On reading Newton's *Principia* at age past eighty

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I first met Daulat Singh Kothari in the company of Meghnad Saha at the 1929 December-Meeting of the Indian Science Congress in Allahabad. Later, we were Research students together in Cambridge (England) for two years (1930–32). Thereafter, our lives took different courses. But we maintained our friendship, meeting once in several years at his home or at the University in Delhi. I had wanted to see him during my last visit to Delhi in December 1992. Even though Kothari was at that time in retirement in Jaipur, he had agreed to come to Delhi. But illness frustrated his intention and my hopes; and we did not realize at that time that it was to be forever.

What follows could very well have been the subject of our conversation, had we met. And I dedicate this essay to his memory.

I

My interest in the *Principia* was stimulated by an invitation in 1986 to contribute a paper at one of the many symposia that were being planned to celebrate, in the following year, the tercentenary of the publication of Newton's *Philosophie Naturalis Principia Mathematica* in 1687. In 1986, I had not more than turned the pages of Cajori's edition of Andrew Motte's English translation of the third 1722-edition on the *Principia* in 1729: I found that studying the *Principia* required considerable concentration to unravel the continuous prose in which Newton had written the geometrical and the analytical equalities and in fact the entire mathematical treatment. Confronted, then, with the prospect of having to prepare a paper with some pretense to substance, I had two choices: either to select and abstract from the extant voluminous Newtoniana or confine myself solely to the *Principia* and personally assess the intellectual achievement that it is. I chose the latter course, recalling the motto that 'to read for oneself a play of Shakespeare is worth a heap of commentaries'. My study of the *Principia*, so begun, continued to smoulder till it became earnest enough to embark, a year ago on 'Newton's *Principia* for the Common Reader' (1995, Clarendon Press, Oxford). And I may add that during the course of my study, I did not find a single Proposition from which I did not learn something new, some things that I did not know, or some things I should have known.

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II

It is common knowledge that Newton proved Kepler's laws of planetary and satellite motions that bodies (idealized as point masses) describe an ellipse, a hyperbola, or a parabola about the focus on the assumption that they are subject to centripetal attraction inversely as the square of the distance. And further, that Newton derived his universal law of gravitation on that basis. That in truth is a caricature of how Newton did, in fact, arrive at his universal law. Instead of elaborating on this theme, I shall give a brief account of the manner in which Newton proved the second law of Kepler and in what mathematical context.

Newton proved in Proposition X that under a law of centripetal attraction proportional to the distance, a particle will describe an ellipse about its centre; and in Proposition XI he proved that under a law of attraction inversely proportional to the square of the distance the particle will also describe an ellipse but about its focus. And he shows how Proposition XI can be deduced from Proposition X; and conversely. In my experience of giving lectures on this topic and talking to students and colleagues, I have rarely found one who had even thought that the two Propositions could be related.

But 'Cut the cackle and come to the hosses!'

III

The theorem that Newton proves can be stated as follows:

Under the action of a central force inversely as the square of the distance, a particle will describe an ellipse about its focus; and a particle with an equal constant of areas (i e angular momentum) will describe the same ellipse about its centre if the law of force is as the distance, and conversely.

The theorem is proved in two steps

(i) We start with the elementary kinematical fact that the normal acceleration (or C.F. for the component of centripetal force in that direction) that a particle describing a curvilinear orbit experiences is

$$\text{C.F. (in the direction } PO) = v^2/\rho, \quad (1)$$

where v denotes the velocity, ρ the radius of curvature, and PO the direction of the inward normal at P (see Figure 1). If this orbit should be described under a centripetal attraction directed towards S, then

$$\text{C.F. (towards } S) = \text{C.F. (in the direction } PO) \times \operatorname{cosec} \varepsilon = \frac{v^2}{\rho \sin \varepsilon}, \quad (2)$$

where ε is the angle of inclination of the direction of motion, PQ, to PS. On the other hand by the law of areas ('Kepler's first law' proved for the first

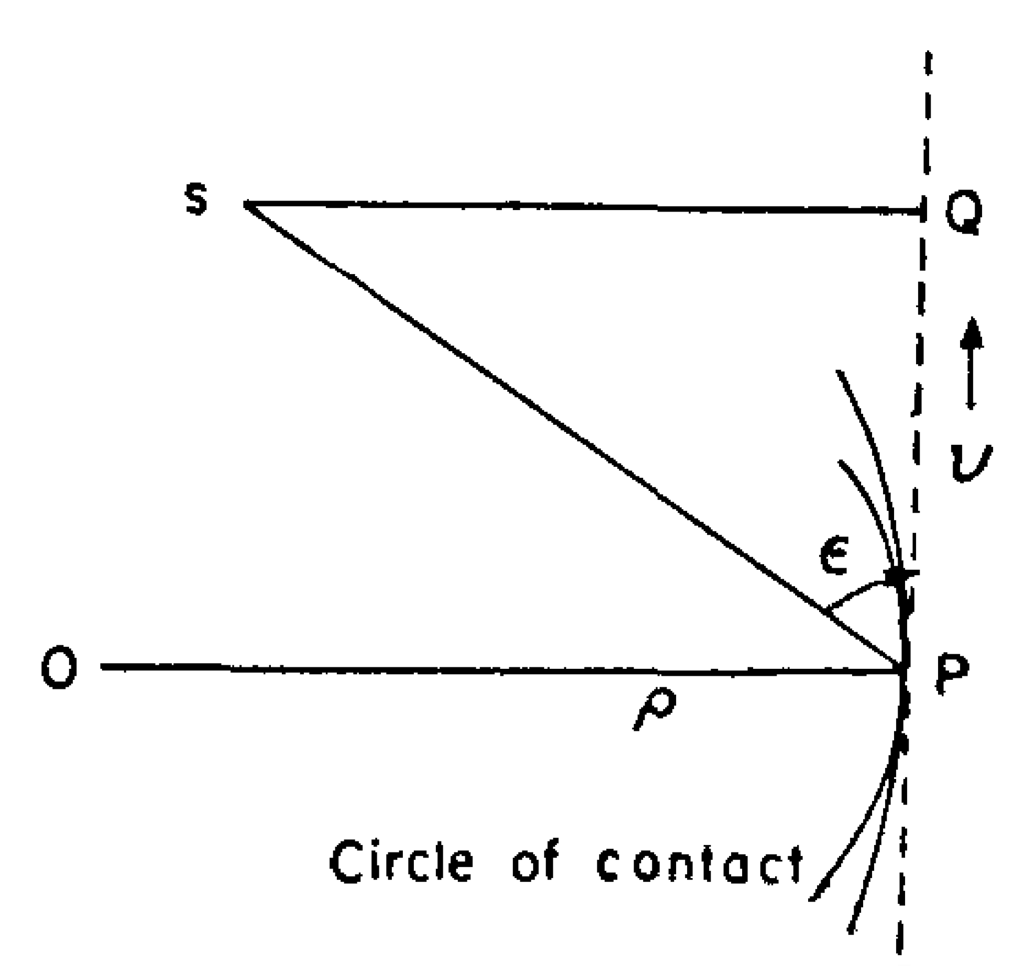

Figure 1.

time in Proposition I) in a short interval of time Δt,

$$(v\Delta t)\times SQ=\tfrac{1}{2}A\Delta t, \qquad (3)$$

where A is the constant of areas. Therefore

$$v=\frac{A}{2SQ}=\frac{A}{2SP\sin\varepsilon}, \qquad (4)$$

a relation proved in Corollary I of Proposition I. By combining equations (2) and (4) we obtain,

$$\text{C.F. (towards } S)=\frac{A^2}{4(SP)^2\rho\sin^3\varepsilon}. \qquad (5)$$

So far no assumption has been made about the nature of the orbit. If we now assume that it is an ellipse, then by the geometry of the ellipse

$$\rho\sin^3\varepsilon=\text{semi-latus rectum}$$

$$=\tfrac{1}{2}\frac{b^2}{a}=\tfrac{1}{2}L, \qquad (6)$$

where a and b are the semi-axes of the ellipse. (If the reader does not know this relation, it will be useful for him to refresh his memory of what he must have learnt in school!) Therefore, in this case

$$\text{C.F. (towards } S)=\frac{A^2}{2L(SP)^2}$$

$$=\text{Constant}\times(SP)^{-2}. \qquad (7)$$

(ii) Now suppose that quite generally the same curve is described by particles with the same angular momentum under two different laws of centripetal attraction towards two different centres S and C. Then it follows from equation (5) that (see Figure 2)

$$\frac{(\text{C.F.})_{\text{towards } S}}{(\text{C.F.})_{\text{towards } C}}=\left(\frac{\sin\varepsilon_c}{\sin\varepsilon_s}\right)^3\left(\frac{CP}{SP}\right)^2. \qquad (8)$$

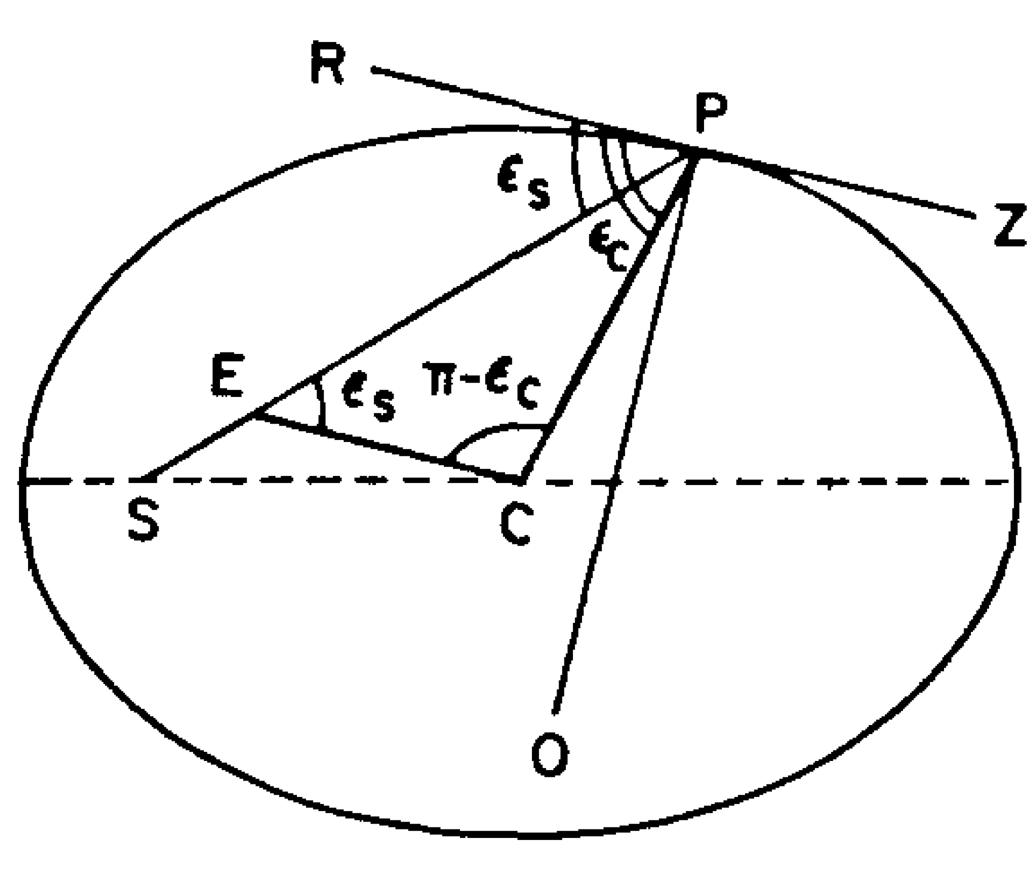

Figure 2.

Table 1. Dual laws of Centripetal attraction

attractive, r^{-2} and attractive, r	(elliptical orbits);
attractive, r^{-2} and repulsive, r	(hyperbolic orbits);
repulsive, r and repulsive, r	(the conjugate branches of a hyperbola);
repulsive, r^{-2} and attractive r^{-2}	(hyperbolic orbits with the centre of force at the focus of the conjugate hyperbola),
attractive, r^{-5} and attractive, r^{-5}	(a circle with the centre of attraction on any point of the circumference).
(The last two are self-dual).	

In Figure 2, RZ is the tangent and PO is the inward normal to the curve at P and ε_s and ε_c are the respective angles of inclination of PS and PC to the direction of motion ZPR.

Now, consider the case when the orbit is an ellipse with centre C and focus S. Draw CE parallel to ZPR. Then

$$\varepsilon_s=\angle RPS=\angle PEC,$$

and

$$\varepsilon_c=\angle RPC=\pi-\angle ECP. \qquad (9)$$

Hence

$$\frac{\sin\varepsilon_c}{\sin\varepsilon_s}=\frac{\sin\angle ECP}{\sin\angle PEC}=\frac{EP}{CP}; \qquad (10)$$

and equation (8) gives

$$\frac{(\text{C.F.})_{\text{towards } S}}{(\text{C.F.})_{\text{towards } C}}=\left(\frac{EP}{CP}\right)^3\left(\frac{CP}{SP}\right)^2$$

$$=(EP)^3\frac{1}{(CP)(SP)^2}. \qquad (11)$$

Again from the geometry of the ellipse (as the reader should be able to verify if he does not know it already),

$$PE=a=\text{the semi major axis of the ellipse.} \qquad (12)$$

Hence,

$$\frac{(\text{C.F.})_{\text{towards } S}}{(\text{C.F.})_{\text{towards } C}}=a^3\frac{1}{(CP)(SP)^2}. \qquad (13)$$

From this relation it follows that if the centripetal attraction towards the focus is proportional to $(SP)^{-2}$, then the centripetal attraction towards the centre is proportional to CP: and conversely

Q.E.D (!)

Could anything be simpler?

I should not be surprised if none of my readers have known of the foregoing demonstration. Yet, they are included between pages 46 and 51 in the 1722-third edition of the *Principia*.

IV

In continuation of Proposition X and XI Newton considers the more general question of when two different laws of centripetal attraction (and/or repulsion), particles with equal angular momenta will describe the same orbit. If we call two such laws as dual to one another, Newton established the following laws of force, with integral exponents, as dual (Table 1)

Newton missed only one other pair (r^{-4} and r^{-7}) which has been shown to be dual only recently.

V

What I have described in § III is only one example from Book I out of many. I could equally well have chosen:

(1) the geometrical and analytical solutions of Kepler's equation (Lemma XXVIII and Proposition XXXI);
(2) the formulation and solution of initial-value problems (Propositions XXXIX, XLI and LXIII);
(3) the variation of the elements of a Kepler orbit caused by an external perturbation and the foundations of lunar theory;
(4) a spherical distribution of matter attracts an external corpuscle as if its entire mass is concentrated at the center (Proposition LXXVI which has been described as a 'superb theorem' by J. L. Glaisher);
(5) the discovery of the method of images, normally credited to Lord Kelvin (Proposition LXXXII);
and many more.

I am convinced that one's knowledge of the Physical Sciences is incomplete without a study of the *Principia* in the same way that one's knowledge of Literature is incomplete without a study of Shakespeare.

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