Application of the Poincaré sphere to radio astronomy

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AS with the other contributions to this issue of *Current Science*, the Editor very kindly invited me to write a piece and suggested the above title. I accepted with mixed emotions. An opportunity to pay tribute to a very dear friend whom I held in such high esteem was not to be missed. On the other hand, the only connection, tenuous at best, that the Poincaré sphere had with Radio Astronomy was that I personally benefited from a study of some of Pancharatnam's work as I shall describe below.

There have been many exciting periods in the development of radio astronomy over the last four or five decades, and one of them around 1960, was the time of search for polarization in the sources of celestial radio emission. The cosmic radio radiation discovered in 1933 and studied intensively by many in the post-war years clearly had a non-thermal component, but its origin was at first obscure. By the early fifties it was generally believed that such radiation could be produced by relativistic electrons gyrating in a magnetic field – the synchrotron mechanism, and that it was the cosmic ray electrons trapped in the galactic magnetic field that produced the diffuse component of the radiation.

Apart from this widespread component, several so-called discrete sources of radio emission had also been discovered. Among those were Taurus A, identified with the Crab Nebula, the visible remnant of the famous supernova of 1054, and Virgo A, an external galaxy. The optical radiations from both these objects were noted to contain a diffuse component whose spectrum was featureless and whose origin was unknown. Shklovsky, the great Russian astrophysicist, advanced the bold hypothesis that both the radio and diffuse optical emission were due to the synchrotron mechanism, and that the optical radiation should be polarized. This was shown to be indeed so in 1954 and 1956 in the two sources respectively. Thus, the most crucial evidence in support of the existence of the synchrotron mechanism in both galactic and extragalactic radio sources was the detection of its *polarization*. But it was extraordinary that the only two known polarized emitters in the sky were in the optical, although they owed their detection to their associated radio properties.

The outstanding pioneers in the search for polarization in radio sources were the group at the Naval Research Laboratory in Washington, DC. They found the first one in 1957, and it was none other than the Crab Nebula. But from then on, for the next five whole years, an incredible situation persisted. Hundreds of radio sources, both galactic and extragalactic, had been discovered, the spectra of most of which clearly indicated that they were non-thermal, and in all probability synchrotron radiators. But not one of them was detectably polarized, excluding the always extraordinary Crab Nebula.

Convinced that all synchrotron sources must show polarization, and that an improvement in measurement techniques should lead to success, my colleagues and I made the most strenuous efforts at the Caltech Observatory, where I happened to work at the time. In this exercise it appeared unavoidable that I should acquaint myself with those strange things called Stokes parameters whose mathematics was laid out in detail in Chandrasekhar's treatise on Radiative Transfer. This was fine if you loved equations. Also if you didn't mind that the simplest antenna – a single straight piece of wire – could respond to all four Stokes parameters, although they were supposed to represent different aspects of the radiation! Worse, there was no way in which you could orient the wire (or rotate your coordinate system) to get it to respond to *less* than three of the four Stokes parameters. I needed another way, preferably pictorial, to look at this difficult problem, and see the answer, so to speak. And this is where Pancharatnam came to the rescue.

I refer to his series of papers on the generalized theory of interference where he used the Poincaré representation to deal with the complex polarization phenomena encountered in crystals involving double refraction, dichroism, optical rotation and partial coherence. Elegance and simplicity characterized his approach, and his theorems and methods made it easy to understand even complicated situations. Further, they aroused an enduring fascination for the subject of polarization in general by revealing its inner beauty. There are several articles in this issue contributed by distinguished physicists which deal in depth with one or other aspect of Pancharatnam's work on polarization. I shall only make a few general remarks in the radio astronomy context.

The original Poincaré sphere representation is shown in Figure 1 *a* and the response of an antenna to radiation of a different polarization is shown in Figure 1 *b*, where

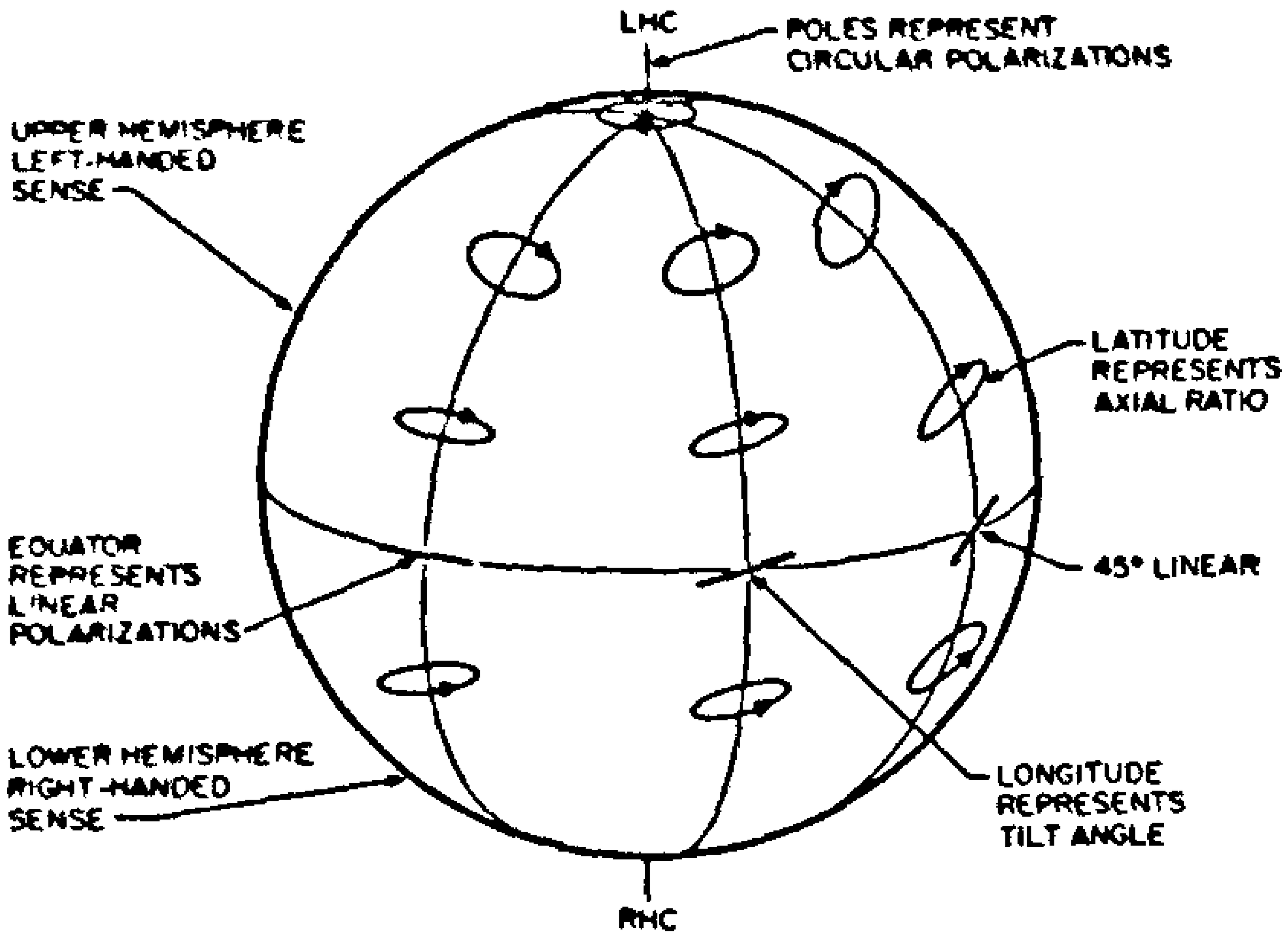

Figure 1. *a*, Polarization states on the Poincaré sphere

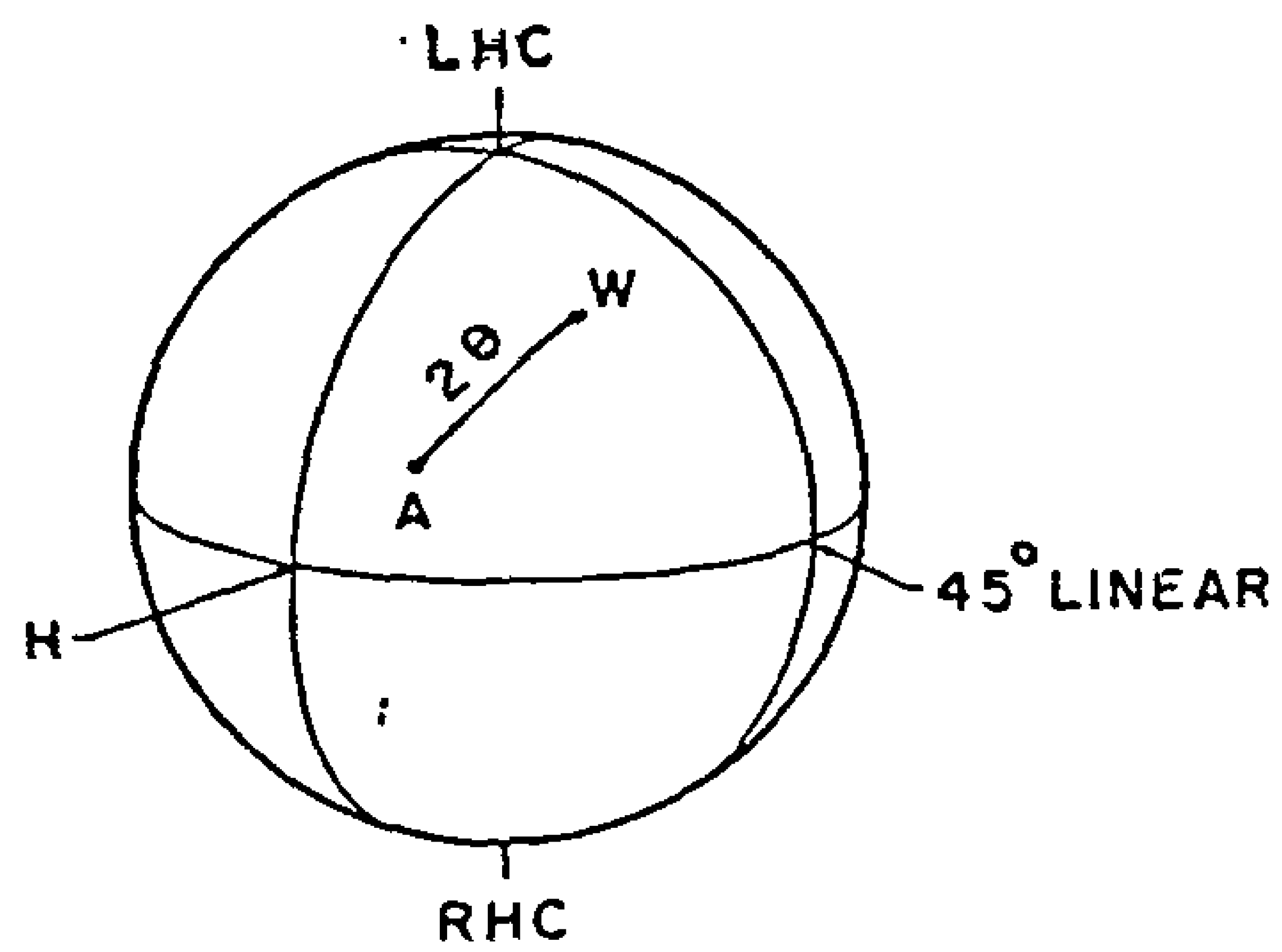

Figure 1 *b*, Receiving polarization of an antenna (analyser) A and polarization of an incident wave W. The fraction of energy received is $\cos^2\Theta$.

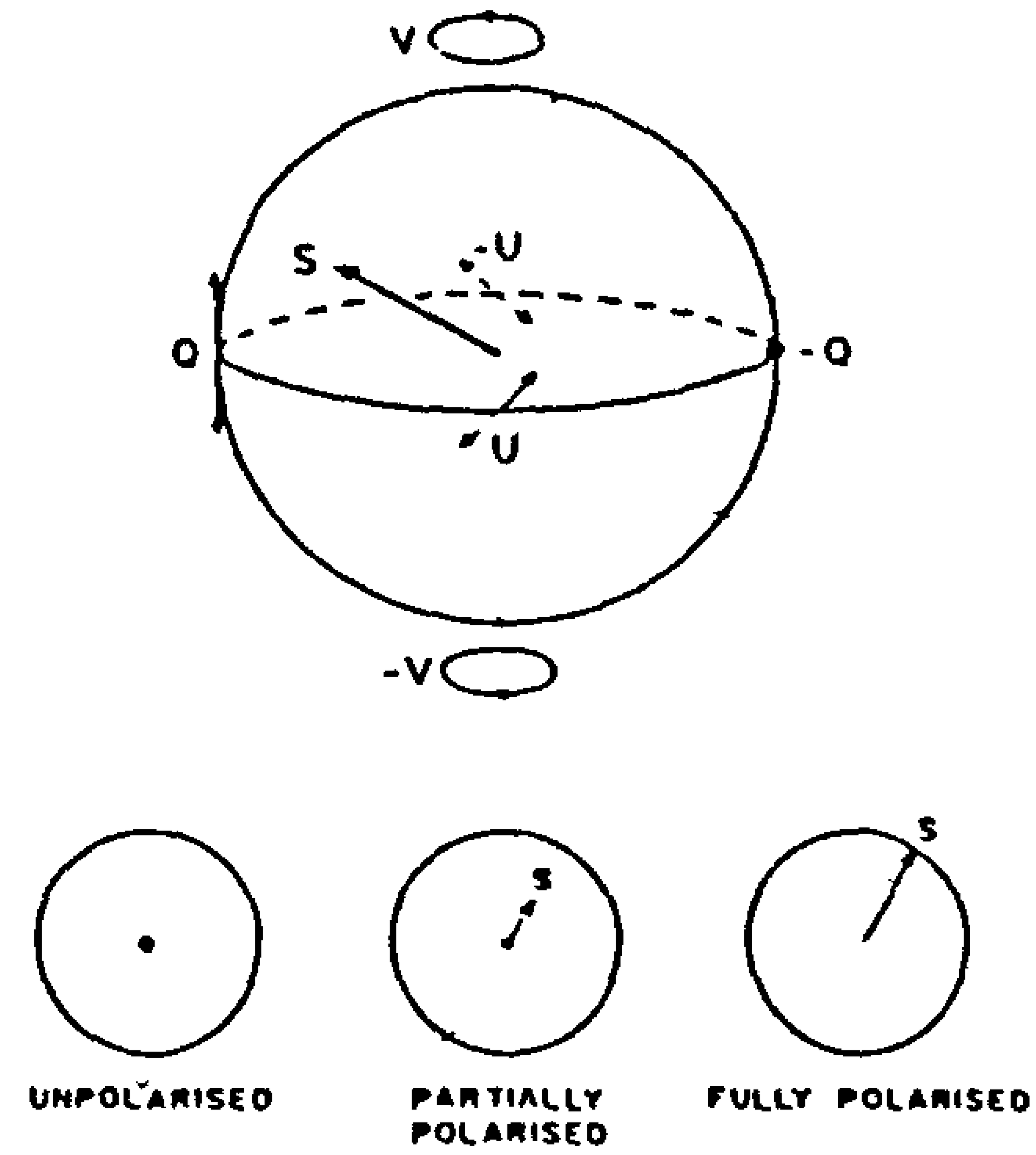

Figure 2. The representation of partial polarization through Pancharatnam's extension of the Poincaré sphere. The radius of the sphere = I, the first Stokes parameter. The vector S is called the Stokes vector, and its length $S = \sqrt{Q^2 + U^2 + V^2}$. The implied zero component is indicated here by the dot at the centre and has a value = $I - S$.

it is assumed that the radiation is fully (100%) polarized. As such this gives no representation for partially polarized radiation, a special case of which is random (0%) polarization. Most natural radiation belongs in this category and to deal with it Pancharatnam made one important addition. The addition[1] was a master stroke, and it was to make the radius of the sphere = I, the first Stokes parameter which corresponds to the total intensity. This is illustrated in Figure 2, as also the representation of unpolarized radiation, partially polarized radiation and totally polarized radiation. The vector S is called the Stokes vector and its length divided by the radius of the sphere represents the degree of polarization of the signal. This makes self-evident the proof of the fact that $[(Q^2 + U^2 + V^2) \leq I]^{1/2}$, which is something that looks like a big deal in earlier treatments.

The new rule for calculating the intensity of any signal picked up by an analyser or antenna which subtends a certain angle with the direction of the Stokes vector is $\cos^2$ half the angle as in Figure 1 *b*, plus one half of a 'zero component' which represents the unpolarized radiation. The value of this component would be truly zero if the radiation were totally polarized, equals the radius of the sphere for totally unpolarized radiation, and is the bit between the tip of the Stokes vector and the surface of the sphere in the case of partially polarized radiation. The important thing to appreciate here is that any analyser or antenna will always pick up one half of the value of the zero component, and for those, like me, who need a picture to understand unpolarized radiation it is as follows[1].

It you look at the electric vector at any instant of time it will have a certain direction which is, of course, perpendicular to the direction of propagation. If you look at its behaviour over a time short compared to $1/\Delta f$, the coherence time of the radiation, it will be 100 per cent polarized in some particular state which is in general elliptic. Over the period of a coherence time, the polarization will gradually change to some other state in a random fashion. If you look at the radiation for a long time, the Poincaré sphere will be covered in a random walk fashion leaving an average value for the polarization which will tend to zero as the number of samples we have tends to infinity. In the case of partially polarized radiation, the sphere will be covered in a non-uniform, or biased, fashion leading to a resultant Stokes vector that is non-zero.

It is now easy to visualize, using the rule for angles given before, why an antenna of any polarization will pick up one half of the available energy in unpolarized radiation, or the zero component, when you integrate over a long time. But there is one more important step, however, to appreciate why oppositely polarized antennas have uncorrelated fluctuations when they are looking at natural or other unpolarized radiation. If you think of the successive states of polarization that the radiation has in successive coherence times, it is clear that it will sometimes be closer to one antenna, and sometimes closer to the other; and therefore if one of them picks up more radiation, the other must pick up less. And indeed one might expect an anticorrelation between these fluctuations for just this reason. What happens is that the intensity of the radiation is also fluctuating in a random way, and while in each coherence time the Stokes vectors will reach all the way to the surface of the sphere, the radius of the sphere itself undergoes oscillations in coherence times and this is such as to precisely decorrelate the signals. (The voltages in the two antennas follow a bivariate Gaussian distribution if you prefer such a description.) Over a long time we get the radius converging to the mean value and the Stokes vector in the middle tending to zero. Pancharatnam's extension of the Poincaré representation thus helps one to *visualize* the intimate connection that exists between partial polarization and the frequency spread (bandwidth) of the radiation.

Partial polarization and partial coherence are phenomena associated only with signals of finite bandwidth and a consequence of the fluctuations natural to such non-monochromatic radiation. True monochromaticity requires sources of infinite temperature and is only an abstraction. In the case of optical radiation the reciprocal of the bandwidth even for the 'narrow' lines chosen for experiments is of the order of 10^{-11} seconds and the mean time between the arrival of photons can be many times larger than this coherence time scale. This is not so at much lower frequencies and radio engineers who are more conscious of the frequency spread of their signals have less difficulty in visualizing their evolution over the time scale of a reciprocal bandwidth. In fact, such evolution of natural, or noise-like signals can be followed in detail and even displayed on an oscilloscope by a suitable choice of instruments and parameters.

The quantities whose variation is the manifestation of the fluctuations are the frequency/phase, the intensity and the state of polarization. If the variation with time of frequency and intensity could be conceived as the result of adding the fields due to the different Fourier components making up the band, the fluctuations in polarization of natural radiation can be thought of as arising from the different polarization states of the Fourier components. Seen this way, the variation of polarization with frequency is a fundamental characteristic of natural radiation and it should cause no surprise that Pancharatnam who probed the subject so deeply turned his attention later to an understanding and description of polychromatic polarization. If the polarization, partial or total, changes in state as a function of frequency the conventional Stokes parameters are no longer adequate and have to be replaced with Stokes spectral functions which Pancharatnam introduced[2] (see Madhusudana, this issue).

Going back to antennas, we saw earlier how unpolarized radiation produces uncorrelated fluctuations in all pairs of oppositely polarized antennas. Interestingly, partially polarized radiation can produce totally uncorrelated fluctuations in non-oppositely polarized antennas. It can be shown that for any antenna of a given polarization exposed to radiation of any given polarization, there is always another antenna with a different polarization that will pick up a signal uncorrelated with that of the first. And the determination of the polarization of the second antenna is a trivial geometrical construction in the sphere as shown in Figure 3, and illustrative of the power of this method in solving problems in polarization. In the limit of the partial polarization becoming total, this construction will lead to no signal being picked up by the second antenna. As an example of an application, this property was used as the basis of an accurate null method for the measurement of very small values of linear polarization[3].

As an example relating to sources rather than antennas, let there be two partially polarized sources within the beam of a telescope and we wish to know what will be the net polarization of the combination taken together. We just draw two spheres, one for each of the sources, each with its own Stokes vector which may be pointing in any direction whatever. To know what we would measure if both were combined, we merely add the radii of the spheres to make a new one, and we add the two Stokes vectors vectorially. The resultant Stokes vector will give you both the fractional polarization and its particular state.

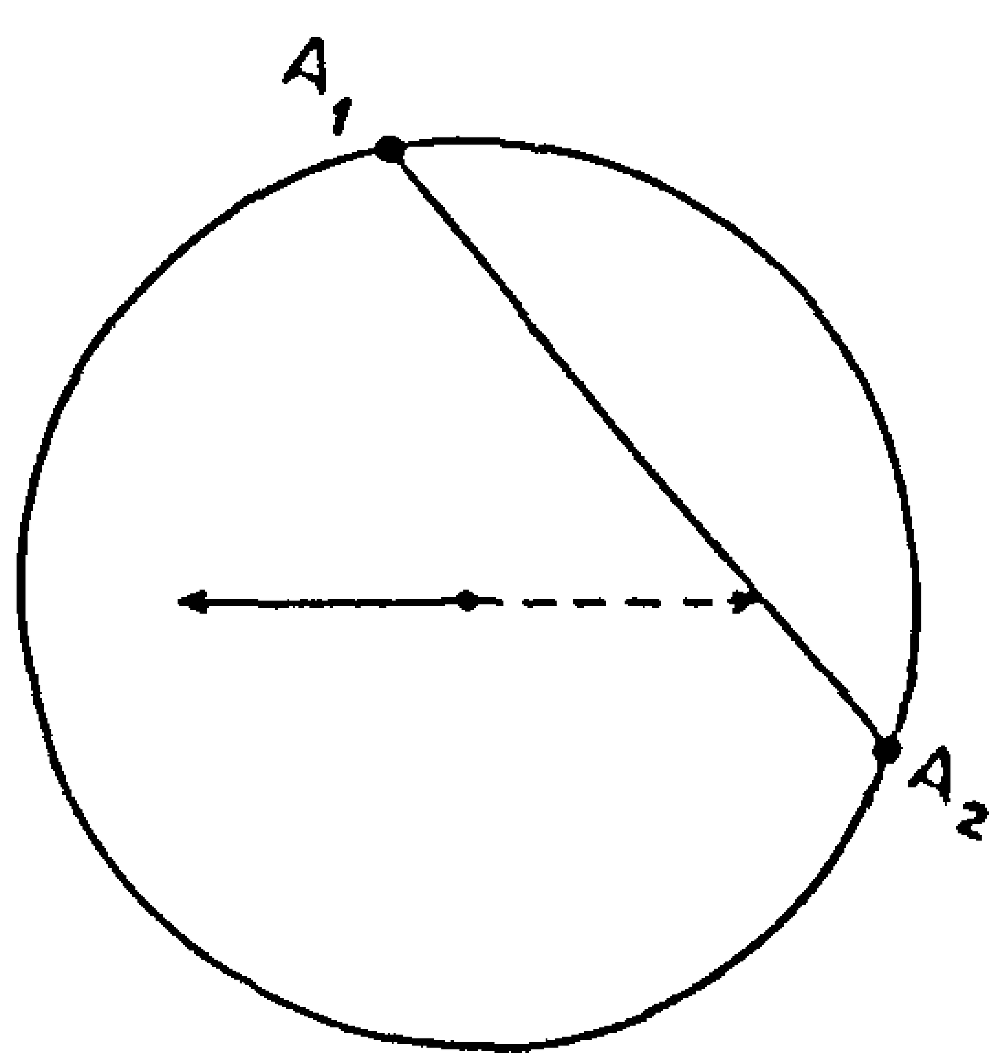

Figure 3. Resolving radiation into incoherent components Let A_1 be any point on the sphere representing the polarization of antenna one, and the Stokes vector the representation of the radiation. The figure shows the great circle through A whose plane contains the Stokes vector. A_2 is the intersection of the line joining A_1 and the tip of the reflected Stokes vector with the great circle, and gives the polarization of antenna two whose signals will be uncorrelated with those of antenna A_1. This construction is valid whatever the degree of *polarization*.

In the foregoing example, I assumed that the two radio sources were incoherent as would be the case for real radio sources separated in the sky. But, if you wish to add coherent polarized beams, phase comes into the picture. As everyone is familiar in connection with understanding Faraday Rotation, one thinks of the observed linear as the resultant from summing two coherent circulars of opposite sense, with the changing phase difference between them leading to a rotation of the linear. On the Poincaré sphere, in this particular case, the two input polarizations are at the poles, with the resultant lying along the equator, and going round it as you change the phase between the two inputs. And because of the symmetry of the sphere this will be true in the appropriately rotated system, whatever the two opposite input polarizations, elliptics in general. When the input polarizations are not opposite, or the intensities not equal, or both, the locus of the resultant with change of relative phase becomes a small circle on the sphere, located easily by simple prescriptions provided by Pancharatnam[1].

As an example of a difficult problem where the Poincaré sphere again lets you see the answer very easily is polychromatic polarization mentioned earlier, namely the situation where the polarization across a band changes as a function of frequency. A familiar and simple case in Radio Astronomy is once again Faraday Rotation. The linear polarization produced by the synchrotron mechanism mentioned earlier gets rotated in its passage through the magnetized interstellar plasma. As the amount of rotation depends on frequency, this leads to a spread in the orientation of the linear *within* the band of reception with consequent apparent depolarization. Instead of a single Stokes vector from the centre of the sphere, we now have a spread of such vectors and what we need to do is a vector addition in this polarization space. It is easy to see that the amplitude will give the net fractional polarization and the orientation will give the state of polarization of the resultant whatever the plane in the Poincaré sphere on which these vectors are spread out.

Further to my earlier remarks on partial polarization and partial coherence it should be pointed out that the 'depolarization', just referred to, is only apparent. By reversing the effect of the interstellar plasma e.g. by splitting the signal into its circularly polarized components, 'delaying' the advanced one by the right amount, and recombining them, one can restore the polarization to its original value and condition. In radio astronomy the loss of fringe contrast, or of constancy of polarization state across the band, caused by path differences is a very familiar phenomenon and recognized as distinctly different from incoherence or the absence of polarization. This was perhaps less well appreciated by experimenters in optics, as Pancharatnam makes a point of stressing the non-equivalence of the two situations and the reversibility of the apparent incoherence by an introduction of the opposite path difference[2].

The ultimate instrument in Radio Astronomy these days is the large aperture synthesis telescope the signals from whose elements are correlated in pairs and subsequently combined and processed to produce remarkable maps. Such large arrays are in operation in many parts of the world and have revealed the polarization and its complicated variations that are to be found in practically every non-thermal source. The high resolution of these instruments both in frequency and angle on the sky has made this possible by removing the smearing suffered by earlier telescopes and receivers. The astronomical part of the story has been told elsewhere[4] and is not relevant here. But as a final remark on the power, or convenience if you like, of the Poincaré–Pancharatnam representation, let me make the following statement regarding the correlations of the interferometer pairs in such arrays. Each element receives radiation which is *partially* polarized and which is *partially* coherent with the radiation received in any other element. Each of the elements is *imperfect* in its polarization in that they differ slightly one from the other. But you could sit down with a globe in front of you, put some chalk marks on it and simply by inspection and without a line of algebra write down the response of an interferometer with arbitrarily polarized antennas to a distribution of arbitrarily polarized radiation[5].

When Pancharatnam came to the US in connection with the Rochester Conference in 1960, he also visited me in Pasadena to my great delight, and I took him to see both the Palomar 200" telescope and the Caltech Radio Observatory in the Owens Valley. Since our previous meeting about a decade earlier he had made more strides than most scientists make in a lifetime. I saw him again in 1964 in Oxford where he took me to his lab and explained some of the very impressive work in an entirely new field that he was doing. But he was still so unassuming and self-effacing that I had to keep reminding myself that while he might be a friend from childhood here was no ordinary scientist. My last recollection of him the one I treasure the most, is when I took him out sailing with some friends from Sussex off the South Coast of England on a blustery day. Panch may have been sickly but it was the others who were sick and he helped cheerfully to wash down the deck while the others looked green and hung on to the railings.

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