Reflecting diffusions*

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Probabilistic models of diffusion processes with 'reflecting boundary conditions' are discussed; the domains considered can be nonsmooth.

THE phenomenon of diffusion in free space (in a dilute solution) can be modelled in a mathematically rigorous fashion by probabilistic methods; in fact, this has been one of the major achievements of probability theory. This has led to a very fruitful interaction between the theory of partial differential equations and the theory of Markov processes; the interface is now known among probabilists as the theory of diffusion processes[1-5].

The aim of this exposition is to indicate how probabilistic methods are applied to model diffusions with 'reflecting boundary conditions'. To keep the length of the article within reasonable limits, we do not consider asymptotic properties, connections with boundary value problems, etc. And for the same reason it is inevitable that we indulge in a bit of probabilistic jargon without explanation; see refs 1, 3, 6–8 for more information concerning unexplained technical terms like submartingales, Markov property, stochastic integrals, ...

Diffusions in $\mathbb{R}^d$

We now briefly review three ways of modelling diffusion processes in $\mathbb{R}^d$. This will help the reader appreciate the approaches taken later to study reflecting diffusions. Our notation will also get fixed in the process.

Let the operator A be defined by

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}, \qquad (1)$$

where the coefficients a_{ij}, b_i are sufficiently smooth real-valued functions on $\mathbb{R}^d$; A is assumed to be uniformly elliptic; $b(\cdot)$ is the *infinitesimal drift* and $a(\cdot)$ the *infinitesimal dispersion*.

Semigroup approach

Let $p(s, x; t, z)$, $0 \le s < t$, $x, z \in \mathbb{R}^d$ denote the (minimal) fundamental solution for the *Fokker–Planck equation*

$$\frac{\partial u}{\partial t}(t,z) = A_z^* u(t,z), \ t > 0, \ z \in \mathbb{R}^d, \qquad (2)$$

where A^* is the formal adjoint of A, and the subscript z denotes differentiation in the z variables; that is, if the initial value $u(0, \cdot) = f(\cdot)$ is specified then u can be given by

$$u(t,z) \equiv (T_t^* f)(z) = \int_{\mathbb{R}^d} f(x)p(0,x;t,z)\,dx. \qquad (3)$$

Since the concentration of a diffusing species satisfies the Fokker–Planck equation, knowledge of p determines the system completely once the initial concentration is known.

It can be shown that $p(s, x; t, \cdot)$ is a probability density function, satisfies the so-called Chapman–Kolmogorov equation, etc.; in other words, p is a transition probability density function (in the sense that $p(s, x; t, z)\,dz$ gives the probability that a diffusing particle starting from x at time s reaches a sufficiently small neighbourhood of z at time t). Hence, by the theory of Markov processes, there is a continuous-time Markov process in $\mathbb{R}^d$ with continuous sample paths for which p is the transition density. This process can be called *diffusion with generator A*. As p satisfies the Chapman–Kolmogorov equation, $\{T_t^* : t \ge 0\}$ given by eq. (3) forms a semigroup of operators. This is the classical approach to diffusions, pioneered by Kolomogorov and Feller, and depends on PDE theory for the guaranteed existence of the fundamental solution.

If $a(\,) \equiv (d \times d)$ identity matrix, $b(\,) \equiv 0$, then $A = (1/2)\Delta = A^*$, eq. (2) is the heat equation, p is the heat kernel and the corresponding diffusion is, of course, the standard Brownian motion (this explains the fondness for the factor 1/2 in eq. (1) among probabilists!). This has been the prototype of diffusion and corresponds to the motion of a diffusing particle purely due to fluctuations caused by bombardment by solvent particles in the absence of external forces, friction, etc.[8].

SDE approach

Levy had suggested that the motion of a diffusing particle can be represented, in differential form, by

$$dY(t) = \sigma(Y(t))dB(t) + b(Y(t))\,dt, \qquad (4)$$

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*This is a slightly enlarged verion of a lecture given at the Department of Mathematics, Indian Institute of Science, Bangalore on 10 February 1995 where $\sigma(x)$ is a matrix such that $\sigma(x)\sigma^*(x) = a(x)$, $\{B(t): t \geq 0\}$ is a standard d-dimensional Brownian motion. The idea is: in a short interval of time the diffusing particle behaves locally as a Brownian particle with appropriate 'standard deviation' σ subject to an 'infinitesimal drift' b. However, it is well known that the Brownian trajectories are nowhere differentiable. It has been a seminal achievement of K. Ito to give meaning for integrals of the form $\int f(s, \omega)\, dB(s, \omega)$ for a large class of processes f; such integrals are called Ito integrals. Then the formal differential expression (4) can be interpreted as the stochastic integral equation

$$Y(t) = Y(0) + \int_0^t \sigma(Y(s))\, dB(s) + \int_0^t b(Y(s))\, ds. \qquad (5)$$

When σ, b are Lipschitz-continuous and satisfy certain growth conditions, it was shown by Ito (essentially by Picard's iteration) that eq. (5) has a unique solution $Y(\cdot)$; also the solution Y is a continuous Markov process. If Y has a sufficiently smooth transition density p, then it can be proved that p is the minimal fundamental solution for eq. (2). Thus, the solution of the *stochastic differential equation* (4) can be called[3, 9] diffusion with generator A.

When $d = 1$, $a \equiv 1$, $b(x) = -\beta x$ with $\beta > 0$, we get the Ornstein–Uhlenbeck process.

Martingale problem

An important feature of Ito integrals is Ito's formula for transformation of variables: for any $f \in C_b^2(\mathbb{R}^d)$,

$$f(Y(t)) - f(Y(0)) - \int_0^t Af(Y(s))\, ds = \int_0^t \langle \sigma(Y(s))\nabla f(Y(s)), dB(s)\rangle. \qquad (6)$$

If σ is bounded, the Ito integral on the right-hand side of eq. (6) is a martingale.

Since Y is a continuous process it induces a distribution (probability measure) on the path space $\Omega = C([0, \infty) : \mathbb{R}^d)$ = the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$. Denote this distribution by P_x, where x represents the starting point (that is, $Y(0) \equiv x$). Let $X(t, \omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$. Then eq. (6) can be expressed as

$$f(X(t)) - f(x) - \int_0^t Af(X(s))\, ds = P_x - \text{martingale} \quad (7)$$

for any $f \in C_b^2(\mathbb{R}^d)$.

The above can be used to characterize a diffusion process. The point of view taken is that the family $\{P_x : x \in \mathbb{R}^d\}$ of probability measures on Ω contains all relevant information about the diffusion process. In their fundamental work Stroock and Varadhan[11] have shown that, if a, b are bounded, continuous, then for each x there is a unique P_x such that eq. (7) holds for any $f \in C_b^2(\mathbb{R}^d)$; also under $\{P_x\}$ the coordinate projections $\{X(t) : t \geq 0\}$ form a continuous Markov process. Moreover, if the coefficients a, b are sufficiently smooth then $\{P_x\}$ agrees with the distribution of diffusion as defined earlier. Thus, the family $\{P_x : x \in \mathbb{R}^d\}$ of probability measures solving the martingale problem given by eq. (7) can be called diffusion process corresponding to A.

One may consult refs 3, 9, 11 concerning the advantages of the various approaches.

Two models

Insulated heat conduction

The problem of heat conduction in a semi-infinite bar (of uniform cross-section) with the end kept insulated is described by

$$\frac{\partial u}{\partial t}(t, z) = \frac{1}{2}\frac{\partial^2 u}{\partial z^2}(t, z), \quad t > 0, \; z > 0 \qquad (8)$$

subject to the boundary condition

$$\frac{\partial u}{\partial z}(t, 0) = 0, \; t > 0. \qquad (9)$$

This is the one-dimensional heat equation with the Neumann condition at $z = 0$. Equations (8) and (9) describe also the concentration of a species undergoing Brownian movement (confined to $[0, \infty)$) in a dilute solution when no mass is allowed to 'escape through' 0 or to 'linger at' 0. It is well known that the fundamental solution for the above is given by

$$q(s, x; t, z) = \frac{1}{\sqrt{2\pi(t-s)}} \times \left[\exp\left\{-\frac{(z-x)^2}{2(t-s)}\right\} + \exp\left\{-\frac{(-z-x)^2}{2(t-s)}\right\}\right] \quad (10)$$

for $0 \leq s < t$, $x \geq 0$, $z \geq 0$.

By arguments analogous to those presented earlier, it can be shown that there is a Markov process (living in $[0, \infty)$) with q as its transition probability density. This process may be called *reflected Brownian motion* (RBM). If $\{B(t)\}$ is a one-dimensional Brownian motion (in $\mathbb{R}$), then it is easily seen that $\{|B(t)| : t \geq 0\}$ is a Markov process with q as its transition density; that is, $\{|B(t)| : t \geq 0\}$ is RBM.

An inventory model

Consider the following model in queueing theory:

$$Z(t) = B(0) + I(t) - [O(t) - L(t)] = B(t) + L(t), \qquad (11)$$

$t \geq 0$, representing the *inventory process* Z in a simple two-stage flow system consisting of input (or production), output (or demand) and an intermediate buffer storage of infinite capacity. Here $I(t)$ = cumulative input up to t; $O(t)$ = cumulative potential output (or demand) up to t, $I(0) = O(0) = 0$ and $B(0) \geq 0$ is the initial inventory level. It is assumed that the demand that cannot be met from the stock on hand is lost with no adverse effect on future demand; thus, $L(t)$ can be interpreted as the demand that could not be met up to t, and $O(t)–L(t)$ is the actual output over $[0, t]$. Clearly, $Z(t) \geq 0$. All are assumed to be continuous stochastic processes; L is called *lost demand process* and B the *netput process*[12].

Under certain conditions of *heavy traffic*, $\{B(t)\}$ can be modelled by one-dimensional Brownian motion: viz. (a) in any short time interval $[t_1, t_2]$, the input $I(t_2)–I(t_1)$ and the output $O(t_2)–O(t_1)$ are quite large but their difference (netput) $B(t_2)–B(t_1)$ is not large; (b) netput increments over nonoverlapping intervals are approximately independent.

With these assumptions the set-up (11) becomes

$$Z(t) = B(t) + L(t), \quad t \geq 0, \qquad (12)$$

satisfying (i) $Z(t) \geq 0$, Z is continuous in t, (ii) $Z(0) = B(0) \geq 0$, (iii) $L(0) = 0$, L is nondecreasing and continuous in t, (iv) L increases only when $Z = 0$, (v) $\{B(t)\}$ is a Brownian motion.

A look at the above indicates that a 'deterministic version' of eq. (12) with the attendant constraints is meaningful. To determine z and l such that

$$z(t) = \alpha(t) + l(t), \quad t \geq 0, \qquad (13)$$

subject to (i)′–(iv)′, where (i)′–(iv)′ are the same as (i)–(iv) with l, z, α replacing, respectively, L, Z, B, and (v)′ $\alpha(\,)$ is a continuous function. (Here α is the known function). For a continuous function α on $[0, \infty)$ with $\alpha(0) \geq 0$ the deterministic problem is solved uniquely by taking

$$l(t) = -\inf\{\alpha(s) \wedge 0 : 0 \leq s \leq t\} \qquad (14)$$

and, of course, $z(t) = \alpha(t) + l(t)$. A hard stare at Figure 1 will convince the reader!

Consequently, the stochastic problem can also be uniquely solved path by path. This problem is called *Skorohod problem* and eq. (12) is called *Skorohod equation.*

The connection and the consequences

What do the above-described models have in common? According to Levy[3], practically everything!

$\{|B(t)| : t \geq 0\}$ and $\{Z(t) : t \geq 0\}$, being continuous processes, induce corresponding distributions in the path space $C([0, \infty) : \mathbb{R})$. Levy has shown that these distributions are identical if the initial values agree. This surprising result is proved[3] by showing that Z is a Markov process with transition density q given by eq. (10). It may be noted that Levy arrived at this result purely by studying the sample path properties of Brownian motion, without any queueing theoretic considerations.

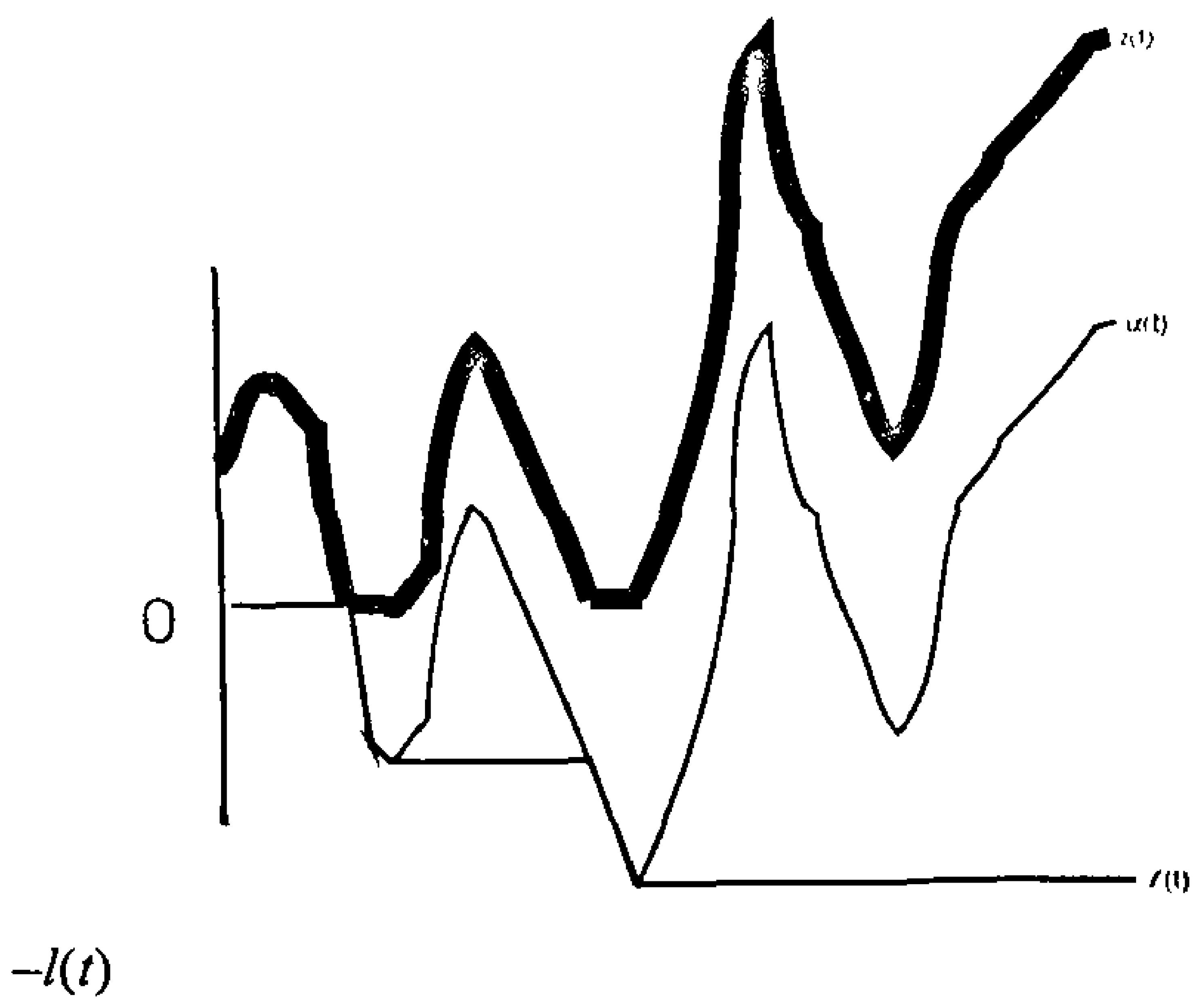

Figure 1.

Since L increases only when Z is on the boundary, L is called the *boundary local time* of Z. Local time process is one of the most esoteric objects associated with Brownian motion; in a sense high irregularity of Brownian paths makes the local time a reasonably well-behaved entity[3].

To indicate how the local time and transition density are tied up, we consider the RBM in the half-space $D = \{x \in \mathbb{R}^d : x_1 \geq 0\}$ with reflection in the inward normal direction; note that the inward normal derivative is just $\partial/\partial x_1$ at the boundary.

Let $\{(B_1(t), B_2(t), \ldots, B_d(t)) : t \geq 0\}$ be the d-dimensional Brownian motion in $\mathbb{R}^d$ (without any boundary condition). From the preceding discussion RBM in D with normal reflection at the boundary can be represented as

$$Z_1(t) = B_1(t) + L(t), \quad Z_i(t) = B_i(t), \quad i = 2, \ldots, d, \qquad (15)$$

where $L(t) = -\inf\{B_1(s) \wedge 0 : 0 \leq s \leq t\}$. Also the transition probability density is given by

$$p(s, x; t, z) = q(s, x_1; t, z_1)\; r(s, \bar{x}; t, \bar{z}), \qquad (16)$$

where $0 \leq s < t$, $x = (x_1, x_2, \ldots, x_d) = (x_1, \bar{x}) \in D$, $z = (z_1, z_2, \ldots, z_d) = (z_1, \bar{z}) \in D$, q is given by eq. (10) and r is the $(d–1)$-dimensional heat kernel. Using the Ito calculus and integration by parts it can be shown that

$$E_x\left[\int_0^t g(Z(r))\, \mathrm{d}L(r)\right] = \frac{1}{2}\int_0^t \int_{\partial D} g(z) p(0, x; \alpha, z)\, \mathrm{d}\sigma(z)\, \mathrm{d}\alpha \qquad (17)$$

for any $x \in D$ and bounded measurable function g on ∂D; here $\mathrm{d}\sigma(\cdot)$ denotes the surface area measure on the boundary (in this case, $(d-1)$-dimensional Lebesgue measure) and E_x denotes taking expectation under the condition $Z(0) \equiv x$.

The case of smooth domains

The importance of the Skorohod equation is that it suggests how reflecting diffusion can be represented as a solution of a stochastic differential equation. Also one may consider reflecting directions other than the normal direction, and possibly depending on $x \in \partial D$.

Let D be a domain with smooth boundary and $\gamma(\cdot)$ a smoothly varying vector field on ∂D; assume that $\langle \gamma(x), n(x)\rangle \geq \beta > 0$, $x \in \partial D$, where $n(x)$ is the inward normal. *Reflecting diffusion* in D with generator A and reflecting field γ can be defined as the process $\{Z(t)\}$ satisfying

$$Z(t) = Z(0) + \int_0^t \sigma(Z(s))\, \mathrm{d}B(s) + \int_0^t b(Z(s))\, \mathrm{d}s + \int_0^t \gamma(Z(s))\, \mathrm{d}\xi(s) \qquad (18)$$

such that $\xi(0) = 0$, ξ is a continuous, nondecreasing process, ξ increases only when Z is on the boundary, and $Z(t) \in \bar{D}$ for all t. This formulation is due to Watanabe[9]. Note, however, the difference between eqs (4) and (18); in eq. (4) there is only one unknown process where (18) involves two unknown processes Z and ξ. It can be shown that eq. (18) has a unique solution which is Markov. In addition, if Z has a sufficiently regular transition probability density p then

$$\frac{\partial p}{\partial s}(s, x; t, z) + A_x p(s, x; t, z) = \delta(t-s)\delta(z-x),$$

$$s < t, \ x \in D \qquad (19)$$

$$\langle \gamma(x), \nabla_x p(s, x; t, z)\rangle = 0, \ x \in \partial D \qquad (20)$$

for fixed t, z. (Observe that eq. (19) is the adjoint of the Fokker–Planck equation, known as the *backward Kolmogorov equation*.) In such a case the analogue of eq. (17) also holds. It may be noted that the notion 'reflecting direction' is justified mainly in view of eq. (20). As the sample paths of a diffusion process are nowhere differentiable, it is meaningless to say that a trajectory gets reflected in a particular direction; however, the boundary is 'reflecting' in the sense of Markov processes, viz. the sample path does not linger on the boundary and it is instantaneously returned to the interior.

Submartingale problem

As in the case of diffusion in $\mathbb{R}^d$, reflecting diffusions in smooth domains can also be characterized as solutions of 'submartingale problem', a development due to Stroock and Varadhan[10] once again. That is, for each $x \in \bar{D}$ we seek a probability measure P_x on the path space such that

(i) $P_x(X(t) \in \bar{D}\ \forall t \geq 0,\ X(0) = x) = 1$,

(ii) for each $f \in C_b^2(\mathbb{R}^d)$ with $\langle \gamma, \nabla f\rangle \geq 0$, on ∂D,

$$f(X(t)) - f(x) - \int_0^t Af(X(s))\, \mathrm{d}s = P_x - \text{submartingale}, \qquad (21)$$

where X is as defined earlier. If a, b are bounded, continuous and γ bounded, Lipschitz-continuous, then the submartingale problem has a unique family $\{P_x : x \in \bar{D}\}$ of solutions; in such a case, under $\{P_x\}$ the coordinate projections $\{X(t)\}$ form a continuous Markov process and

$$E_x\left[\int_0^\infty I_{\partial D}(X(s))\, \mathrm{d}s\right] = 0, \ x \in \bar{D}, \qquad (22)$$

where E_x denotes the expectation with respect to P_x; eq. (22) means that the time spent on ∂D is zero. When a, b, γ are sufficiently smooth, $\{P_x\}$ agrees with the distribution of the reflecting diffusion. Thus, we have another way of looking at reflecting diffusions: the family $\{P_x : x \in \bar{D}\}$ of probability measures solving the above submartingale problem can be called[10] the reflecting diffusion process corresponding to A, γ.

Another inventory model

Consider the three-stage flow system (also called the *tandem buffer system*) outlined in Figure 2. Each buffer satisfies the earlier assumptions. Here $I_1(t)$ = cumulative input into buffer 1 up to t, $I_2(t)$ = potential transfer from buffer 1 to buffer 2 up to t = potential output from buffer 1 (potential input into buffer 2) up to t, $I_3(t)$ = potential output from buffer 2 up to t. The netput process (B_1, B_2) is defined by

$$B_1(t) = B_1(0) + I_1(t) - I_2(t),$$

$$B_2(t) = B_2(0) + I_2(t) - I_3(t). \qquad (23)$$

Let $Z_i(t)$ = content of buffer i at time t, $i = 1, 2$. We may write $Z_1(t) = B_1(t) + L_1(t)$, where $L_1(t)$ = amount of potential output lost up to t because of buffer 1 being empty. Consequently, $I_2(t) - L_1(t)$ = cumulative input into buffer 2 up to t. Therefore, applying a similar argument to buffer 2 we get $Z_2(t) = B_2(t) - L_1(t) + L_2(t)$, where $L_2(t)$ = amount of potential output lost up to t because of buffer 2 being empty. Clearly, $Z_i \geq 0$.

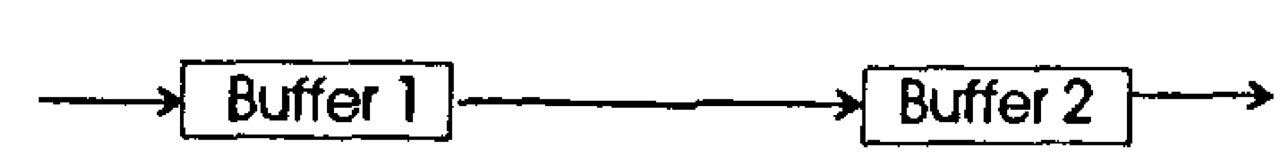

Figure 2.

In the case of heavy traffic, when the netput process can be assumed to be a two-dimensional Brownian motion $\{(B_1(t), B_2(t))\}$, the above suggests the problem: to find continuous processes $Z = (Z_1, Z_2)$, $L = (L_1, L_2)$ such that for $t \geq 0$,

$$\begin{aligned} Z_1(t) &= B_1(t) + L_1(t) \geq 0, \\ Z_2(t) &= B_2(t) - L_1(t) + L_2(t) \geq 0, \end{aligned} \tag{24}$$

with $Z_i(0) = B_i(0) \geq 0$, L_i continuous, nondecreasing; L_i increases only when $Z_i = 0$, $i = 1, 2$. Note that Z takes values in the positive quadrant $\{(z_1, z_2): z_1 \geq 0, z_2 \geq 0\}$; the 'direction of reflection' on the boundary $\{z_2 = 0\}$ is the normal direction, whereas on $\{z_1 = 0\}$ it is the 'diagonal' direction $(-1, 1)$.

The above model is due to Harrison[12] and has given a lot of impetus to the study of reflecting diffusions in nonsmooth domains, especially to RBM in quadrant, wedge, orthant, etc. with oblique reflection at the boundary. Mathematical interest is due to the presence of corners and the reflecting direction field having 'discontinuity', and the consequent nonavailability of suitable results from the theory of partial differential equations. Here 'discontinuity' is to be understood as the difference between the reflecting direction and the normal direction being discontinuous.

Normal reflection in nonsmooth domains

What is meant by normal direction at a boundary point when the boundary is nonsmooth? Following Tanaka[13] and Lions and Sznitman[14], Saisho[15] has taken the approach given below.

For a domain D, $x \in \partial D$, $r > 0$, put $\mathcal{N}(x, r) = \{n \in \mathbb{R}^d : |n| = 1, B(x-rn, r) \cap D = \emptyset\}$, where $B(y, r)$ denotes the ball with centre y and radius r; the set $\mathcal{N}(x) = \cup_{r>0} \mathcal{N}(x, r)$. $\mathcal{N}(x)$ is defined to be the *set of all inward normal unit vectors* at x.

If ∂D is smooth at x, than $\mathcal{N}(x) = \{n(x)\}$. When D is the positive quadrant, $\mathcal{N}((0, 0)) = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, \quad x_1^2 + x_2^2 = 1\}$. Let $D = \{(-1, 0) \times (-1, 1)\} \cup \{(0, 1) \times (-1, 1)\}$; that is, D is a rectangle with a 'fibre' removed; note that for any x on the fibre $\mathcal{N}(x) = \emptyset$.

If D satisfies a uniform exterior sphere condition then $\mathcal{N}(x) \neq \emptyset$ for $x \in \partial D$; for such domains the deterministic Skorohod problem can be formulated as follows: Let $\alpha \in C([0, \infty) : \mathbb{R}^d)$ with $\alpha(0) \in \overline{D}$. Find a pair of continuous functions z, k such that

$$z(t) = \alpha(t) + k(t) \in \overline{D}, \quad k(t) = \int_0^t n(s)\, \mathrm{d}|k|(s), \tag{25}$$

where $|k|(t)$ is continuous, nondecreasing, and increases only when z is on the boundary ∂D, and $n(s) \in \mathcal{N}(z(s))$ if $z(s) \in \partial D$. Note that k is $\mathbb{R}^d$-valued and is of bounded variation in every bounded interval.

Under a further technical assumption which may be called a uniform interior cone condition, it has been shown by Saisho[14], based on the work of Lions and Sznitman[16], that a unique pair of solutions exists for any α.

Moreover, if σ, b are bounded, Lipschitz-continuous, $Z(0) \in \overline{D}$, and $\{B(t)\}$ is a d-dimensional Brownian motion, then there exist unique continuous processes $\{Z(t)\}$, $\{K(t)\}$ such that Z is $\overline{D}$-valued, K is $\mathbb{R}^d$-valued with bounded variations,

$$Z(t) = Z(0) + \int_0^t \sigma(Z(s))\, \mathrm{d}B(s) + \int_0^t b(Z(s))\, \mathrm{d}s + K(t), \tag{26}$$

$$K(t) = \int_0^t n(s)\, \mathrm{d}|K|(s), \tag{27}$$

where $|K|(t)$ is a continuous, nondecreasing process, and increases only when Z is on the boundary, and $n(s) \in \mathcal{N}(Z(s))$ if $Z(s) \in \partial D$. The process Z may be called reflecting diffusion in $\overline{D}$ with generator A and normal reflection at the boundary[13–15].

If Z is a quadrant/rectangle/orthant, $\sigma \equiv$ identity matrix, $b \equiv 0$, then RBM with normal reflection can be easily got from the d-dimensional Brownian motion by the method of images; in such a case Z constructed above agrees with RBM in law.

RBM with oblique reflection

Heavy-traffic approximation in queueing theory has led in recent years to an intensive study of RBM in domains like quadrant/wedge/orthant with oblique reflection. Not much is known about other diffusion processes; and even about RBM the last word is not said. We will confine ourselves to a brief discussion of RBM in a wedge; because of the 'discontinuity' at the corner, there are a few surprises around the corner!

Let D denote the wedge with angle ξ; $\partial_1 D$, $\partial_2 D$ are the two sides of the wedge. We assume that the direction of reflection is constant on each side. Let v_i denote the reflecting direction on $\partial_i D$, θ_i denote the angle v_i makes with the normal, $i = 1, 2$ (θ_i is taken to be positive if v_i points towards the corner) (Figure 3).

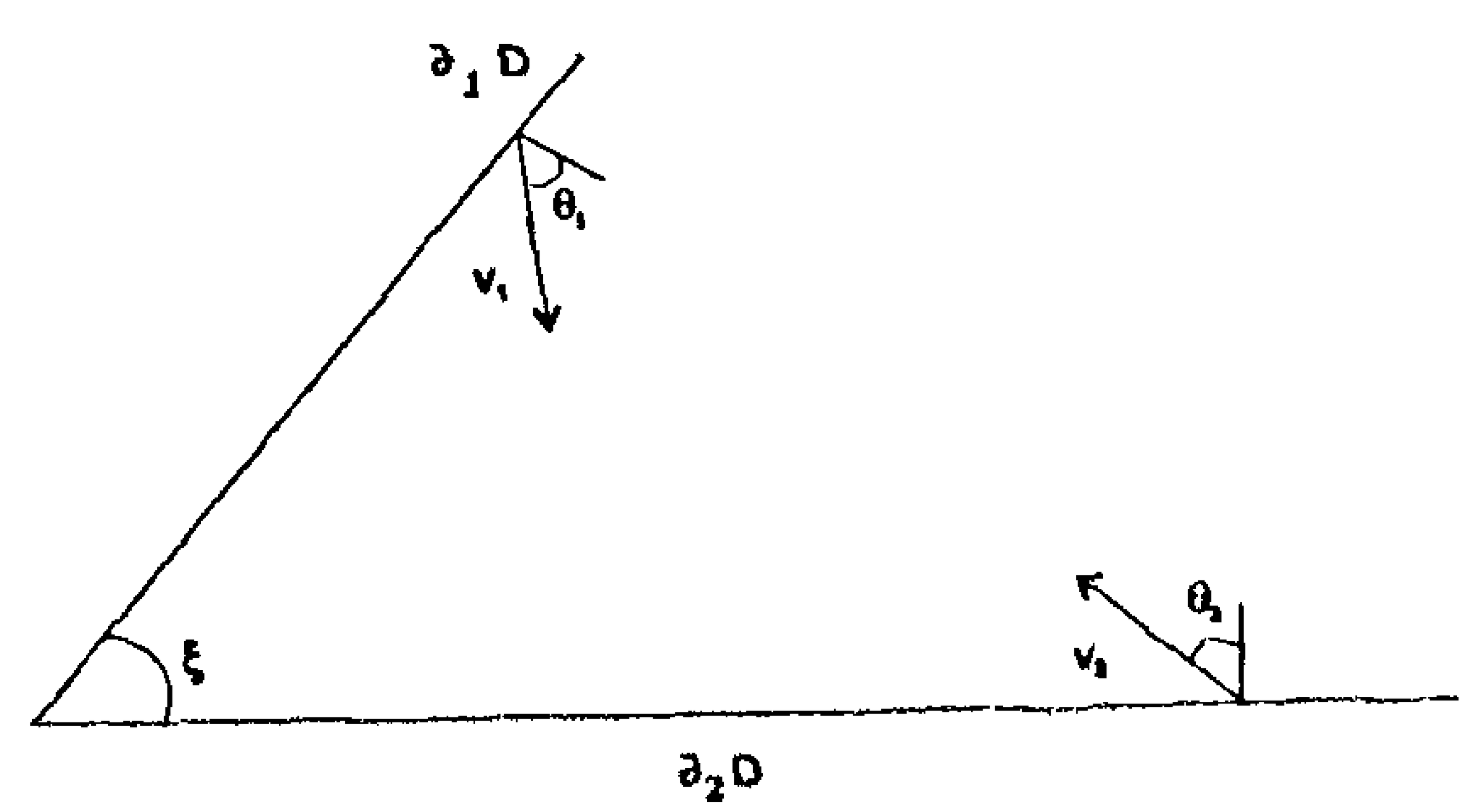

Figure 3.

In their fundamental paper Varadhan and Williams[16] defined RBM in a wedge as the solution to the following submartingale problem. Let $\mathfrak{S}$ be the collection of all smooth functions f on $\mathbb{R}^2$ such that f is constant near $(0, 0)$ and $\langle v_i, \nabla f\rangle \geq 0$ on $\partial_i D$, $i = 1, 2$. Let Ω, X be as before. A family $\{P_x : x \in \bar{D}\}$ of probability measures on Ω is a solution to the submartingale problem corresponding to $((1/2)\Delta, v_1, v_2)$ if the following hold for each $x \in \bar{D}$.

(i) $P_x(X(t) \in \bar{D} \;\; \forall t \geq 0, \; X(0) = x) = 1$;

(ii) for any $f \in \mathfrak{S}$

$$f(X(t)) - f(x) - \int_0^t \frac{1}{2} \Delta f(X(s))\, ds$$

$$= P_x - \text{submartingale}; \tag{28}$$

(iii) $$E_x \int_0^\infty I_{\{0\}}(X(s))\, ds = 0, \tag{29}$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. Set $(\theta_1 + \theta_2)/\xi$. The parameter α determines crucially the behaviour of the submartingale problem[16], viz.:

(a) $\alpha \leq 0$. There is a unique solution to the problem, and the corner is never visited.
(b) $0 < \alpha < 2$. There is a unique solution. The process hits the corner with probability one, instantaneously leaves the corner.
(c) $\alpha \geq 2$. There is *no* solution satisfying (i)–(iii) above. However, for each $x \in \bar{D}$ there is a unique P_x satisfying (i) and (ii); this P_x is supported on those paths which reach the corner and terminate there; that is, the process is *absorbed* at the corner.

Note that if $\theta_i = 0$, $i = 1, 2$, we have the case of normal reflection, and the solution of the submartingale problem is the distribution of the process considered in the previous section with, of course, $\sigma \equiv$ identity matrix, $b \equiv 0$.

The condition $\alpha > 0$ roughly means that there is a net push towards the corner. In such a case the corner is hit, and if the push is beyond a critical level (viz. $\alpha \geq 2$), the corner becomes an absorbing barrier! This is a surprising development as diffusions in $\mathbb{R}^d$ and in smooth domains (with $d \geq 2$) do not hit any specified point.

Proof of the result of Varadhan and Williams[16] centres around the function

$$G(r, \theta) = \begin{cases} r^\alpha \cos(\alpha\theta - \theta_2), & \alpha \neq 0, \\ \log r + \theta \tan \theta_2, & \alpha = 0. \end{cases} \tag{30}$$

which is the function (in polar coordinates) satisfying $\Delta G = 0$ in D, $\langle v_i, \nabla G\rangle = 0$ on $\partial_i D \backslash \{(0, 0)\}$, $i = 1, 2$.

Another surprising result, due to Williams[17], is that RBM in a wedge (that is, the solution of a submartingale problem) cannot be realized as a semi-martingale for $1 < \alpha < 2$; in other words, representation in terms of a stochastic differential equation (s.d.e) is not possible! This again is in contrast with diffusions in $\mathbb{R}^d$ or in smooth domains.

It may be mentioned that when D = quadrant, $\theta_i \geq 0$, $i = 1, 2$, $\theta_1\theta_2 < 1$, Harrison and Reiman[12, 17] have constructed RBM as the solution of an s.d.e by considering a modified Skorohod problem. Of late there has been considerable interest concerning the Skorohod problem with oblique reflection in nonsmooth domains.

For more information about RBM with oblique reflection the interested reader may refer to the forthcoming survey article by Williams[17].

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