Numerical integration IV. Proceedings of the conference at the Mathematical Research Institute, Oberwolfach, November 8–14, 1992, ISNM. H. Brass and G. Hammerlin, eds. Vol. 112. Birkhauser Verlag, Besel-Boston-Berlin. 1993. pp. 382

The need often arises for evaluating the integral of a function which has no explicit antiderivative or whose derivative has values that are not easily obtained. So we have to approximate the integral numerically, when analytic techniques fail. Numerical integration is the study of how the numerical value of an integral can be found.

Consider the integral

$$I[f] = \int_\Omega f(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n$$

$$= \int_\Omega f(X)\,\mathrm{d}X,\ X = (x_1,\dots,x_n),$$

where Ω is a region in $\mathbb{R}^n$ and $f:\Omega \to \mathbb{R}^1$. The numerical approximation of $I[f]$ is often done by the formula of the type

$$I[f] \simeq Q_N[f] = \sum_{j=1}^{N} a_j f(X_j).$$

Here X_j are points in Ω and are called points (nodes) of the formula; $a_j \in \mathbb{R}^1$ and are called the coefficients (weights) of the formula. If $n = 1$, then Q_N is called N-Point quadrature formula. If $n \geq 2$, then Q_N is called N-point cubature formula. By $R_N[f]$, we denote the error term of the formula, $R_N[f] = I[f] - Q_N[f]$. The associated degree, deg (R_N) is defined by

$$\deg(R_N) = m = \sup\{k \mid R_N[P_k] = 0\},$$

where P_k denotes the space of all polynomials in $x_1,\dots,x_n$ of degrees $\leq k$, m is called the degree of precision (order of the formula) of Q_N.

One of the approaches for finding better formulas that has been actively followed is to look for formulas which achieve a given degree of precision using fewest possible points $X_1, X_2, \dots, X_N$. With invention of the calculus, the subject began to evolve systematically; the well known trapezoid rule and Simpson's rule are members of an infinite sequence of formulas due to Newton and to Cotes. From seventeenth century to now, a great deal of effort is going into the development of the subject. However, the earliest interesting integration formulas for multidimensional integrals seems to have appeared only in 1877 by J C. Maxwell. Even though a number of new methods have been devised, practical consideration have led to problems of ever increasing complexity, so that even with current computing speeds, numerical integration may be a difficult task. As mentioned by the editors in the preface of the book, higher dimension and complicated structure of the region of integration and singularities of the integrand are the main source of difficulties.

This book is a collection of 27 research papers, presented at the conference which was held at Oberwolfach in November 8–14, 1992. These articles are devoted to the study of existence, construction and error analysis of the numerical integration formulas. Some of them contain a brief survey of the subject and most of them have a good reference list. Nine open problems are given at the end of this volume with precise formulations.

Let us first start the review of the articles which are devoted to one-dimensional case. In one dimension, since all the finite intervals are equivalent under affine transformation it is enough if we consider the following intervals of the type (a, b) (a and b are finite), (a, ∞) and $(-\infty, \infty)$. It is frequently convenient to consider an integral of the form $\int w(x) f(x)\,\mathrm{d}x$ instead of the integral $\int f(x)\,\mathrm{d}x$.

Consider the approximation of the integral

$$I_w[f] = \int_a^b w(x) f(x)\,\mathrm{d}x, \qquad (2)$$

where weight function w is continuous and non-negative on (a, b) and $I_w[p_k]$, $p_k(x) = x^k$, $k = 0, 1, 2, \dots$ are assumed to exist. Let Q_N be the quadrature formula for equation (2), i.e.

$$I_w[f] \simeq Q_N[f] = \sum_{j=1}^{N} a_j f(x_j),$$

$$x_j \in [a, b]. \qquad (3)$$

If x_j's in $[a, b]$ are fixed in advance, then we know that we can find a_j's s.t. Q_N is of order $N - 1$. If we treat both x_j's and a_j's as unknowns, then we can find a quadrature formula which is of order $2N - 1$. The formula of Gauss type is an N-point formula with degree of precision $2N - 1$. Now consider the integration formulas of Gauss type with a certain number of preassigned nodes By this we mean a formula of the type

$$I_w[f] \simeq Q_{N,M}$$

$$= \sum_{r=1}^{M} \alpha_r f(\eta_r) + \sum_{\mu=1}^{N} \beta_\mu f(\xi_\mu),$$

where η_r are fixed and prescribed in advance and $\alpha_r, \beta_\mu, \xi_\mu$ are to be determined so that quadrature formula is of order $2N + M - 1$. In the paper of Ehrich, an iterative method and its implementation to compute the nodes ξ_μ numerically is proposed. It is shown that the iterative method is of order two. His numerical experiments show the suitability of this method. Sometimes, it is preferable to use lower degree piecewise polynomial approximants for f with more knots than simple polynomials of higher degree. By this motivation, explicit quadrature formulas of Gauss, Radau and Lobatto type for spaces of polynomial splines of degree 1 (arbitrary knots) and 2 (the case of equidistant knots) are presented in the paper of Nikolov. It is also shown that the Gauss type quadrature for splines with equidistant knots are asymptotically optimal. In the paper of Schneider, rational Hermite interpolation is used to derive and analyse the quadrature formulas in two different ways One approach gives quadrature of Gauss type whereas the other one generalizes Engel's dual quadratures. It is well known that the set of orthogonal polynomials plays an important role in the theory of numerical integration. In the paper of Peherstorfer, he has shown how to get in a simple and unified way, the characterization of positive quadrature formulas by using interlacing properties of the zeros of the orthogonal polynomials. In the paper of Locher, he relates the stability criteria of linear difference equation and the interlace properties of the zeros of Chebyshev polynomials of the first and second kind (these are orthogonal polynomials) Hence the stability test is also a test for the existence of certain positive quadrature formula.

For integrands having poles (outside the interval of integration) it would be more natural to include also rational functions among the functions to be exactly integrated. In the paper of Gautschi, he derives N-point quadrature rule that exactly integrates M rational functions (with prescribed location and multiplicity of the poles) as well as polynomials of degree $2N - M - 1$, $0 \le M \le 2N$. Rational functions are to be chosen so as to match the most important poles of the integrand. In the paper of Hasegawa and Torii, they considered the integral of functions having poles near the interval of integration. After the smooth part of the integrand is expanded in terms of the Chebyshev polynomials, the integral is approximated and evaluated by using recurrence relation and extrapolation method. An automatic quadrature method is given.

It has been known for a long time that the trapezoidal rule, under certain conditions gives a very accurate approximation to integrals over the entire real line, see e.g., Davis and Rabinowitz[1]. In the paper of Gustafson, he considers the integral over the entire real line and approximates, first the integral with a trapezoidal sum, then applies a linear convergence acceleration scheme to approximate this trapezoidal sum. He has developed efficient quadrature formulas that are applicable for both decaying and oscillating integrands.

An important tool for the discussion of the error term $R_N[f]$, is the Peano representation

$$R_N[f] = \int_{-1}^{1} f^{(s)}(x) K_S(x)\,\mathrm{d}x,$$

$$s = 1, 2, \ldots, \deg(R_N) + 1$$

where K_s is the Peano kernel and $f^{(s)}$ denotes the sth derivative of f. In the paper of Brass, he has considered the inequality

$$\|R_N[f]\| \le \|K_S\| \int_{-1}^{1} |f^{(s)}(x)|\,\mathrm{d}x,$$

$\|\ \|$ is a sup norm

and studies the asymptotics of $\|K_S\|$ for sequence $Q_1, Q_2, \ldots,$ of quadrature formulas. Let $R_N[P_{r-1}] = 0$, where P_{r-1} is a space of all polynomials of degree $\le r-1$. With

$$\|R_N\|_j = \sup\left\{\frac{|R_N[f]|}{\|f^{(j)}\|_\infty} \,|\, f \in C^j[a,b], \|f^{(j)}\|_\infty \ne 0\right\}$$

for $j = 0, 1, \ldots r$, the error estimates

$$|R_N[f]| \le \|R\|_j \|f^{(j)}\|_\infty$$

hold. In the paper of Köhler, he expresses for $j = 1, \ldots, r-1$, $\|R_N\|_j$ in terms of $\|R_N\|_0$ and $\|R_N\|_r$. If such estimates are available, then it is not necessary to compute all the intermediate error constants $\|R_N\|_j$ ($j = 1, \ldots, r-1$) separately. When a_j in equation (3) are not all positive, the paper of Mastroianni and Vertesi gives error estimates for product quadrature formulas in the weighted L_1-norm. In the paper of Hunter and Smith, they generalize an expression due to Stenger for the error in the associated Gaussian quadrature formula and consider Rodrigue's function U_n associated with the polynomial P_n in the orthogonal sequence over $(-1, 1)$ w.r.t. weight function $w(x)$. Some properties of U_n are proved and some conjectures are made.

Brass has constructed a quadrature rule $\{Q_N\}_{N\in\mathbb{N}}$ for which an error of order $O(N^{-2})$ can always be guaranteed for convex functions. So all further quadrature rules have to compete with such a method. Let $a = -1$ and $b = +1$ in equation (2). For convex integrands, trapezoidal and midpoint formulas have an error of order $O(N^{-2})$ if derivative of f at ± 1 exists. Brass has posed the problem to extend this result to Gaussian quadrature in 1982. In ref. 2, Förster and Petras, proved a much more generalized result. In the paper of Petras, he has given an overview of these results as well as some new results on the quadrature theory of convex functions. It is natural to think that, it is possible to reduce error by using adaptive methods, i.e. the node x_i depends on the previously computed values $f(x_1), \ldots, f(x_{i-1})$, i.e., $x_i = x_i(f(x_1), \ldots, f(x_{i-1}))$. In the paper of Novak, he considers the function f in a nonsymmetric convex class of functions and proved that adaption cannot help in the worst case but considerably helps in the case of Monte Carlo methods. And also he has given examples where adaptive methods are better than nonadaptive ones.

The question of small variance and high algebraic degree was first considered by Chebyshev. Since that time several investigations on this subject can be found in the literature. In the paper of Förster, a survey of these results and some open problems are given with a good reference list. Suppose we are given a convergence sequence of interpolatory quadrature formulas $\{Q_N\}_{N\in\mathbb{N}}$. What can we say about the distribution of the points x_j? In the paper of Bloom *et al.*, they review previous results, which shows that half the points in the formulas behave like zero's of appropriate orthogonal polynomials. In the case of the interval $(a, b) = (-1, 1)$, this usually means that half the points have arcsin distribution. They also present a result relating the rate at which half the points converge to the arcsin distribution.

Variable transformations are used to enhance the performance of lattice rules for multidimensional integration. These transformations are either polynomial or exponential type. In the paper of Sidi, a short survey of this is given and he has also proposed a new transformation, $\sin^m$-transformation which is trigonometric in nature. His results indicate that $\sin^m$-transformation can be more advantageous than known polynomial transformation and have less underflow, overflow problems than exponential ones.

Let us now consider the review of the articles which are on multiple integrals. If the dimension $n \ge 2$, there are infinitely many distinct bounded connected regions, which are not equivalent under affine transformations. In $\mathbb{R}^2$, for example, the square, circle and triangle are not equivalent under an affine transformation. Integration formulas for any one of these regions are different from formulas for the others. Let $\Omega = [0, 1]^n$, a unit cube. Suppose we want to obtain an asymptotic error estimate of the form $O(2^{-q\alpha})$ ($\alpha > 1$, q a positive integer) via multivariate product midpoint rule it is necessary to evaluate f at $2^{n(q-1)} = O(2^{nq})$ points. It is well known that this number can be reduced drastically by using efficient lattice rules for multivariate numerical integration, see e.g., Sloan[4]. In the paper of Baszenski and Delvos, they have constructed Boolean mid point rules for multivariate numerical integration which are based on the ideas of multivariate Boolean interpolation and in that number of functional evaluation is $O(q^{n-1}2^q)$ and they have obtained asymptotic error estimates of the form $O(q^{n-1}2^{-q\alpha})$ as $q \to \infty$. One can also think of cubature formulas of trigonometric degree in the place of algebraic degree. In the paper of Beckers and Cools, they have given a summary of results on cubature formulas of trigonometric degree that have appeared in the Russian literature. They showed that such formulas can be approached with the tools used to construct formulas of algebraic degree and also from the field of lattice rules. For a unit square, a new family of cubature formulas of trigonometric degree with the lowest possible number of points is given. For a square, construction of fully symmetric cubature formula is proposed in the paper of Haegemans and Verlinden. In this method one does not have to solve large nonlinear systems Here one has to compute the solution of a linear system and a nonlinear system of two polynomial equations of lower degree with two unknowns. Advantage and disadvantage of the method are discussed. In the paper of Niederreiter and Sloan, for n-dimensional unit cube they modify quasi-Monte Carlo method in which only one vertex is a quadrature point, by distributing the corresponding weight between all the vertices. And also they show that among all the vertex modified rules, the rule that minimizes the L_2 version of the discrepancy is the rule that integrates all multilinear functions exactly. They have also explained the important role played by vertex variance. In the paper of Guessab, he considers the problem of approximating a double integral on a convex compact set K as a minimal linear combination of integrals on the real line and also obtains cubature formulas which are exact on the space $Q_{2k+1}(K)$ of all polynomials of degree $\leq 2k+1$ in each variable x_1, x_2.

When the integrand has singularities in the region of integration, most standard formulas are inefficient. Lyness[3] gives error functional expansion in multidimensional quadrature with a singular integrand function. To compute singular integrals effectively, knowing the existence of such expansions is essential. Once it is known, one can develop extrapolation scheme based on this error expansion. After Lyness[3], there are several papers in that direction and all are based on a uniform subdivision. In the paper of Espelid, he describes an extrapolation scheme based on the error expansion produced through a nonuniform sub-division of the region. This strategy can be applied to vertex singularities, line singularities and more general sub-region singularities. When $\Omega = S$, the n-sphere with radius R, in the paper of Genz two subdivision methods, radial subdivision and simplex subdivision methods are analysed and an adaptive algorithm for multiple integration is considered. And also some comparisons are made.

In the paper of Lyness, finding a canonical form of a lattice rule is defined by a general t cycle D–Z form. A step-by-step approach that parallels the group theory is described, leading to an algorithm to obtain a canonical form of a rule of prime order. The number of possible distinct canonical forms is derived and this is used to determine the number of integration lattices having specified invariants. For integrals with circular symmetry, Verlinden and Cools[5], derived necessary and sufficient condition such that cubature formulas of degree $4k+1$ attain Möller's lower bound (i.e. $N = k(2k+4)+1$) and also they showed that these conditions do not hold for some integrals of circular symmetry and some $k \in \mathbb{N}$. In the paper of Cools and Schmid, they showed for two classes of integrals that the number of nodes of cubature formulas of degree $4k+1$ will not attain Möller's lower bound. Thus in this case that bound has to be increased by 1. In the paper of Ritter *et al.*, they considered multivariate integration for stochastic processes and obtained bounds on the minimal average case errors of cubature formulas. The error bounds are derived in terms of smoothness properties of the covariance function. For some special case of covariance functions, these bounds are sharp, sometimes modulo a logarithmic power.

This volume does provide a valuable reference for the workers in the field and makes a fine addition to the library.

The authors and their titles are: G. Baszenski and F. J. Delvos, Multivariate Boolean mid point rules; M. Beckers and R. Cools, A relation between cubature formulas of trigonometric degree and lattice rules; T. Bloom, D. S. Lubinsky and H. Stahl, Distribution of points in convergent sequences of interpolatory integration rules: The rates; H. Brass, Bounds for Peano kernels; R. Cools and H. J. Schmid, A new lower bound for the number of nodes in cubature formulas of degree $4n+1$ for some circularly symmetric integrals; S. Ehrich, On the construction of Gaussian quadrature formulas containing preassigned nodes; T. O. Espelid, Integrating singularities using non-uniform subdivision and extrapolation; K. J. Förster, Variance in quadrature – a survey; W. Gautschi, Gauss-type quadrature rules for rational functions; A. Genz, Subdivision methods for adaptive integration over hyperspheres; A. Guessab, Formules de quadrature dans $\mathbb{R}^2$ avec 'réseau' minimal de droites; S. A. Gustafson, Quadrature rules derived from linear convergence acceleration schemes; A. Haegemans and P. Verlinden, Construction of fully symmetrical cubature rules of very high degree for the square; T. Hasegawa and T. Torii, Numerical integration of nearly singular functions; D. B. Hunter and H. V. Smith, Some problems involving orthogonal polynomials; P. Köhler, Intermediate error estimates for quadrature formulas; F. Locher, Stability tests for linear difference forms; J. N. Lyness, The canonical forms of lattice rule; G. Mastroianni and P. Vértesi, Error estimates for product quadrature formulas; H. Niederreiter and I. H. Sloan, Quasi-Monte Carlo methods with modified vertex weights; G. Nikolov, Gaussian quadrature formulas for splines; E. Novak, Quadrature formulas for convex classes of functions; F. Peherstarfer, On positive quadrature formulas; K. Petras, Quadrature theory of convex functions; K. Ritter, G. W Wasilkowski and H. Wozniakowski, On multivariate integration for stochastic processes; C. Schneider, Rational Hermite interpolation and quadrature, A. Sidi, A new variable transformation for numerical integration.

1 Davis, P J and Rabinowitz, P , *Methods of Numerical Integration,* Academic Press, New York, 1984.
2 Förster, K. J and Petras, K , *Numer Math* , 1990, 57, 763–777
3. Lyness, J N , *Math Comp* , 1976, 30 (133), pp 1–23

4 Sloan, I H, *J. Comput Appl. Math*, 1985, **12–13**, pp 131–143.
5. Verlinden, P and Cools, R., *Numer Math*, 1992, **61**, pp. 395–407.

G. D. VEERAPPA GOWDA

T.I F.R. Centre
P B. No 1234
Indian Institute of Science
Bangalore 560 012, India

China Bound, A Guide to Academic Life and Work in the PRC. by Anne F. Thurston, Karen Turner-Gottschang and Linda A. Reed. National Academy Press, Washington, DC, USA. 1994. pp. xii + 252. Price $24.95 (paperbound). [ISBN:0-309-04932-6]

Cross-cultural exchanges call for certain amount of getting to know the other culture even before one leaves one's home country. The Americans, with their penchant for standardizing, usually hold a programme lasting for a few days for overseas scholars, especially students, wherein the visitors are told how to make their stay in the USA productive and enjoyable, how to get along happily in society, what to do and what not to do under different circumstances, etc. For most countries in the world, a few days of preparation is all that one would need. But China is different, for various reasons. As one researcher says, being prepared in China can mean 'the difference between a headache and a productive day'!

This revised version of a much acclaimed book, originally published in 1981 and rewritten in 1987, is a comprehensive guide to scholars and scientists visiting China – first-time visitors and 'old China hands' alike.

To the American mind, China, though fascinating, is curiously ambivalent. It at once evokes the images of Marco Polo and Genghis Khan, rapid economic development and Tiananmen Square, friendly people and a frustrating bureaucracy. Added to these contrasting images is the language barrier, the vastly different social and cultural mores, and the totally different ways in which academic and scholarly institutions function and are administered.

It is to facilitate overcoming these barriers for the ever-increasing number of American scholars who are visiting China since the 'opening up' began in 1979 that this book was written. However, the material provided will be of great use to visitors from other lands as well to those who want to go and live, work and learn in China. This friendly and practical book offers all the details academic and other visitors need to make long-term stays in China productive, comfortable and fun. It covers everything from how to obtain the correct' travel documents to the things to take care of before one leaves China after one's assignment there is over. Frank discussions on the research and academic environment in China, aspects of human relations (including possible romance with a Chinese citizen!), negotiating the costs of services, and a host of other equally important things make this book a truly invaluable guide.

The book provides useful information on science and social science field work, living costs, health care, addresses and fax numbers of important institutions and services, currency, transportation, communication, children's education, etc.

Yet another good point of the book is the large number of quotes from people who had lived and worked in China in the recent past.

There are 17 appendices and an index.

Anyone going to China will find this volume enormously useful. The Committee of Scholarly Communication with China deserves appreciation for commissioning this book as well as the two earlier versions.

SUBBIAH ARUNACHALAM

Central Electrochemical Research Institute
Karaikudi 623 006, India