Selection rules for quasilattices

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Theory of tiling in general and of quasiperiodic tiling in particular has been studied extensively since the discovery of quasicrystals. Several algorithms have been proposed to generate quasiperiodic tiling in two-dimensional plane as well as in three-dimensional space. A new algebraic approach to generate such tilings is developed and discussed. The four-integer representation of a two-dimensional quasiperiodic system is explored to arrive at the analytical conditions for generating quasiperiodic tiling.

THE novel procedure of decorating a plane quasiperiodically with a minimum of two types of rhombii as proposed by Penrose[1] and the computation of diffraction patterns of such decorations have been studied extensively even before the discovery of a quasicrystalline phase. A close resemblance between the diffraction patterns[2] of quasicrystalline phase[3] and the computed diffraction patterns of three-dimensional (3D) quasiperiodic tiling[4] (a 3D extension of Penrose's idea) aroused a general interest among physicists and mathematicians to explore further the theory of tiling in general and quasiperiodic tiling in particular.

During the past one decade, several interesting and elegant algorithms have been proposed to construct quasiperiodic tilings. Some of the algorithms are based on theorems of higher-dimensional geometry ($D > 4$)[5–7], while the others make use of geometrical constructions along with certain selection-making steps[8]. The higher-dimensional approach, while being elegant and capable of yielding the geometric structure factor, suffers from the drawback of an indirect choice of hyperspace and hypercell. It is silent over the growth process of tiling and works at the cost of visualization. The other group of methods involves selection-making steps of two or more legal choices. Amongst all these legal choices, only one is correct and the others lead to an incorrect tiling, which comprises frustrations and violates the quasiperiodic transitional order. Also, it does not provide any further input to the physical properties of such configurations. Moreover, both these conjectures fail to establish unambiguously a definite mathematical correlation amongst all the vertices generated by such algorithms, which is necessary to provide a lattice-like definition to the generated tilings.

In the present communication, we will briefly discuss a newly proposed algorithm by Ramachandrarao *et al.*[9] and generalize it further to obtain purely analytical conditions for generating quasiperiodic lattices.

The algorithm proposed by Ramachandrarao *et al.*[9] involves cyclic repetition of a given set of geometrical operations, very similar to normal crystallographic operations, on a rhombus with included angles of 72° and 144° and edge length a. The pattern generated after six cycles of operations is shown in Figure 1. Mathematical simplifications establish that the coordinates of the vertices of 2D Penrose tiling (2DPT), may be expressed in a generalized form as[9]

$$X = a_N(m + n\tau)\sin(\pi/5)$$

$$Y = a_N(p + q\tau)\cdot\tfrac{1}{2}$$

a_N is the edge length of the rhombii in the tiling, τ is the golden mean, and m, n, p and q are integers. The existence of a global fivefold rotational centre in 2DPT invokes the following parity conditions on these integers.

m	n	p	q
Even	Even	Even	Even
Odd	Even	Even	Odd
Odd	Odd	Odd	Even
Even	Odd	Odd	Odd

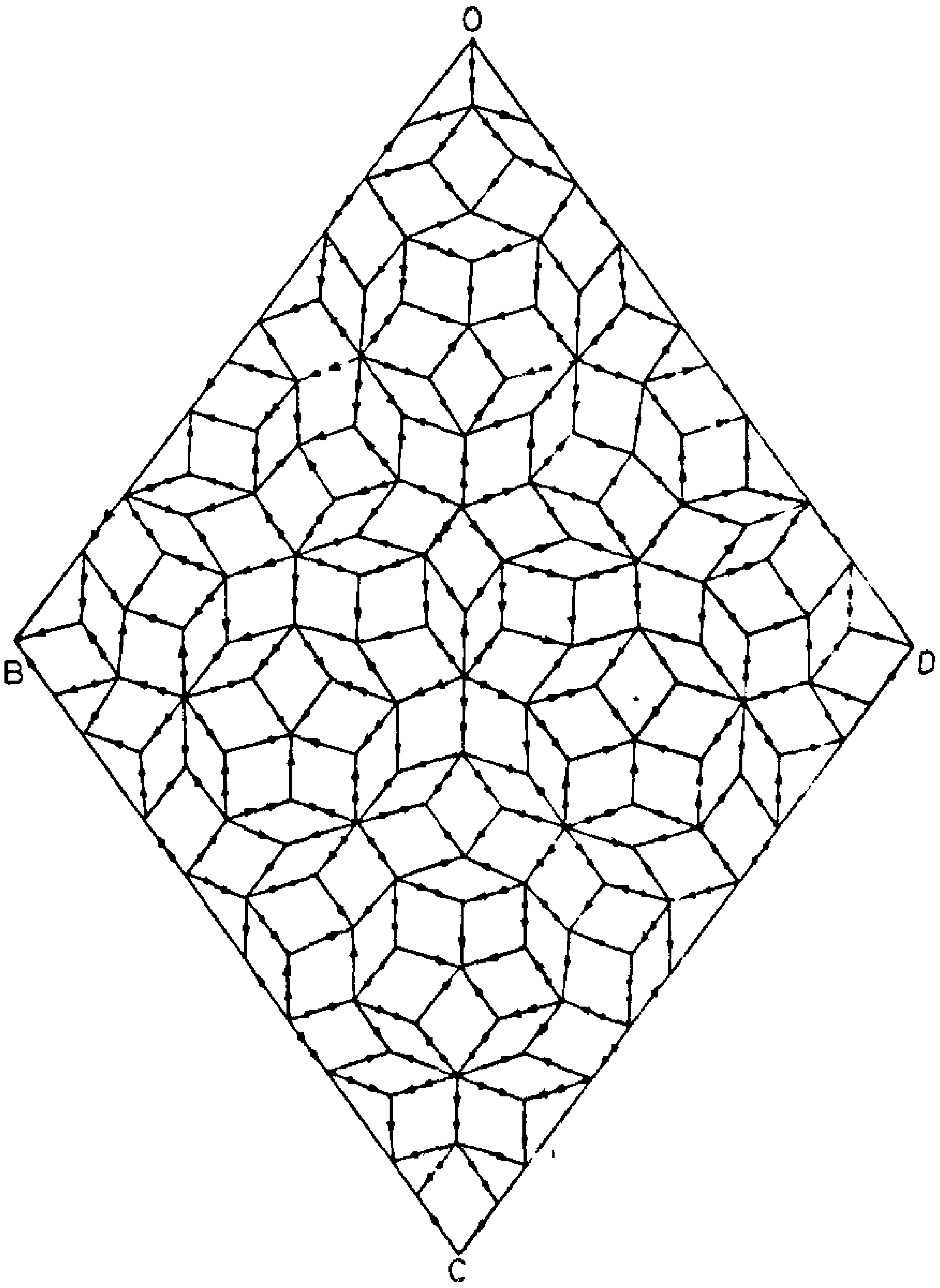


Figure 1. The tiling generated in the sixth generation

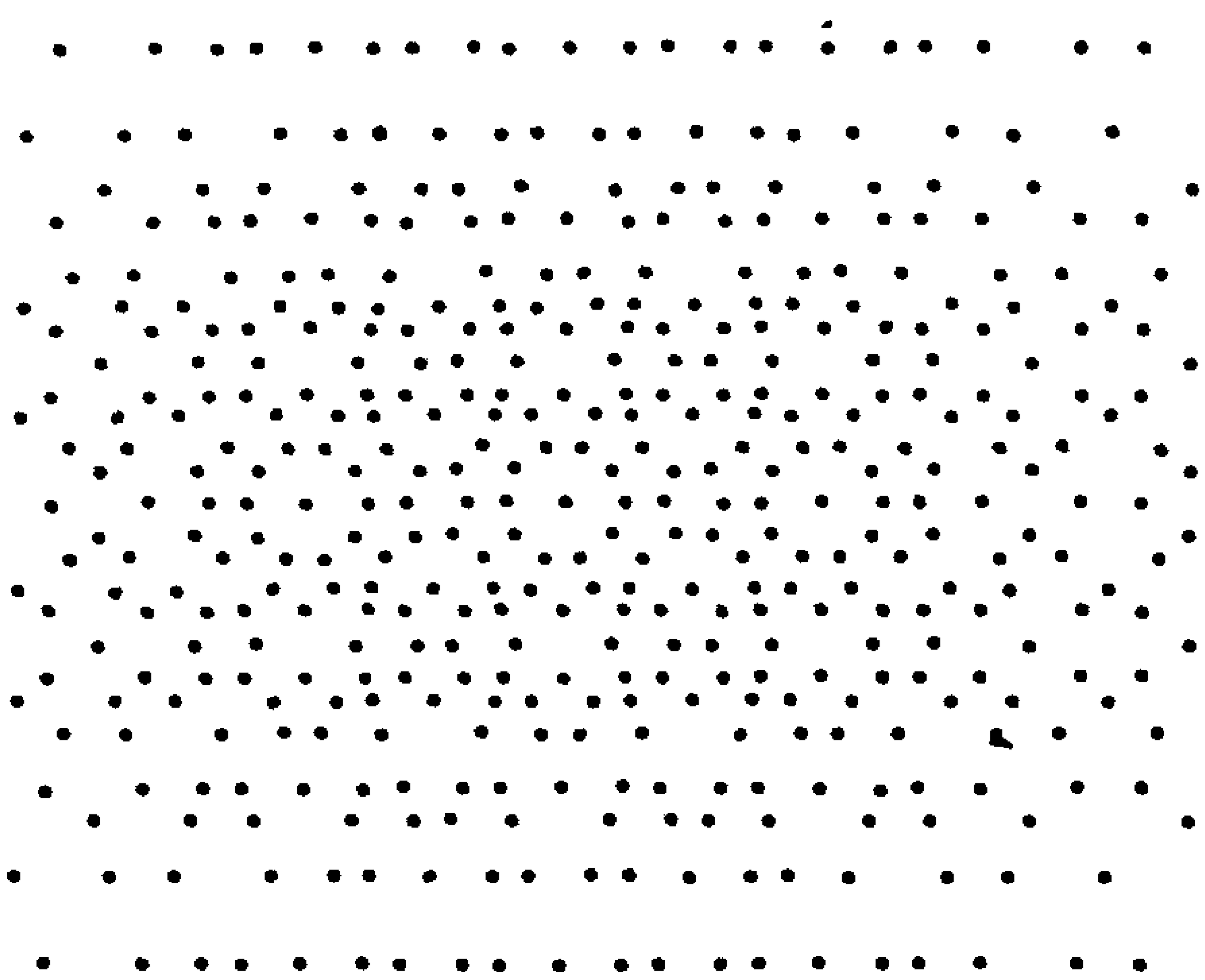


Figure 2. Quasilattice with local non-crystallographic rotational centre

These generalized coordinates and the parity conditions provide a new possibility to generalize algebraically the above algorithm to generate 2DPT, which may further throw some light on several important aspects of a quasiperiodic tiling.

The generalized form of coordinates suggests that quasilattices may be described by integer domains (closed set of integers) of m, n, p and q. These domains are obtained by simple mathematical considerations. Let the area which is to be quasiperiodically decorated be bounded by

$$(0, 0), (-X_{max}, -Y_{max}), (X_{max}, -Y_{max}) \text{ and } (0, -Y_{max}).$$

We can write

$$X_{max} = a_N (F_{N-1} + F_N \tau) \sin(\pi/5),$$

$$Y_{max} = (a_N/2)(2F_N + 2F_{N+1} \tau),$$

where F_{N-1}, F_N and F_{N+1} are the $(N-1)$th, Nth and $(N+1)$th elements of Fibonacci series.

We can also write

$$X_{max} = a_N (m_{max} + n_{max} \tau) \sin(\pi/5),$$

$$Y_{max} = (a_N/2)(p_{max} + q_{max} \tau).$$

A comparison of the above equations simply dictates that the domains of integers are given by

$$|m| \le F_{N-1},$$

$$|n| \le F_N,$$

$$|p| \le 2F_N,$$

$$|q| \le 2F_{N+1}.$$

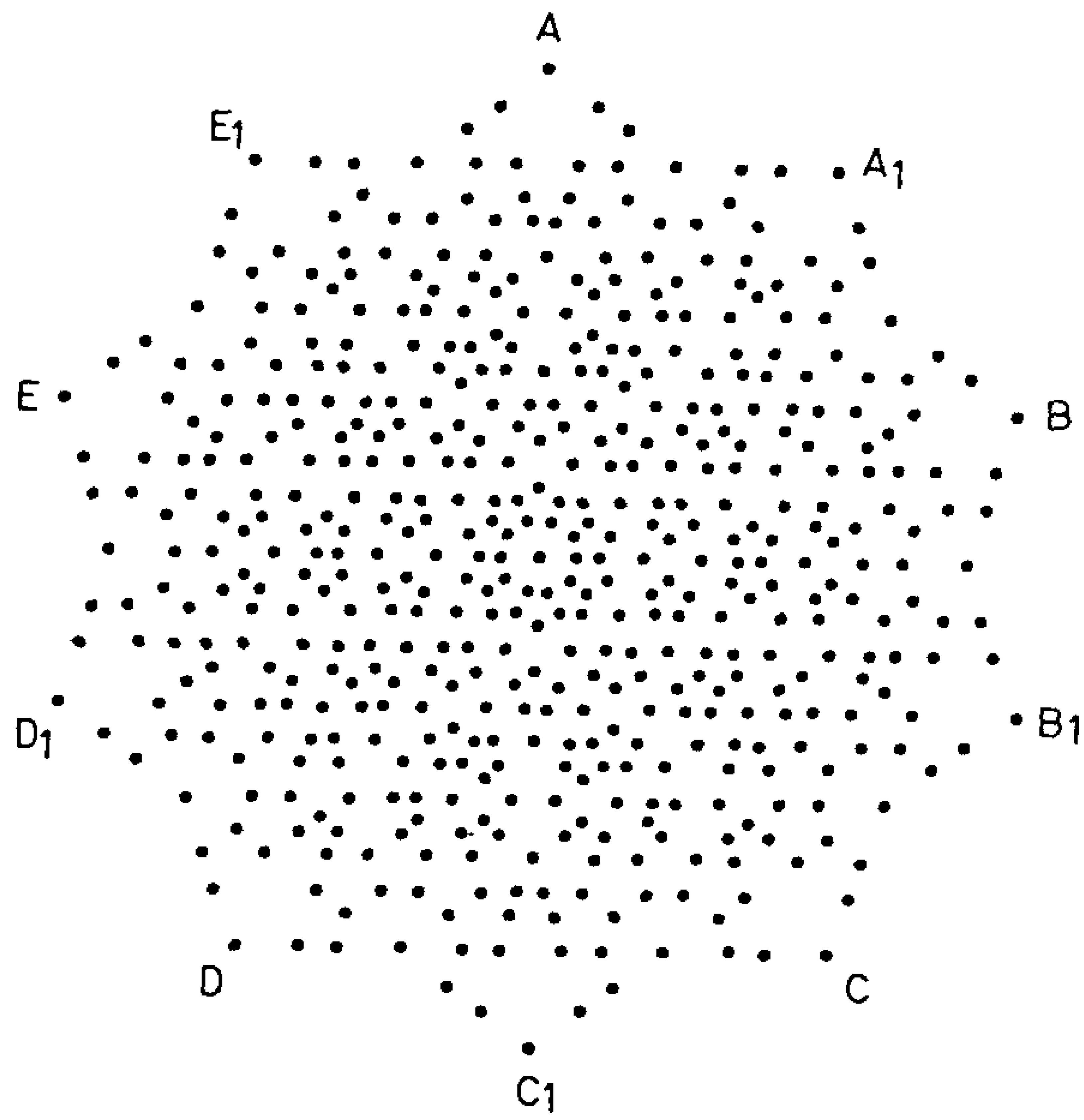


Figure 3. Quasilattice with tenfold rotational symmetry

Hence, one can fix up the range of integers for a given area of interest. If all possible sets of the four integers within these domains are accepted and plotted in a cartesian frame of reference, the generated pattern will be a superimposition of several periodic lattices shifted with respect to each other by a linear multiple of τ. Now, we may put a window in terms of parity conditions over the sets of integers and choose the sets which qualify any of the four parity conditions. When plotted together, the points that qualify yield a quasiperiodic point lattice with a local non-crystallographic rotational centre (Figure 2). However, the global rotational symmetry is crystallographic. Thus, it establishes that the parity conditions are necessary but not sufficient to generate a quasilattice with non-crystallographic rotational symmetry.

The quasilattice with tenfold rotational symmetry (Figure 3) is generated by subjecting the parity-satisfying sets of integers to the following further conditions:

$$3m^2 + 2n^2 + p^2 + q^2 - 2mn > F_{N-1}, \quad (1a)$$

$$-m^2 + n^2 + q^2 + 4mn + 2pq > F_N. \quad (1b)$$

The above two conditions are derived based on the fact that the dimensions of the area of interest are described as a function τ in such a way that the extreme points are located at a radial distance R_{max} from the centre, where

$$R^2_{max} = F_{N-1} + F_N \tau.$$

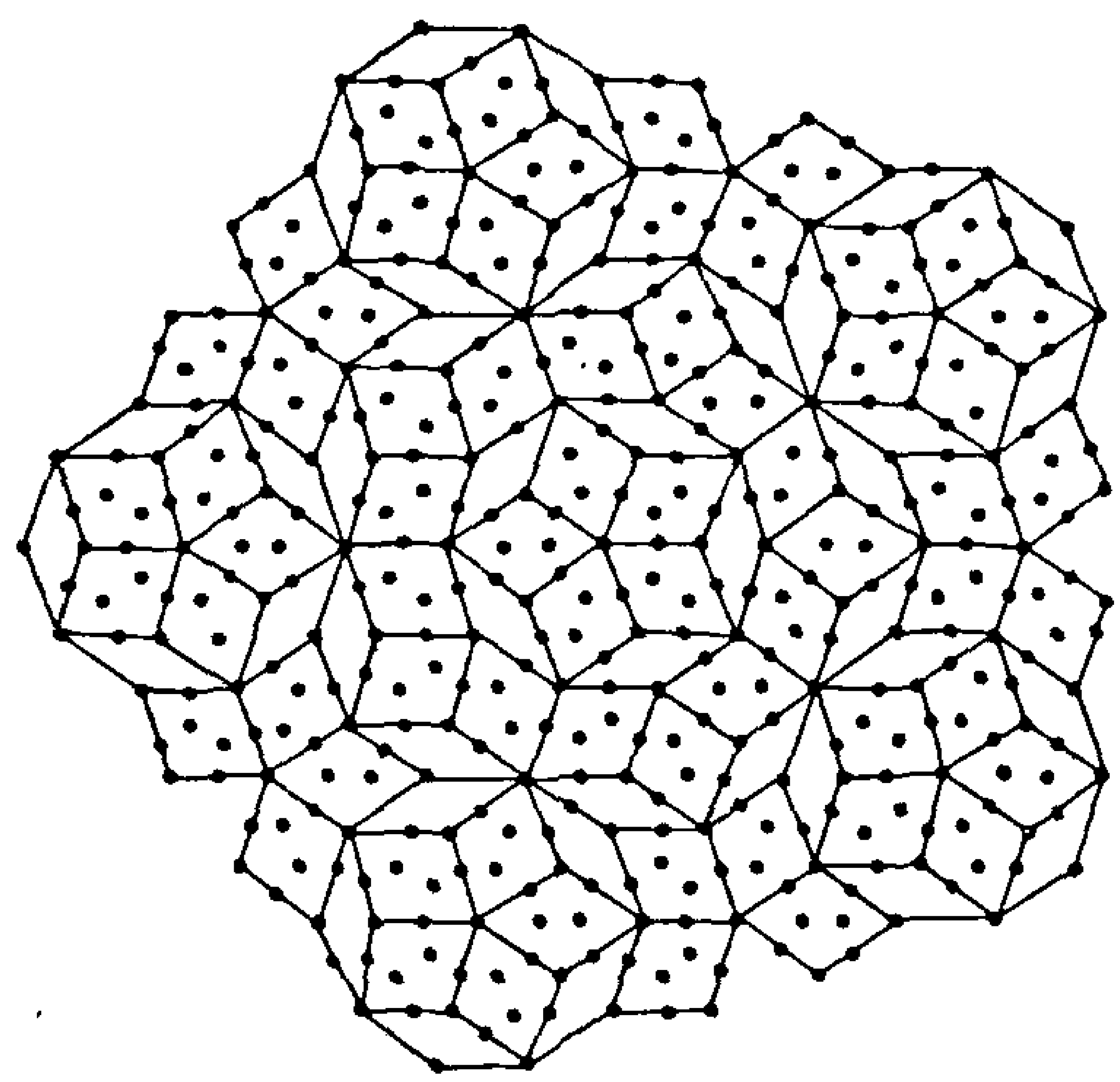


Figure 4. Penrose tiling – a subset of decagonal quasilattice.

A necessary and sufficient condition for a point (m, n, p, q) to lie within the area may now be derived as

$$R^2 < R^2_{\max},$$

where the radial distance (R) of a point from the origin $(0, 0)$ is given by

$$R^2 = (3m^2 + 2n^2 + p^2 + q^2 - 2mn) + \tau(-m^2 + n^2 + q^2 + 4mn + 2pq). \quad (2)$$

It ultimately yields the conditions given above (eqs. (1*a*) and (1*b*)). A violation of any of these two conditions along with parity conditions destroys the basic characteristic features of the quasiperiodic lattice from the generated pattern.

One can note that the quasiperiodic lattice generated by these algebraic conditions possesses all the vertices of 2DPT of a definite edge length as a set (Figure 4). The remaining lattice points will also become the vertices of the tiling at a higher order of deflation. Selection of 2DPT vertices only needs some more conditions on the parity-satisfying sets of m, n, p and q. Such conditions are discussed below.

To derive the analytical conditions for the selection of 2DPT vertices, let us consider the 2DPT within a frame of reference of five oblique axes, each at 72° separation with its successor. Any vertex in such a frame of reference can be defined as

$$R = \sum_{i=1}^{5} M_i V_i,$$

where M_i are integers and V_i are the unit vectors along the vertices of a regular pentagon. If a be the magnitude of unit vectors, any arbitrary vector R will be given by

$$R = a[M_1V_1 + M_2V_2 + M_3V_3 + M_4V_4 + M_5V_5].$$

Resolving it into x and y components, we get

$$R_x = a[(M_3 - M_4) + \tau(M_2 - M_5)] \sin(\pi/5),$$

$$R_y = (a/2)[(2M_1 - M_2 - M_5) + \tau(M_2 - M_3 - M_4 + M_5)].$$

M_i being integers, we can write

$$m = M_3 - M_4, \quad (3a)$$

$$n = M_2 - M_5, \quad (3b)$$

$$p = 2M_1 - M_2 - M_5, \quad (3c)$$

$$q = M_2 - M_3 - M_4 + M_5. \quad (3d)$$

Since the set of M_i also represents the vertices of a 5D cube when associated with five linearly independent orthogonal vectors, the obtained relations (Eqs. (3*a*)–(3*d*)) unambiguously establish a link between 2D indices (m, n, p, q) and 5D indices (M_i). It further makes it possible in the following manner to compute directly the 5D indices for a given set of four integers.

We know that in 2DPT

$$0 < \Sigma M_i < 4.$$

Substituting the above-derived relations (Eqs. (3*a*)–(3*d*)) in the condition on ΣM_i, we get

$$0 < -(2p_i + q_i) + 5M_i < 4, \quad (4)$$

where p_i and q_i are the values of p and q after a rotation of $2i\pi/5$ $(i = 1\text{–}5)$. Thus, for each value of i one can determine the values of M_i for a given set of m, n, p and q. This complete mapping of two sets of integers enables us to perform the projection mechanism (5D onto 2D) algebraically in terms of integers of 2D cartesian system. It also establishes that the coordinates in pseudospace can also be directly determined by the following relations:

$$m^\circ = -n,$$

$$n^\circ = m,$$

$$p^\circ = -(p + q),$$

$$q^\circ = -q,$$

where m°, n°, p° and q° are the indices of a point M_i when projected in pseudospace and possess the indices m, n, p and q in physical (real) space. A substitution of the above relations in the projection mechanism formulates a similar approach in 2D space only, where the area of interest is called true plane and is characterized by m, n, p and q. Similarly, there is a test plane characterized by m°, n°, p° and q°. The area of the test plane differs for each set of the value of $-(2p + q)$, which determines the height (ΣM_i) of the projected point in pseudospace.

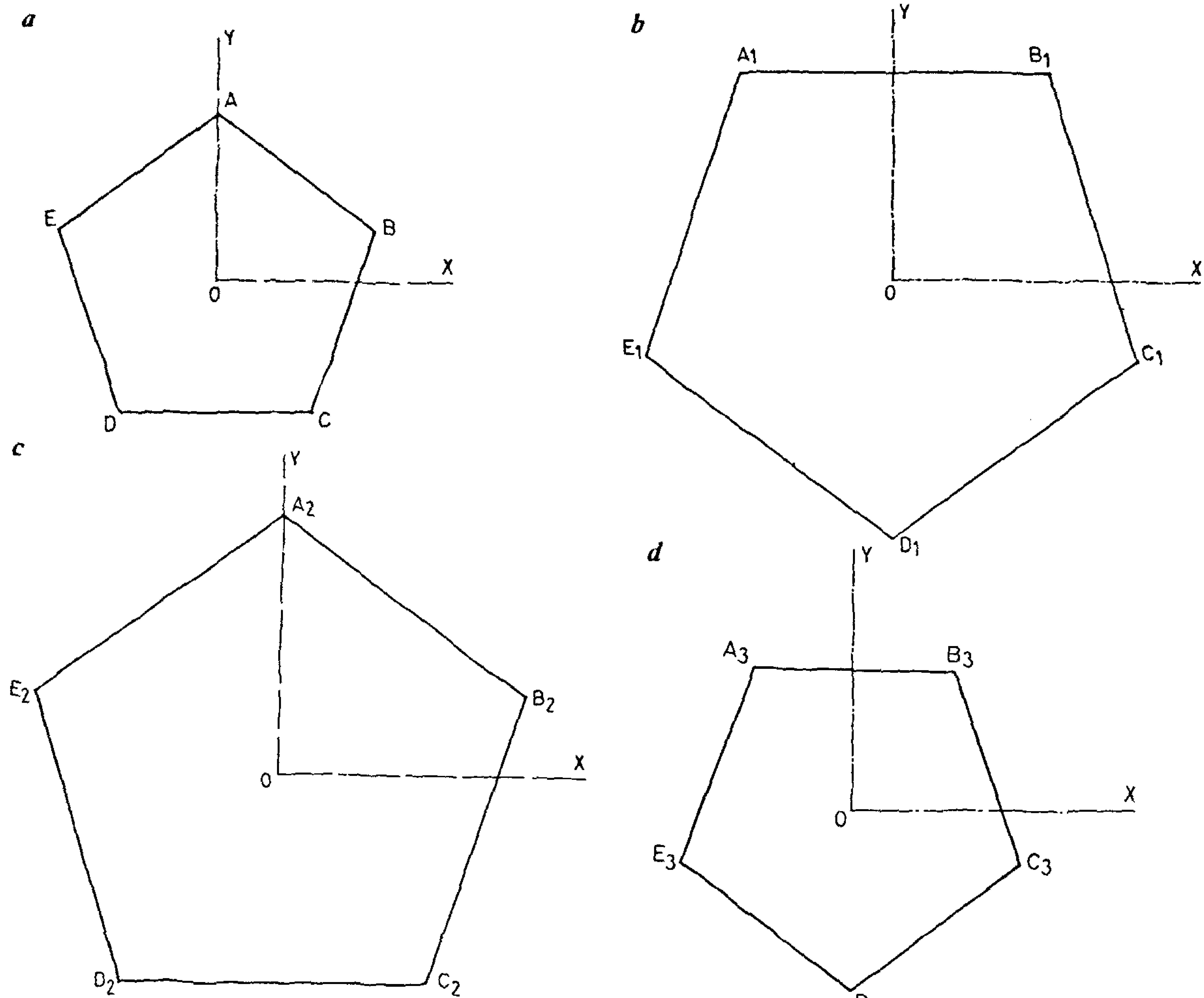
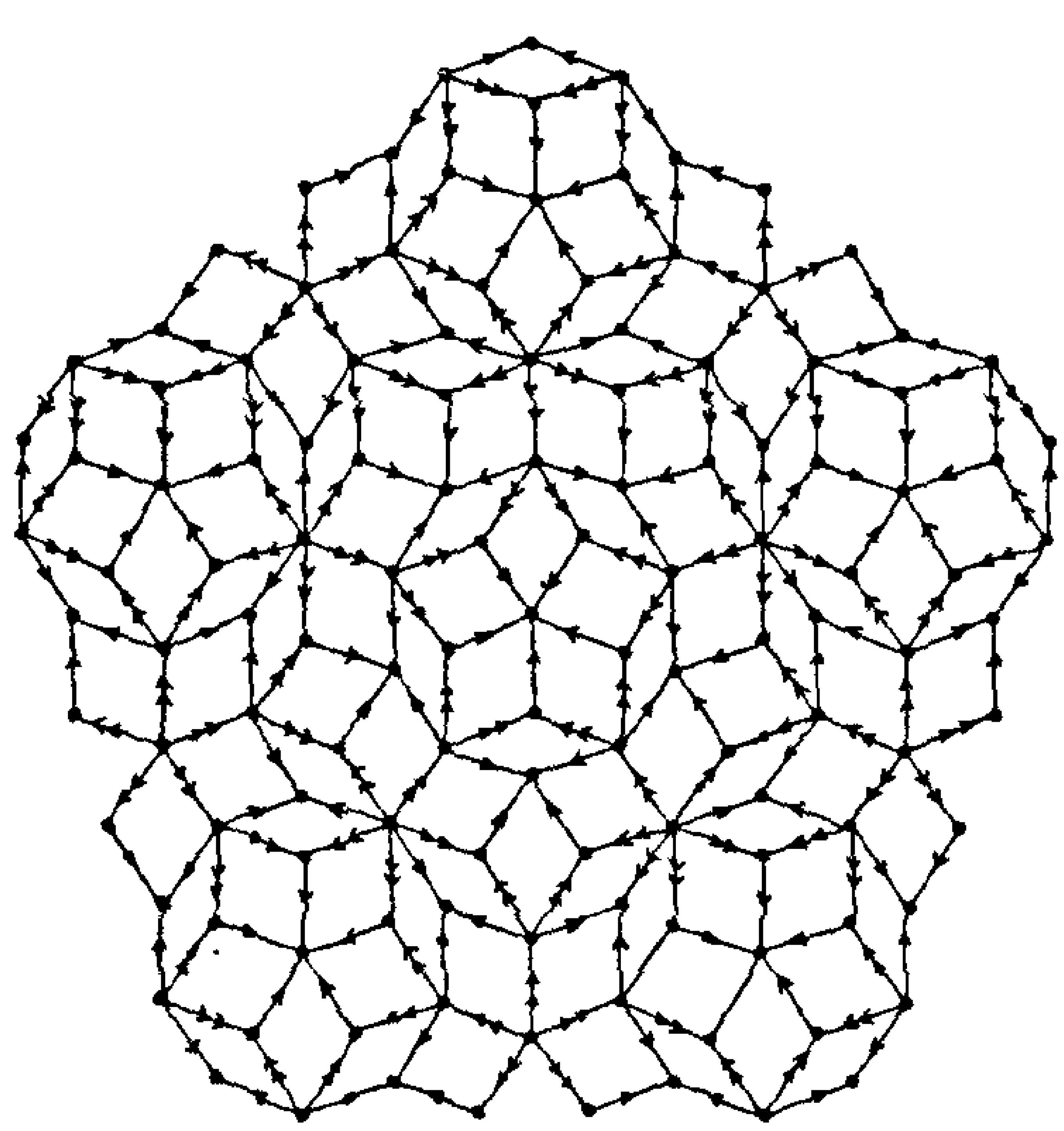


Figure 5. ***a***, Test plane for the first set of $-(2p + q)$, ***b***, test plane for the second set of $-(2p + q)$, ***c***, test plane for the third set of $-(2p + q)$; ***d***, test plane for the fourth set of $-(2p + q)$



Our conditions for selecting a point for 2DPT dictates that its test plane counterpart should be within the bonds of that test plane. Test planes are constructed with the following ranges (Figure 5):

$m^\circ = -1$ to 1,

$n^\circ = -1$ to 1,

$p^\circ = -2$ to 2,

$q^\circ = -2$ to 2.

◀ **Figure 6.** Penrose tiling generated by analytical conditions

Applying the concept of test plane for the selection, we get the following conditions on a point to belong to 2DPT:

$$\pm(p_i + q_i) + \tau(1 + q_i) > 0,$$

$$(1 + p_i + q_i) + (1 \pm q_i) > 0.$$

The upper signs are used for the first and second test planes and the lower signs are used for the third and fourth test planes. The pattern generated by the above conditions on a given range of m, n, p and q is shown in Figure 6. It comprises the 2DPT vertices and a global fivefold rotational centre.

Starting with the novel geometrical algorithm proposed by Ramachandrarao *et al.*[9], we have derived analytical conditions for the generation of 2DPT. We establish an entirely new algorithm to generate 2DPT and make the geometrical algorithm only a mental aid. The present approach highlights effectively certain interesting aspects of quasiperiodic lattices. A relation between the indices (m, n, p, q) with the indices in higher-dimensional space (M_i) is also rewarding in considering the geometrical algorithm in 5D space. This study, for the first time, establishes a long-range definite mathematical order among the vertices selected by a window (results to be published elsewhere).

1. Penrose, R., *Bull Inst Math Appl*, 1974, **10**, 266–271
2. Makay, A L., *Sov. Phys Crystallogr.*, 1981, **26**, 517–522.
3. Schechtman, D, Blech, I, Gratias, D and Cahn, J. W, *Phys Rev Lett*, 1984, **53**, 1951–1953
4. Levine, D and Steinhardt, P J, *Phys Rev Lett*, 1984, **53**, 2477–2480
5. Conway, J. H and Knowles, K. M, *J. Phys A*, 1986, **A19**, 3645–3653.
6. Bruijn, N G de, *Proc K Ned. Akad Wet Ser*, 1981, **A43**, 39–66
7. Janic, M V, *Phys Rev B*, 1986, **B36**, 4685–4698.
8. Sasisekharan, V, *Pramana*, 1986, **26**, L283–294
9. Ramachandrarao, P, Sastry, G V. S, Pandey, L. and Sinha, A, *Acta Crystallogr*, 1991, **A47**, 206–210
10. Sinha, A, Ph D Thesis, Banaras Hindu University, 1993

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Application of drag-reducing polymers in agriculture

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Drag-reducing polymers reduce the drag in a turbulent flow (by a mechanism not yet fully understood) while increasing the drag in a laminar flow, due to an increase in the shear viscosity. This feature of drag-reducing polymers has been utilized in reducing the energy requirements of sprinkler irrigation systems. Their use also increases the water throughput and the area of coverage of the sprinkler irrigation system. The water containing drag-reducing polymers percolates slowly in the soil, thus reducing the percolation losses of water. Utilizing this aspect, a slow-release urea has been developed by blending urea with guargum. The present paper outlines the results of a detailed investigation carried out at the Indian Institute of Technology, Kharagpur, on the application of drag-reducing polymers in agriculture.

EXTREMELY minute concentrations of large polymer molecules, fibres or particles when present in a fluid cause reduction in the friction resistance in a turbulent flow compared to that of the fluid alone. This phenomenon is called drag reduction and was first reported by Toms[1] in 1949. The historical perspectives of drag reduction phenomenon have recently been illustrated in the review of Singh[2]. In 1961, US Naval Laboratories began an investigation into drag reduction. The creditable studies of their scientists have given new dimensions to research in the field of turbulent drag reduction and drew worldwide attention. The practical implications of this phenomenon were first realized by Savins[3], who also applied the term drag reduction for the first time. A large number of papers and reports on an international scale confirming and extending the details of the drag reduction effect have appeared since these early investigations. The recent papers by Hoyt[4], Singh[2] and Shenoy[5] review the various aspects of this phenomenon and are good general references. Ever since the discovery of drag reduction with additives, many practical applications have been suggested and a few have already materialized. The list of possible areas of applications has increased enormously to include oil well fracturing, crude oil and refined petroleum product transport, fire fighting, irrigation, sewage and flood water disposal, hydrotransport of solids, water heating circuits, jet cutting, hydraulic machinery, marine applications and biomedical applications. A review by Sellin *et al.*[6] covers the progress in all these application fields except for biomedical applications. Sellin's review paper[7] discusses the applications of drag reduction with more emphasis on sewage and flood water disposals. Pollert[8] discusses its applications in the field of hydrotransport of solids, water heating circuits and agricultural irrigation in detail.

A programme of drag reduction investigations was initiated at the Indian Institute of Technology, Kharag-