over times much longer than the period $2\pi/\omega$. Further, the effect of propagation through crystals polarizers, etc. was all to form *linear* combinations of these two complex numbers. Thus, the result of any one intensity measurement would be given by the sum of expressions like

$$I = |(c_1 z_1 + c_2 z_2)|^2 = (c_1 z_1 + c_2 z_2)\overline{(c_1 z_1 + c_2 z_2)}, \quad (2)$$

(using bar for complex conjugation). The coefficients c_1 and c_2 *vary from one experiment to* another. For example, a polarizer placed at 45° to the x axis would have $c_1 = 1/\sqrt{2}$, $c_2 = 1/\sqrt{2}$. If the same polaroid were preceded by a quarter-wave plate with the fast axis along y, then the coefficients would be given by $c_1 = i/\sqrt{2}$, $c_2 = 1/\sqrt{2}$. This latter arrangement is called a right circular analyser since it annihilates the 90° phase lead which the x *component of right circular light has, the result being linear polarization along 45° which is then transmitted fully*. All the information about the properties of the *light* is contained in the four combinations which occur in equation (2), namely $z_1\bar{z}_1, z_2\bar{z}_2, z_1\bar{z}_2$ and $z_2\bar{z}_1$. Notice that these quantities do not have the rapid time variation at frequency ω and any phase common to both z_1 and z_2 will not affect them, while the phase *difference* between the two vibrations is of course all-important. These four quantities are naturally displayed as a Hermitian matrix, given by

$$\psi\psi^\dagger = \begin{pmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & z_2\bar{z}_2 \end{pmatrix}. \quad (3)$$

The famous Stokes parameters are four real numbers, I, Q, U, V which contain the same information as the elements of the matrix above

$$-\cos\delta = \frac{(-I_3 + I_2 + I_1)}{2\sqrt{I_1 I_2}\cos(c/2)}$$

$$= \frac{(-1 + \cos^2(c/2) + \cos^2(a'/2) + \cos^2(b'/2))}{2\cos(c/2)\cos(a'/2)\cos(b'/2)}, \quad (5)$$

6) Now one would think the problem is solved. Here is an explicit expression for the cosine of the required phase δ, in terms of the parameters which describe the mutual disposition of the points A, B and C (actually C′) on the Poincaré sphere. What more could one want? It is intrinsic in that it makes no reference to any coordinate system. But what comes at this stage in the paper is the statement that, 'The expression on the right-hand side is the cosine of half the solid angle subtended by the triangle C′BA at the centre of the sphere (see McClelland and Preston, 1897, Part II Ch. 7, p. 50, Ex. 1). [!!] Since the Poincaré sphere has unit radius, we arrive at the following unexpected geometrical result. When a beam of polarization C is decomposed into two beams in the states of polarization A and B respectively, the phase difference δ between these beams is given by $|\delta| = \pi - \frac{1}{2}|E'|$ where the angle E' is numerically equal to the area of the triangle C′BA which is colunar to ABC.' Colunar is a quaint-sounding but classical term for the relationship of two triangles sharing the same *lune*, i.e. crescent enclosed between two great circles. What I find remarkable is that Pancharatnam either (i) was a black belt in spherical trigonometry and instantly recognized the right hand side of equation (5) or (ii) felt instinctively that the expression must really be something more meaningful than some arbitrary combination of trigonometric functions, and hence searched the available books on the subject. My own guess is (ii)!

The paper goes on to develop the analysis and synthesis problem in detail, and the case of A and B orthogonal, which would be conventionally regarded as easier, actually needs some limiting processes because the phase convention breaks down. For our purposes, I will take a short cut and show how the 'unexpected geometric result' stated above is only one step away from the geometric phase. Remember the argument

Henri Poincaré (1854–1912)

Poincaré is acknowledged by all to have been the greatest living mathematician at the turn of the century. During the three decades he was at the University of Paris, he authored 500 scientific papers, scores of monographs and books. He gave lectures at the University every year in different subjects covering all physics and all mathematics. It is fortunate that many of these remarkable lectures have appeared in print, for they are not just reviews but contains gems of original thought. In one such treatise Theorie Mathematique de la Lumiere (3 vols) he introduces a concept of remarkable beauty – of representing polarized light on a sphere (now called the Poincaré sphere).

but have a clearer physical interpretation. One writes the same matrix as

$$\frac{1}{2}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}. \quad (4)$$

Clearly, I is the total intensity $z_1\bar{z}_1 + z_2\bar{z}_2$ and Q is a measure of linear polarization since it is the difference between what is transmitted by an analyser along x and one along y, $Q = z_1\bar{z}_1 - z_2\bar{z}_2$. Positive values of Q correspond to polarization along x and negative values to polarization along y. Less obviously, U measures linear polarization along 45° when positive and 135° when negative, since it is the difference between the intensities transmitted by analysers aligned along these two directions, $U = z_1\bar{z}_2 + z_2\bar{z}_1$. Finally, V is the difference between intensities recorded in two measurements made with a right and left circular analyser respectively. Therefore V is given by $V = i(z_1\bar{z}_2 - z_2\bar{z}_1)$. Thus V is positive for right circular light and negative for left circular light. (The convention used here is that right circular carries angular momentum parallel to the direction of propagation, i.e. has positive helicity, while left circular carries angular momentum opposite to the direction of propagation. Thus, for right circular, the electric field at a fixed point in space rotates in the positive sense, i.e. from x to y, which in turn implies that E_x leads E_y, $z_1 = -iz_2$ and therefore V as defined by equation (4) is positive. Be warned that the opposite convention was once in vogue and is followed in Pancharatnam's papers! According to this convention, the pattern of electric vectors at a fixed instant of time makes a right-handed screw in space for right circular light.)

A look at equations (3) and (4) shows how brilliant and prescient Stokes' analysis was. Like a sleepwalker, he is constructing the density matrix of a quantum mechanics still eighty years leading to equation (2) above, which required the cancellation of beams 1 and 2 when resolved along C′. This means that their phase difference, after resolving along C′, must be π. But their phase difference δ as it stands was just derived to be $\pi - \frac{1}{2}E'$. Clearly, an additional phase of $E'/2$ has appeared in the process of resolving A and B along C′, i.e. half the solid angle subtended by C′BA. This same result, but stated for the triangle CBA, actually appears a few pages later in the paper but is already implicit at this earlier stage, as applied to the triangle C′BA.

The generalised theory of interference

Having dwelt in some detail on how Pancharatnam was led to discover the geometric phase for polarized light, it is time now to briefly place this in a broader context. This paper is just no. I in a series of papers with the common title 'The generalized theory of interference'. The first paper, only part of which has been discussed above, carries the subtitle 'Coherent pencils' and the second deals with partially polarized light, generalizing the Poincaré representation and supplying solutions to the problems of analysis and synthesis in this case as well (see Radhakrishnan's article). The article by Ranganath in this issue goes over the investigations on the optics of absorbing anisotropic crystals with which Pancharatnam started his research career. In principle, he could have interpreted his results on the conoscopic figures (see box accompanying Ranganath's article) in a straightforward but prosaic way in terms of the analytical approach, with which he was quite familiar. Clearly, he had set himself a higher goal, which in his words (Paper III of the series) was to give a 'unified and physically intelligible approach to the interference phenomena exhibited by crystalline plates in parallel or convergent light – under general conditions when the polarizing and analysing states are linear, circular, or elliptic in form': The Poincaré sphere suited his purpose admirably, especially after he had generalized it. It is clear on reading his papers on the interference figures

ahead and expanding it in terms of Pauli matrices still to be born! (Actually, Stokes' contemporary and competitor, Hamilton had already invented the Pauli matrices via his quaternions but the story of this even greater sleepwalker does not belong here.)

The use of *four* real parameters seems an overkill because we started with two complex numbers and then threw away one overall phase, so we should need only three. In fact, the four parameters as defined earlier obey the identity

$$I^2 = Q^2 + U^2 + V^2, \qquad (5)$$

which is valid only for completely polarized light. However, the true power of Stokes' creation is in situations in which the relative amplitudes and phases of the two components fluctuate in a statistical fashion. One then defines the four parameters I, Q, U, V as the time averages of the expressions given above. Once one averages, the relation (5) above is no longer true – after all, the average of Q^2 is not the same as the square of the average of Q! In fact, for unpolarized light, I continues to be the total intensity, while Q, U, V all average to zero. In a general, possibly partially polarized situation, the relation (5) is replaced by an inequality,

$$I^2 \geq Q^2 + U^2 + V^2.$$

Having defined the Stokes parameters, one route to the Poincaré sphere is that given by Perrin and followed and generalized to the partially coherent case by Pancharatnam. Q/I, U/I, V/I

S. Pancharatnam (1934–1969)

can, in view of equation (5), be used as rectangular coordinates of a point on a sphere of unit radius. For example, circularly polarized light corresponds to $Q = 0$, $U = 0$, $V/I = \pm 1$, i.e. the two poles of the sphere. Box 2 in Ranganath's article gives a geometrical visualization of these features. It also presents the methods of working out the effect of a linearly birefringent element on an incident state of polarization.

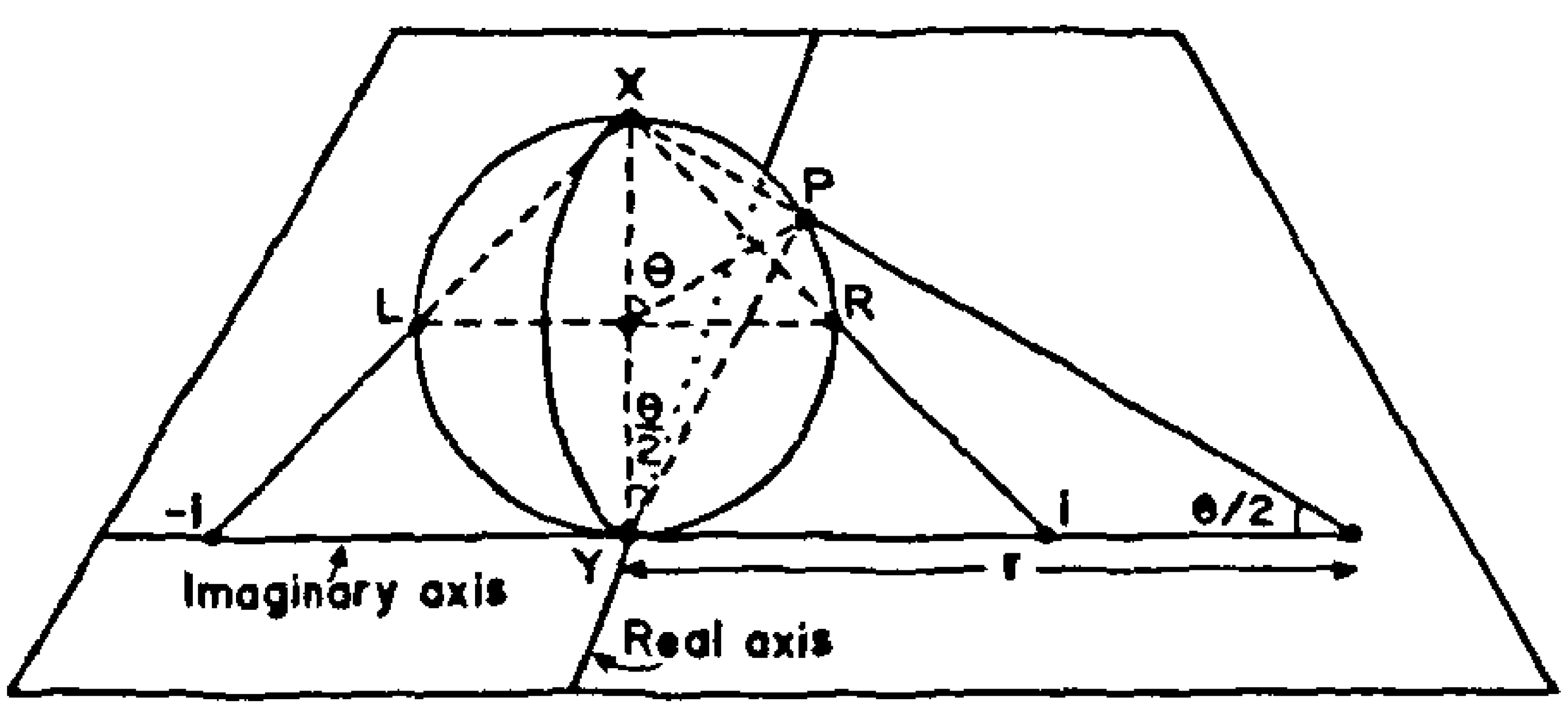
A more mathematically elegant and economical but possibly less physically transparent way of reaching the Poincaré representation is to form the ratio $z = z_2/z_1$. This single complex number has now lost information about the absolute intensity and phase while still encoding the value of the amplitude ratio a_2/a_1 and the phase

(Papers III and IV of the series) that the geometric phase was very central to his thinking. As an example, in many cases spiral interference figures were observed (Papers III and IV). Without going into details of his explanation of the observations, one can make the following general remark. If one naively thinks of a fringe as a contour of constant phase difference where this has some simple dependence on the direction of propagation, a spiral is not possible since the contours of a single-valued function should be closed curves. However, he points out that '... the sum of the phase differences ϕ_1 and ϕ_2 which are introduced by the process of resolution and analysation respectively [these are geometric phases] is not a constant but itself a function of the direction of propagation...'. And later he points out that these phases can 'increase continually' as the direction of propagation traverses a circle. The kinks in the interference figures are due to the non-uniform, and in extreme cases infinitely rapid variation of these phases with the direction of propagation. Since the unbounded nature and singularities of the geometric phase, as the parameters of the path are varied, have come in for considerable recent investigation, (see Bhandari's article in this issue) it is interesting to find these developments foreshadowed in Pancharatnam's papers. This shows that his discovery of the phase was not an isolated or chance circumstance and that he fully appreciated its value and properties. If it was forgotten later, even in optics circles in Bangalore, the blame must lie with others and that is the theme of the last section of this article.

Misadventures on a voyage of rediscovery

I cannot resist adding a note on how I became involved in problems relating to polarization optics in general and Pancharatnam's work in particular. This will serve both as an acknowledgement and as an interesting illustration of how strange and winding a path one sometimes has to take to reach understanding which in retrospect seems so straightforward.

As research students working with Ramaseshan, both Ranganath and I had occasion to refer to his review



difference $\phi_2 - \phi_1$. This is all that is needed to determine the shape and sense of the ellipse described by the electric vector in the $x-y$ plane. It would thus be quite natural to use the complex plane of z to represent this ellipse, but for one snag. When we approach the case of linear polarization along y, the ratio tends to infinity and further, its phase becomes irrelevant since the amplitude of the x-component in the denominator of z finally vanishes. One way of achieving a more faithful representation is to use Riemann's stereographic projection of the complex z-plane onto a sphere, which is shown in the figure. The sphere has unit diameter, and the angular coordinates of the projected point on the sphere corresponding to z are given by $z = re^{i\phi}$; $r = \cot(\theta/2)$.

It is not obvious, but verifiable, that this construction leads to the same Poincaré sphere as the method is based on the Stokes parameters. For example, one can see that the two circular states have gone to opposite poles, and all the linear states, corresponding to the real axis of the z plane, go into the equator with respect to these two poles. Passing light through any transparent linear anisotropic medium leads to a linear transformation of the two basic complex amplitudes z_1 and z_2, and hence also to a linear transformation of the Stokes parameters, which preserves the intensity I and hence the sum $Q^2 + U^2 + V^2$. Such a linear transformation preserving the sum of the squares of Cartesian coordinates can only be a rigid rotation (or an inversion corresponding to time reversal). Thus, passage through any such lossless linear anisotropic medium or device will rotate the state of polarization rigidly on the sphere, leaving two opposite points invariant. In terms of the original complex numbers z_1 and z_2, the transformation is linear and preserves the intensity $z_1\bar{z}_1 + z_2\bar{z}_2$. This is a unitary transformation in two complex variables. It has two orthogonal *eigenvectors* ψ_1, ψ_2 satisfying $\psi_1^\dagger\psi_2 = 0$. Thus these two orthogonal states of polarization translate into opposite points on the *Poincaré sphere. The physical* meaning of orthogonality is clear when we calculate the intensity of a superposition of two waves $(\psi_1 + \lambda\psi_2)$. *The interference term*, which occurs in addition to the individual intensities is proportional to $\psi_1^\dagger\psi_2$ and its complex conjugate and vanishes if the two states are orthogonal.

These transformations are important for another reason which can be called 'democracy on the Poincaré sphere'. Take for example a state of linear polarization P, inclined at an angle article on crystal optics, appearing in the *Handbuch der Physik* and coauthored with G. N. Ramachandran. Pancharatnam's work is clearly described there, but, speaking for myself, I think only the analytical part seeped in. Some years later, I attended a lecture at the Raman Institute given by Radhakrishnan which apparently did go over the topic of polarization and the Poincaré representation. I say apparently because a friend assured me that I was (quite literally) asleep during most of the talk. My third opportunity came also by chance when I attended another lecture by Radhakrishnan, this time on pulsars, and actually saw him write down one equation '$\varepsilon = 1 - \omega_p^2/\omega^2$'. When I later remarked on his excursion into higher mathematics, he jokingly referred me to an old paper of his with Morris and Seielstad, in which, he assured me, there was an equation occupying no less than four lines of the *Astrophysical Journal* that he had derived. Naturally, I took it as a challenge and tried to work it out on a train trip to Madras, using the only tool I was used to, representing polarized radiation by two complex numbers (see box). It took most of the journey and after deriving the final expression (for the visibility of the fringes obtained when correlating signals from two arbitrarily polarized antennas looking at an arbitrarily polarized sky), I would have forgotten all about it but for one point which left me uneasy. What kind of phase convention was being used to compare two different polarizations? My first few attempts failed (and one of them was a particular case of the Pancharatnam convention, resolving all states along right circular, which gives a 2π discontinuity at left circular). I did not even know what kind of three-dimensional object one got when one added phase to the Poincaré representation. Once this was constructed, and closed orbits drawn to represent variation of overall phase for a fixed direction of polarization, it became clear that there could be no globally valid continuous convention[5]. I was too pleased with this result to see its broader implications, in spite of receiving plenty of help. R. Rajaraman showed me how my mysterious three-dimensional space including polarization and phase was just the constant energy sphere living in the four-dimensional phase space of two harmonic oscillators. I had missed that because of dissecting it into two solid tori instead of two solid spheres. Two relativists visiting India in connection with the Einstein centenary, Martin Walker and Ted Newman, told me more about the general mathematical phenomenon one was encountering, a fibre bundle with no global section. Still no realization of any connection to Pancharatnam! Seven years later, after a journal club talk by my colleagues Shukre and Samuel on two geometric phase papers, Ramaseshan insisted that there must be a connection to Pancharatnam's work. Since I wrote a joint paper with him on this, it was not possible for me to admit then my initial resistance to such a sweeping identification but I can do so now! Subsequent developments were rapid and systematic and my role more that of an interested spectator. They are reviewed in Bhandari's article in this issue. It was particularly illuminating, from my point of view, to learn from J. Samuel about the deep connections between the geometric phase and ideas from differential geometry such as gauge invariance. But I must confess that my appreciation of the way Pancharatnam himself derived his results was incomplete until the opportunity to write this article. Reading him in the original is an experience I would

$c/2$ to the x axis, and hence $\pi/2 - c/2$ to the y axis. The points P, X, and Y represent these three states on the Poincaré sphere. X and Y are opposite while the arc PX equals c. It is elementary that the components of P transmitted by analysers along X and Y are proportional to $\cos(c/2)$ and $\sin(c/2)$ respectively. But, by changing from X and Y to some other basis, it is clear that the result is general, and the component of any state resolved along any other is the cosine of half the angular separation between the two on the sphere. Similarly, when two opposite circular states are superposed, it is easy to verify that advancing the phase of the right circular component by α will advance the major axis of the resultant ellipse in the positive sense by the angle $\alpha/2$ in the $x-y$ plane, which means by α on the Poincaré sphere. Again, one can change the coordinate system and generalize this result to the superposition of any two orthogonal states A and A'. A phase change by α in one of them rotates the resultant state by the same angle α about the diameter AA' joining the two states. Pancharatnam's own thinking was certainly rooted in the intrinsic, basis-free description which the Poincaré sphere provides. This is shown not only by the words he wrote but also by the fact that all the Poincaré spheres in his papers are unlabelled with any equator or poles unless required for the discussion of some specific problem! One would be tempted to call this underlying property 'Poincaré invariance', except that the term is rightly reserved to remind us of Poincaré's all too often forgotten contribution to the birth of special relativity.

recommend to anyone who aspires to walk through Plato's door. Not only are the papers, especially I and II, completely geometric in content, but the style is that of the geometry texts of a bygone era, one proposition following the other inevitably till the edifice is complete.

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