Consequences of one spring researching with Chandrasekhar

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Starting from Chandrasekhar's work on ellipsoids and energy principles we follow a trail of questions which leads to (i) all possible flows that are attributable to vortex lines; (ii) the theorem that all steady inviscid flows are states of stationary energy at fixed circulation, and (iii) a discussion of energy principles for barotropic and non-barotropic inviscid flows.

Introduction

Yerkes 1962

Chandrasekhar's work is pervaded by a strong sense of mathematical style. A paper should not merely be relevant and right, it should be elegant and exact. To attain this elegance Chandra concentrated on problems abstracted from the real world, but nevertheless simple enough to allow analytical solution. These he used to illustrate the application of more general methods such as energy principles and higher virial theorems. I dedicate this paper to Chandra in memory of the months that I spent at Yerkes working with him in the spring of 1962.

At that time Chandrasekhar and Lebovitz[1,2] were developing the Virial Tensor theorem into a powerful method for treating the problems of equilibrium and stability of homogeneous ellipsoidal configurations and P. H. Roberts[5] was generalizing their methods to inhomogeneous bodies. I had just completed a work on the collapse of the Galaxy with Eggen and Sandage (ELS)[6] and was interested to see whether the simplified model in which the collapsing Galaxy was taken to be a homogeneous rotating and dynamically collapsing spheroid[7] would be stable or unstable to changes of shape. Chandra's method had to be generalized to dynamically collapsing bodies. In what follows I first analyse why it is that Chandrasekhar's Tensor Virial theorem[8] works so well for these problems. I shall then consider underlying principles that arose from these studies and show how they are connected with the fundamental energy principles of inviscid fluid mechanics. On the way we shall be led to study the vortex lines and to find the general motion that can be attributed to them. We also find connections to Moffat's work on helicity. My aim in writing this paper is not to explore the great field of work that Chandrasekhar[8–15] covered himself, but rather to illustrate how my brief period working with him led to the exploration of various interesting aspects of the fluid mechanics of gravitating systems and their relationships to observed phenomena.

The basic theorem on dynamical ellipsoids

Theorem. If an homogeneous ellipsoidal body of inviscid fluid initially has velocities that are linear functions of position, then it will remain ellipsoidal and the internal velocities will remain linear functions of position for all time provided that the density remains spatially uniform. The density may be prescribed function of time in which case the pressure distribution within the expanding or contracting 'liquid' is deduced from the equations of continuity and motion, or the pressure may be zero, in which case the uniformity of the density follows from the initial homogeneity and the theorem.

To prove the theorem it is easiest to start with the special case of zero pressure which is especially simple. We need Gauss's result (see Chandrasekhar[1]) that the *potential within an homogeneous ellipsoid* is a quadratic function of position. It follows that the gravitational field vector **g** is a linear function of position. If the velocities are linear functions of position at any time t they will remain linear functions at $t + \delta t$ since the only accelerations are due to gravity and *they* are linear. The ellipsoidal shape at time $t + \delta t$ follows because the ellipsoid is subjected to a linear transformation by velocities that are linear functions of position, so it remains a quadratic closed figure which can only be another ellipsoid. So the theorem is proved.

When the uniform density is prescribed, the same result will hold provided we can show that the pressure is a quadratic function of position at each time, because then the accelerations will again be linear as in the purely gravitational case. Now, if the velocities are linear functions of position at time t, then the pressure distribution at that time obeys $\nabla^2 p = C$, where C is independent of position. This follows from the divergence of the equation of motion. However, p must also obey the boundary condition $p = 0$ on the bounding ellipsoid $\mathbf{x}' \cdot A \cdot \mathbf{x} = 1$. From this it follows that

$p = -C/2A\,(1 - \mathbf{x}' \cdot A \cdot \mathbf{x})$, where A is the trace of A. Hence the pressure is quadratic in $\mathbf{x}$ at time t. It follows that the velocities are linear and the body is ellipsoidal at time $t + \delta t$, from which it follows in turn that the pressure remains quadratic then too. The theorem is true also when instead of $\rho(t)$, the central pressure p_c is a prescribed function of time. The theorem would also hold if p_c, were prescribed as a function of ρ and t.

We are now in a position to see why the tensor virial theorem is so successful in describing the second harmonic perturbations of these homogeneous ellipsoids. Firstly, the prescribed knowledge of the density $\rho(t)$ contains the information of the equation of continuity once we ensure that the total mass is conserved. Secondly, there are nine independent components in the Virial Tensor theorem when we include the three antisymmetric ones that give angular momentum conservation. This is exactly the number of independent unknowns in the linear transformation of the initial position of a fluid element that turns it into its current position. Thus once we write-in mass conservation and the linearity we have demonstrated above, then the Virial Tensor theorem extracts all the rest of the information that remained in the equations of motion. In this sense, for time-dependent homogeneous ellipsoids, the Tensor Virial theorem is equivalent to the equations of motion. But what of the second order deformations? Here it is only necessary to note that there are many different possible initial velocities that are linear functions of position and that we can choose motions close together. One of these can be considered as a perturbation of the other by a displacement that is a linear function of position – these are precisely the displacements due to second harmonic deformations. Thus, while it appeared miraculous to me at the time that the second order perturbed Virial equations formed a *closed* set of moment equations, a more intuitive investigator might have seen that it *must* be so without doing all the algebra required to prove it.

Shape instability of collapsing spheroids and ellipsoids

Briefly returning to the problem of the Galaxy's collapse that brought me into this field, the methods of Chandrasekhar and Lebovitz worked beautifully for collapsing bodies but I would have made an error had I not been corrected by Chris Hunter who pointed out that I should not look for changes in the amplitudes of the disturbances of a collapsing body but rather for changes of shape. After much algebra this led me to consider the complex dimensionless virial quantity

$$Z = (M\,a_1^2)^{-1} \exp\left(-i\int_0^t \Omega \mathrm{d}t\right) \int (\xi_1 + i\xi_2)(x_1 + ix_2)\rho \mathrm{d}^3x.$$

Here ξ is the displacement vector of the disturbance and $\Omega(t)$ is the rotation rate of the collapsing body whose equatorial radius is a_1. Conservation of angular momentum ensures that Ωa_1^2 is constant. When no external tides act on the system and $\phi = \int_0^t \Omega \mathrm{d}t$ is the total angle turned through, the equation governing Z takes the very simple form (Lynden-Bell[16], equation (68))

$$\mathrm{d}^2Z/\mathrm{d}\phi^2 + \beta_0 Z = 0,$$

where $\beta_0 = 4\pi G\rho B_{11}\Omega^{-2} + 2a_1^{-1}\Omega^{-2}\ddot{a}_1 - 1$, and B_{11} is as defined in Chandrasekhar[1]. Now for a freely falling collapse the equation for the major axis a_1 reads

$$\ddot{a}_1 - \Omega^2 a_1 = 2\pi G\rho A_1 a_1$$

so

$$2a_1^{-1}\Omega^{-2}\ddot{a}_1 = 2 - 4\pi G\rho\Omega^{-2}A_1$$

and then

$$\beta_0 = 1 + 4\pi G\rho\Omega^{-2}(B_{11} - A_1),$$

where B_{11} and A_1 are the dimensionless coefficients de- fined by Chandrasekhar[1]. Evaluating β_0 for collapsin spheroids I found it to be negative in all cases, so that Z grows 'exponentially'. Now, if we rotate our axes for ward through an angle λ, $\int(\xi_1 + i\xi_2)\,(x_1 + ix_2)\rho \mathrm{d}^3x$ get an extra factor $e^{-2i\lambda}$, thus the phase of the complex Z corresponds to a disturbance that travels at half the an gular velocity of the fluid. So while one equatorial axi elongates and the other contracts, those axes rotate an the fluid rotates forward through them. Thus when th system collapses to form a disc, it will make a Rieman ellipsoid in the form of a rotating bar with the fluid ro tating forwards within the bar. It may be no coincidenc that such motions are those of the predominant family c stellar orbits in the bars of barred spirals. The rotatin case has therefore some extra interest as compared t the non-rotating one in which the orientation of the i stability is unchanged.

Nevertheless it is the non-rotating case that has so f seen major applications to astronomy. It is particular easy to explain the instability in the non-rotating ca

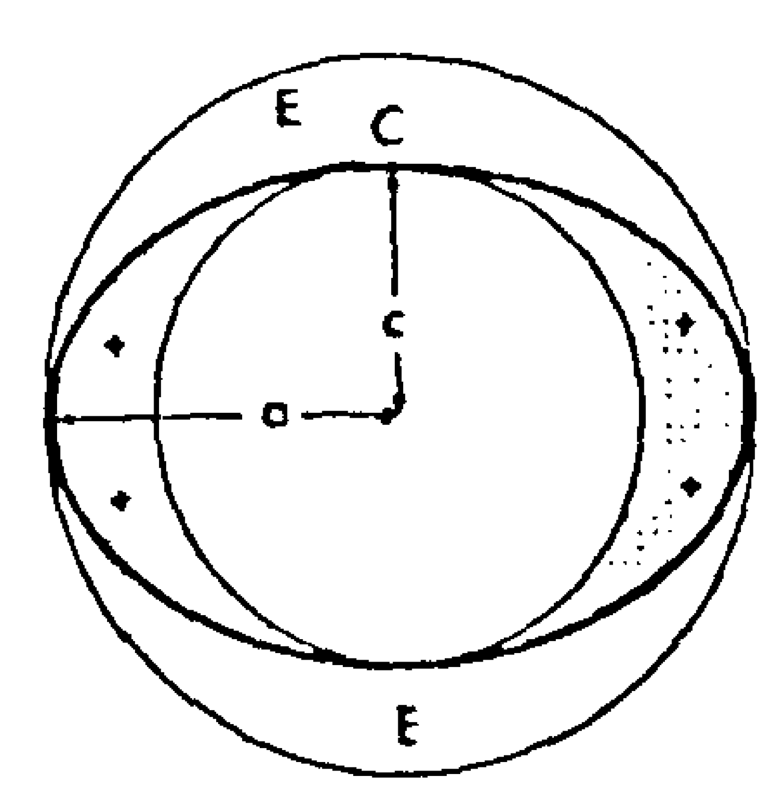

Figure 1. Cross-section in the *ac* plane of a generalized ellips (heavy line) with inscribed and circumscribed spheres. See text.

studied by Lin, Mestel and Shu[17]. The pictorial treatment follows Lynden-Bell[18] where it is also shown that the instability is suppressed if pressure reduces the inward acceleration to less than two fifths of its free fall value.

Consider the gravitational field at C, the top of the minor axis. This is the sum of the contribution $\frac{4}{3}\pi G\rho c$ from the inscribed sphere and from the remainder labelled '+' in Figure 1. Hence the inward acceleration, $-\ddot{c}/c$, of the minor axis per unit length is $>\frac{4}{3}\pi G\rho$. Similarly, considering the circumscribing sphere we find that the acceleration $-\ddot{a}/a$ of the major axis per unit length is $<\frac{4}{3}\pi G\rho$ since the acceleration due to the filled sphere is reduced because the regions E are empty. Hence $-\ddot{c}/c > -\ddot{a}/a$ at all times. This implies that the final collapse time of the c-axis starting from rest is less than the collapse time of the a-axis; the system will become flat – the axial ratio is unstable.

Lin, Mestel and Shu[17] explored the nonlinear development of these instabilities and deduced that it was the shortest axis that collapsed fastest leading to pancake instabilities as the generic case. Zel'dovich was excited by my quasar paper[19] on disc accretion by black holes, so he explored my other papers and applied these large-scale instabilities very fruitfully to cosmology[20]. It was the realization that the shortest and fastest collapsing axis might not be the angular momentum axis, that led me to consider the possible precession of galaxies whose 'planes' were not normal to their angular momenta[21]. Such considerations were stimulated by Kerr's discovery of the Galaxy's warp[22].

Although Chandrasekhar and Lebovitz have given a full delineation of major families of rotating ellipsoids and their stability, we shall see presently that their relationships to hydrodynamical energy principles are not fully understood[23–25]. In spite of pioneering work by Fujimoto[26] the full exploration of the dynamically collapsing rotating ellipsoids is still incomplete and their behaviour is unexplained. An important paper by Rosensteel and Huy[27] puts the basic equations in Hamiltonian form and I feel this should be a good starting point for further development.

Two ellipsoidal conundrums

I shall finish this section with two conundrums encountered while working in this field that led me to a better understanding of the different roles of angular momentum conservation and conservation of circulation around closed curves (Kelvin's theorem).

Imagine an inviscid Maclaurin spheroid whose density is slowly increasing as it contracts at fixed angular momentum $\mathbf{J}$. Its moment of inertia is $2/5Ma_1^2$ so $\mathbf{J} = 2/5Ma_1^2\Omega$. Its equatorial circulation is $C = \int \mathbf{\Omega} \times \mathbf{r} \cdot d\ell = \mathbf{\Omega} \cdot \int \mathbf{r} \times d\ell = 2\pi a_1^2\Omega$. Thus both conservation of angular momentum and conservation of equatorial circulation (flux of vorticity through the equatorial plane) lead to conservation of $a_1^2\Omega$ as the system slowly shrinks. Now following my note[28], apply the same arguments to a slowly shrinking Jacobi ellipsoid with $a_2 \neq a_1$. Then $J = \frac{1}{5}M(a_1^2 + a_2^2)\Omega$ while $C = \mathbf{\Omega} \cdot \int \mathbf{r} \times d\ell = 2\pi\Omega a_1 a_2$. Dividing these two relationships we find that a_1/a_2 is constant, i.e. the equatorial eccentricity of the Jacobi ellipsoid must be fixed as the system shrinks. But this cannot be true! The increasing density increases the dimensionless parameter $(J^2/GM^3)[(\rho/M)]^{1/3}$ which governs the position along the Jacobi sequence. And a_1/a_2 certainly increases along that sequence starting from unity at the Maclaurin bifurcation. It is certainly possible to imagine an inviscid shrinking fluid currently in the shape of a Jacobi ellipsoid and the correct uniform rotation, so what is the way out of this apparent contradiction?

There is another very similar conundrum. Imagine a liquid ellipsoid or prolate spheroid to be momentarily static with its long axis at 45° to a distant body that acts upon it with a tidal field. Then imagine that the tidal field is turned off and the ellipsoid is left to evolve. Circulation is conserved in time-dependent gravity fields so it remains zero around every closed curve within the liquid so the vorticity ω remains everywhere zero. However, the action of the tidal field on the elongated body at 45° certainly gives it angular momentum. We have therefore generated a body with angular momentum but with no vorticity anywhere. How can this be?

The escape from both these conundrums comes by breaking the concept that the bodies must rotate rigidly. Inviscid fluids readily slip into more complicated flows. The shrinking Jacobi ellipsoid is only momentarily a Jacobi ellipsoid. As its density increases its figure starts rotating at a different pattern speed Ω_p than the fluid's rotation. Put differently, the body remains ellipsoidal but with an internal circulation within the ellipsoid which itself rotates. The first conundrum led straightway to the understanding of why an inviscid shrinking Maclaurin spheroid cannot enter the Jacobi sequence as it crosses the bifurcation[28]. The only inviscid uniformly rotating sequences that can be traversed by a shrinking body must have a_1/a_2 constant. Only the Maclaurin sequence has that property. To enter the lower energy Jacobi sequence the body must break Kelvin's theorem. Whereas this can be done by viscosity or by internal weak magnetic fields, the rate of entry will be determined by the strength of such effects and for molecular viscosity the times involved are often prohibitively long. To an inviscid fluid the Jacobi sequence is not a possible evolutionary sequence. This conclusion of course raises the question of how possible inviscid evolutionary sequences should be defined. We find that Chandrasekhar's sequences of Riemann ellipsoids defined for good mathematical reasons to have ω/Ω_p fixed along each sequence are *not* possible evolutionary sequences in the sense described above. More suitable sequences through the Riemann plane of solutions are those traversed by a gradually shrinking mass. These are defined by $J/(MC)$ being constant along each sequence. For some recent explorations of such sequences see Christodoulou *et al.*[29,30]. However, there are other physically defined problems for which still other sequences are important; e.g. a body with constant density is gradually given more angular momentum by a gravitational tide. For that problem J/MC varies and each such sequence is defined by a constant value of $C^2/GM(\rho/M)^{1/3}$.

Energy principles

Background

Barely a hundred meters from where I write, towering over those who enter Botanic park in Belfast is a fine statue of William Thomson, Lord Kelvin, a famous son of the city who made his career in Glasgow. It was he who first developed energy principles for the flows of inviscid fluid and showed how multiply connected irrotational flows should be analysed using them. His basic developments are well described in Lamb's great book *Hydrodynamics*[31]. Kelvin sought a new view on atomic physics through his knotted vortices which he hoped would be the basic atoms of the real world. I first learned about energy principles and their power in finding both equilibrium and stability from Jeans's book *Cosmogony and Stellar Dynamics*[32] where the method was developed using K. Schwarzschild's energy principle for uniformly rotating bodies at fixed angular momentum. However, my interest was reawakened by a fine lecture by Martin Kruskal on magnetohydrodynamic stability at Les Houches in 1959. (It was at that school also that I first met Bob and Vera Rubin.) I at once applied the method to differentially rotating axially symmetrical equilibria where $M(h)$, the total mass with specific angular momentum less than h, is some given function of h. The mass with specific angular momentum between h and $h + dh$ is then $M'(h)dh$. If the fluid is polytropic with $p = \kappa\rho^{\gamma}$ the internal energy is $\int \rho\varepsilon d^3x$ where $\varepsilon = \kappa\rho^{\gamma-1}/(\gamma - 1)$. It is useful to remember that differentially rotating polytropes rotate on cylinders. This is because

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -u_\phi^2 \hat{\mathbf{R}}/R = -\rho^{-1}\nabla p + \nabla\psi = \nabla(\psi - \gamma\varepsilon), \tag{2.1}$$

here $\mathbf{R} = (x, y, 0)$ and $\hat{\mathbf{R}}$ is the corresponding unit vector that points away from the rotation axis. Evidently $\nabla(\psi - \gamma\varepsilon)$ points in the $\hat{\mathbf{R}}$ direction, so $\psi - \gamma\varepsilon$ is a function of R. This with equation (2.1) implies that u_ϕ is a function of R at equilibrium. Now $h = Ru_\phi$ and we may specify the inverse function $h(M)$ rather than $M(h)$. Now give a trial density distribution $\rho(R, z)$ corresponding to the correct total mass M_T. We define $M(R) = \int_0^R \int_{-\infty}^{\infty} \rho(R, z)dz2\pi RdR$ and calculate its inverse function $R(M)$. Then R the velocity is $h(M)/R(M)$ so the kinetic energy is

$$\frac{1}{2}\int_0^{M_T}\left[\frac{h(M)}{R(M)}\right]^2 dM.$$

Adding this to the internal and gravitational energies we find the energy principle that

$$W = \frac{1}{2}\int_0^{M_T} \frac{h^2}{R^2} dM + \int \frac{\kappa}{\gamma - 1}\rho^{\gamma} d^3x - \frac{G}{2}\iint \frac{\rho\rho'}{|\mathbf{x} - \mathbf{x}'|} d^3x\, d^3x' \tag{2.2}$$

should be stationary at any axially symmetrical equilibrium. The proof that variation of ρ with the total mass fixed leads to the equations of axially symmetrical equilibria is best done by realizing that any such ρ can be obtained by axially symmetrical displacement ξ which is so made that every fluid ring preserves its angular momentum. This energy principle was proved in my thesis but as far as I know it has never been used apart from an early unsuccessful attempt with Ostriker. Of course if W is a minimum, then the system must be stable to all axially symmetrical modes because there are no neighbouring lower energy axially-symmetric states towards which it might fall. However, the principle says nothing about stability to non-symmetrical displacements and these are some of the most interesting – given that Jacobi ellipsoids, Riemann ellipsoids, normal spirals, barred spirals, etc. are all known. Thus we need a more powerful principle capable of determining stability to such modes and if possible one that is capable of finding non-axially symmetrical equilibria. In my thesis[33] I pointed out that the function $M(h)$ must come from some cosmogonical considerations. This may have aroused my examiner's interest since Crampin and Hoyle[34] set about finding it for a number of galaxies and Mestel, my advisor, made it part of his fine discussion of the Galactic Law of rotation[35]. In fact, if one takes the same simple law $M(h) \propto h$ as occurs for the uniformly rotating cylinder, then crudely ignoring flattening and balancing gravity against rotation, $GM(h)R^{-2} = h^2R^{-3}$, one obtains $h \propto R$, i.e. V = constant. It is not hard to show that this still holds exactly for the infinite disc that is perfectly flat.

In 1964–65 a young post-doc J. P. Ostriker, who had been a graduate student of Chandra's at the time I visited Yerkes, came and worked with me at Cambridge. Earlier a paper by Chandrasekhar[11] and another by Clement[36] had demonstrated energy principles for gaseous bodies either at rest or in uniform rotation which gave the frequencies and the stability of all modes not just the axially symmetrical ones. They were not true nonlinear energy principles but rather quadratic expressions for the variation of the energy for systems close to a given known equilibrium. Thus they could not be used to search for equilibria but could be used to find the frequencies of modes.

They were not at all tied to the special forms of displacement for which the Virial method worked so well (in homogeneous bodies). Ostriker and I set out to generalize these results for general flows so that we could apply them to differentially rotating bodies such as galaxies. After making false starts we found it all came out beautifully[37] provided one used time derivatives that followed the motion. We showed that the perturbed equations of motion for the displacement vector ξ could be written in axes rotating with angular velocity Ω as

$$\rho_0 D_0^2\, \xi/Dt^2 + 2\rho_0\Omega \times D_0\xi/Dt + \rho_0\Omega \times (\Omega \times \xi) = -\boldsymbol{V} \cdot \xi - \boldsymbol{P} \cdot \xi \tag{2.3}$$

where $D_0/Dt = \partial/\partial t + \mathbf{u}_0 \cdot \nabla$, $\mathbf{u}_0$ is the unperturbed velocity and $\boldsymbol{V}$ and $\boldsymbol{P}$ were Hermitian operators related to the gravitational energy and the internal energy changes. Expanding the $D_0 . Dt$ terms gave

$$\rho_0\partial^2\xi/\partial t^2 - i\boldsymbol{B} \cdot \partial\xi/\partial t + \boldsymbol{C} \cdot \xi = 0, \tag{2.4}$$

where $\boldsymbol{B}$ and $\boldsymbol{C}$ are Hermitian with $\boldsymbol{C}$ real and $\boldsymbol{B}$ pure imaginary. $\boldsymbol{C} = \boldsymbol{T} + \boldsymbol{V} + \boldsymbol{P}$ where

$$\boldsymbol{T} \cdot \xi = \rho_0(\mathbf{u}_0 \cdot \nabla)(\mathbf{u}_0 \cdot \nabla)\xi + 2\rho_0\Omega \times (\mathbf{u}_0 \cdot \nabla)\xi + \rho_0\Omega \times (\Omega \times \xi)$$

which is the kinetic energy change and

$$-i\boldsymbol{B}\partial\xi/\partial t = 2\rho_0(\mathbf{u}_0 \cdot \nabla)\partial\xi/\partial t + 2\rho_0\Omega \times \partial\xi/\partial t.$$

Now $\int \partial\xi/\partial t \cdot i\boldsymbol{B} \cdot \partial\xi/\partial t\, d^3x = 0$. This follows because the integrand may be written $-\nabla \cdot [\rho_0\mathbf{u}_0(\partial\xi/\partial t)^2]$ which may be converted to a surface integral and $\rho_0\mathbf{u}_0 \cdot d\mathrm{S} = 0$, since there is no unperturbed flow through the surface. Multiplying (2.4) by $\partial\xi/\partial t$ and integrating in space and time, we find what appears to be the energy equation of the perturbation. This is not in fact true!

$$\int\left[\frac{1}{2}\rho_0(\partial\xi/\partial t)^2 + \frac{1}{2}\xi \cdot \boldsymbol{C} \cdot \xi\right] d^3x = \text{const.}$$

Clearly, if $\boldsymbol{C}$ were positive definite, a small initial ξ would have to remain small, so a sufficient condition for stability is that $\boldsymbol{C}$ should be positive definite. While this is true it is also dull since $\boldsymbol{C}$ is hardly ever positive definite owing to the existence of trivial displacements that allow ξ to grow while leaving the density and velocity fields unchanged. While we recognized the existence of some trivial displacements, the importance of others that allow $\boldsymbol{C}$ to attain negative values was first noticed by Schutz and Sorkin[30] and exploited by Friedman *et al.*[39,40] and Bardeen *et al.*[41]. Although $\boldsymbol{C}$ is not the energy, the variational principle for the frequencies, derived in our paper is correct, so it can be used to study normal modes and demonstrate their stability. A second problem with the criterion that C be positive definite is that it looks for stability in a reference frame that rotates with fixed angular velocity rather than treating the problem at fixed angular momentum, i.e. it is analogous to Poincaré's minimizing of $(V + \frac{1}{2}I\Omega^2)$ rather than K. Schwarzschild's $(V - J^2/(2I))$. While these give the same equilibria they do not give the same criteria for secular stability and Schwarzschild's is the correct one for a freely rotating body. The point is that a small increase in I can lead to a slower net rotation of the perturbed body which for a Jacobi ellipsoid would lead to ξ growing as the figure rotated further and further from the orientation of the unperturbed one. Whereas Hunter's[33] discussion removes that trouble with C, it does not deal with the other 'trivial' displacement problem to which we now turn.

The trouble with the trivials arises because we may convert a flow into itself by associating each fluid element with a displaced one. If we do this across the lines of the mean flow, then the associated ξ will grow without limit although no instability is involved because we have merely mapped the flow into itself. Friedman and Schutz show that the removal of such trivial parts of ξ is equivalent to the requirement that

$$\Delta\oint \mathbf{u} \cdot d\ell = 0$$

around every closed curve – i.e. that the displacement should preserve the circulation so that the new circulation around the displaced curve is the same as the old circulation around the undisplaced one. For the non-barotropic case the same requirement holds for the restricted class of curves on surfaces of constant entropy.

When do two flows have 'the same' circulations?

Any functional of the density and velocity fields that is constant as a result of the equations of motion is a first class invariant. Good examples are the total energy and the total angular momentum. Although Kelvin's circulation theorem gives us invariants, they need the specification of a path that moves with the fluid. Thus the circulations are invariants but are only second class invariants because without further knowledge we cannot tell which path to take at a later time in order that it should correspond to a given path at the earlier time. How can we find out whether a flow has the 'same' circulations in it as some other flow specified earlier? By the same circulations we do not mean that we calculate the circulations along the same paths, rather we mean that *there exists a continuous mass preserving mapping of the flows into each other such that the circulation* *around each closed path in one flow is equal to the circulation around the corresponding closed path in the other.* The closed paths in the first flow can be chosen at will. It follows from Kelvin's theorem that any barotropic flow taken at one time is isocirculational to whatever flow it develops into at later times. As a corollary it follows that if a flow taken at time t_1 is isocirculational to another flow taken at time t_2, then they are isocirculational at whatever times we consider them. But how do we specify a given state of circulation in an invariant manner? To do this we need to find first class invariants that carry the same information as Kelvin's 'second class' circulations. There are hints that this is possible. Helicity, $\int \mathbf{u} \cdot \omega \mathrm{d}^3x$, is a first class invariant which as we shall see can be expressed in terms of linked circulations. When all the circulations are zero, the isocirculational condition may be expressed in the special but first class form *Curl* $\mathbf{u} = 0$. We shall be particularly interested in steady states or at least states that are steady when viewed from rotating axes. Writing $\mathbf{u} = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}$ then $\omega = 2\Omega + Curl\, \mathbf{v}$ where Ω is the angular velocity of the axes and

$$\omega \times \mathbf{v} = \nabla\chi \qquad (3.1)$$

where

$$\chi = \int \rho^{-1}\mathrm{d}p + \tfrac{1}{2}\mathbf{v}^2 + \tfrac{1}{2}|\Omega \times \mathbf{R}|^2 - \psi.$$

The surfaces $\chi = \text{const.}$ are known as Bernoullian surfaces and both ω and $\mathbf{v}$ must lie on them. If we consider the situation in which the vorticity field is nowhere zero; then by the hairy sphere theorem these surfaces cannot have spherical topology because ω must lie in the surface and is non-zero. The condition that ω be nowhere zero can be considerably relaxed because we are not interested in just one surface but in a nested set of surfaces. The condition $\omega = 0$ gives us three equations so in three dimensions the general case is that ω is zero only at isolated points – the solutions for x, y, z of the three equations $\omega_x = 0$, $\omega_y = 0$ and $\omega_z = 0$. A typical member of a nested set of Bernoullian surfaces all with supposed spherical topology will have no point where $\omega = 0$ so the hairy sphere theorem shows us that the nested Bernoullian surfaces cannot have spherical topology under even this relaxed constraint on ω. The simplest nested surfaces on which the hair can be brushed are topologically either cylinders or tori. The former occur naturally for any differentially rotating axially symmetrical configuration whereas the latter are found in bodies with meridional currents such as Hill's spherical vortex. When azimuthal circulation is superposed on the latter, the Bernoullian surfaces remain toroidal with the vortex lines twisting around them. However, let us begin with the simple case in which the vortex lines pass through the fluid from a 'bottom' B, to a 'top' T. We follow Lynden-Bell and Katz[43] in considering a narrow vortex tube surrounding our chosen vortex line. The mass in this vortex tube is $\Delta M = \int_B^T \rho \Delta \mathbf{S} \cdot \mathbf{d}\ell$ where $\Delta \mathbf{S}$ is the tube's cross section and $\mathbf{d}\ell$ is the infinitesimal increment along our vortex line. The strength of the vortex tube is the circulation around it, $\Delta C = \omega(\ell) \cdot \Delta \mathbf{S}(\ell)$ which is independent of ℓ since the narrow tube is a vortex tube. Without loss of generality we may choose $\Delta \mathbf{S}$ to be a normal cross section perpendicular to $\mathbf{d}\ell$, which by definition lies along ω since the latter defines the vortex line. Then

$$\lambda \equiv \Delta M / \Delta C = \int_B^T \rho / (\omega \cdot \hat{\ell}) \mathrm{d}\ell, \qquad (3.2)$$

where we have substituted for $\Delta \mathbf{S} \cdot \hat{\ell}$ in the expression for ΔM. $\hat{\ell}$ is the unit vector along $\mathbf{d}\ell$.

In equation (3.2) $\Delta \mathbf{S}$ has disappeared so as we shrink our narrow vortex tube to a line, we obtain a property of our chosen vortex line itself. Now by a well-known theorem, vortex lines and tubes may be chosen to move with the fluid, so for a general barotropic flow ΔC is conserved as is ΔM the mass in the tube. Thus λ, the load is a conserved property of each vortex line. Notice that this is true of any barotropic flow steady or time dependent, in which the vortex line concerned intersects the surface. When the flow is a steady flow we may consider the vortex lines as either fixed or moving with the fluid. In the latter interpretation they must flow into other fixed lines of the same λ, since λ is conserved following the motion. Thus in any steady flow the surfaces of constant λ contain both ω and $\mathbf{u}$ and are therefore the Bernoullian surfaces (Figure 2). Furthermore, we may now use the time-dependent surfaces of constant λ as generalizations of the Bernoullian surfaces to time-dependent flows. There is a second invariant associated with any given fluid element, E, on a vortex line, the

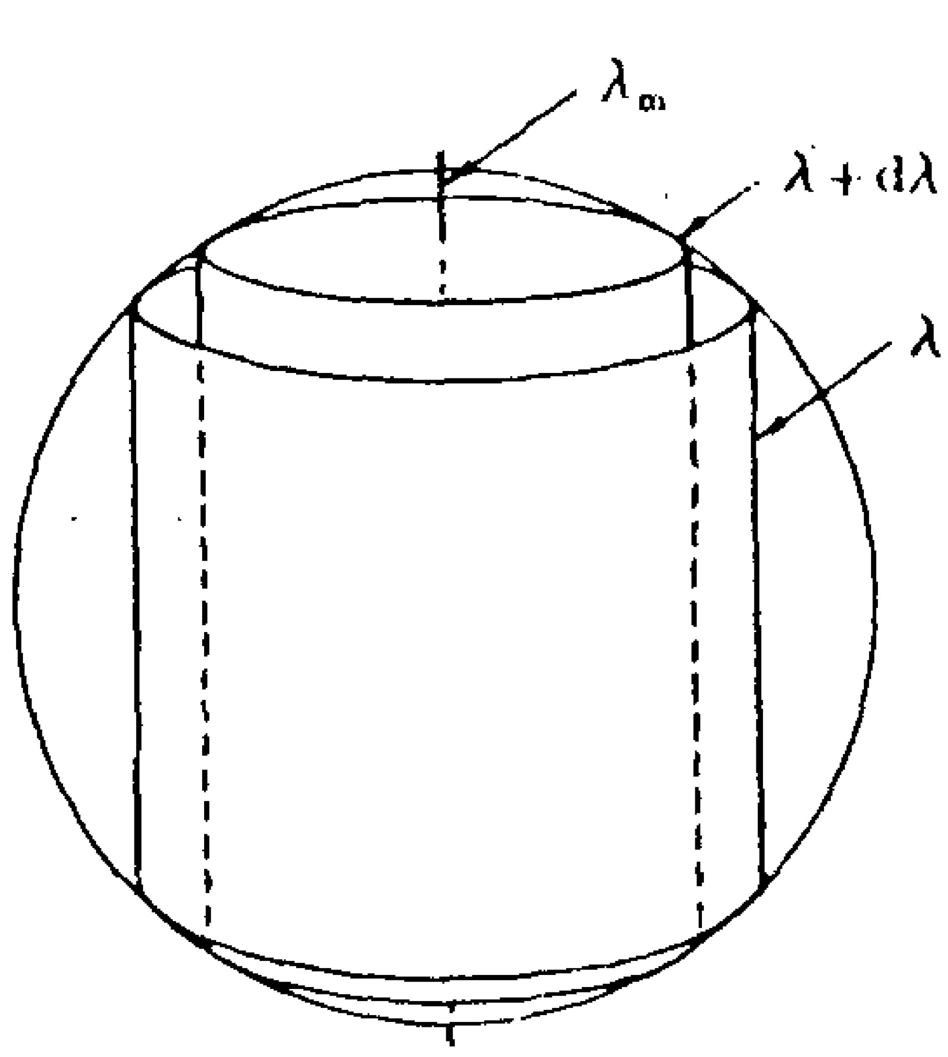

Figure 2. Nested cylinders of constant load in a uniformly rotating sphere; λ_m is the vortex line of maximum load.

partial load that measures how much of the load lies 'below' the chosen fluid element. This quantity is called the 'metage' (pronounced meat age, the 'measure' of the partial load). The metage, μ, is also conserved since the mass in the narrow tube and below the small ΔS through the chosen element is also conserved. Specifically

$$\mu = \int_B^E \rho/(\omega \cdot \hat{\ell}) \mathrm{d}\ell . \tag{3.3}$$

Notice that

$$\omega \cdot \nabla \mu = \rho . \tag{3.4}$$

Thus by construction both $D\lambda/Dt$ and $D\mu/Dt$ are zero so in steady states the flow is along the intersections of the surfaces of constant λ and constant μ surfaces. Notice that the surfaces of constant λ and constant μ can be constructed from the density and velocity fields specified at any time, thus λ and μ are first class comoving coordinates. Let $M(\lambda)$ be the mass enclosed by the 'cylinder' of load λ and let $C(\lambda)$ be the circulation around the large vortex tube made by that 'cylinder'; then by splitting the region between λ and $\lambda + \mathrm{d}\lambda$ into infinitesimal vortex tubes it is simple to show that

$$\frac{\mathrm{d}M}{\mathrm{d}C} \equiv \frac{\mathrm{d}M/\mathrm{d}\lambda}{\mathrm{d}C/\mathrm{d}\lambda} = \lambda . \tag{3.5}$$

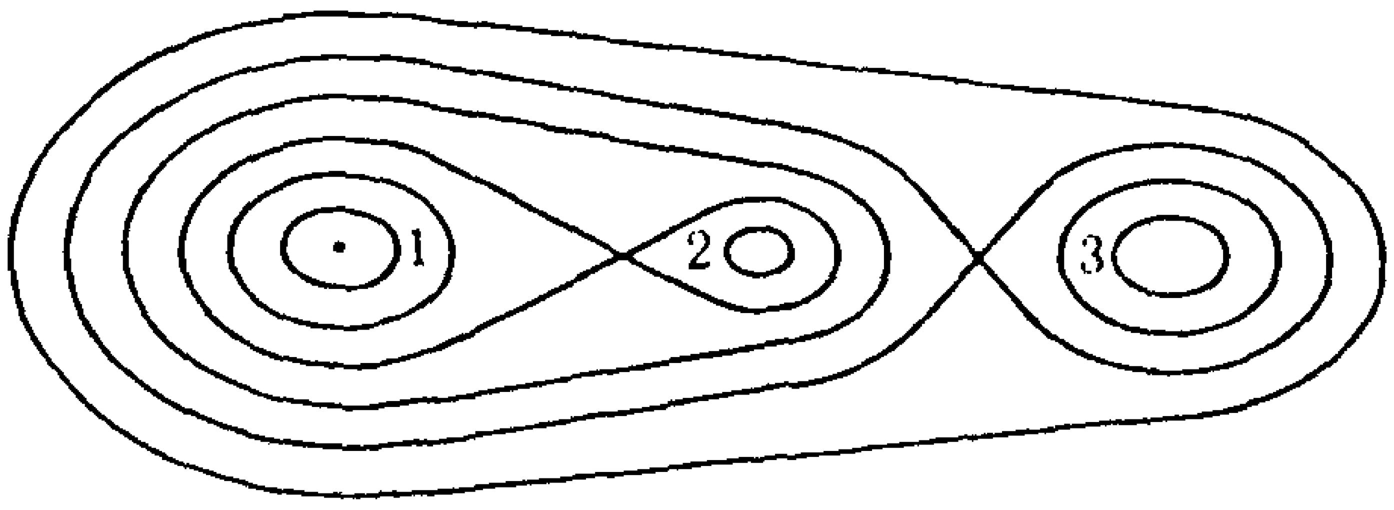

Figure 3. Cross section of 'cylindrical' surfaces of constant load with three distinct pieces to some surfaces of constant load.

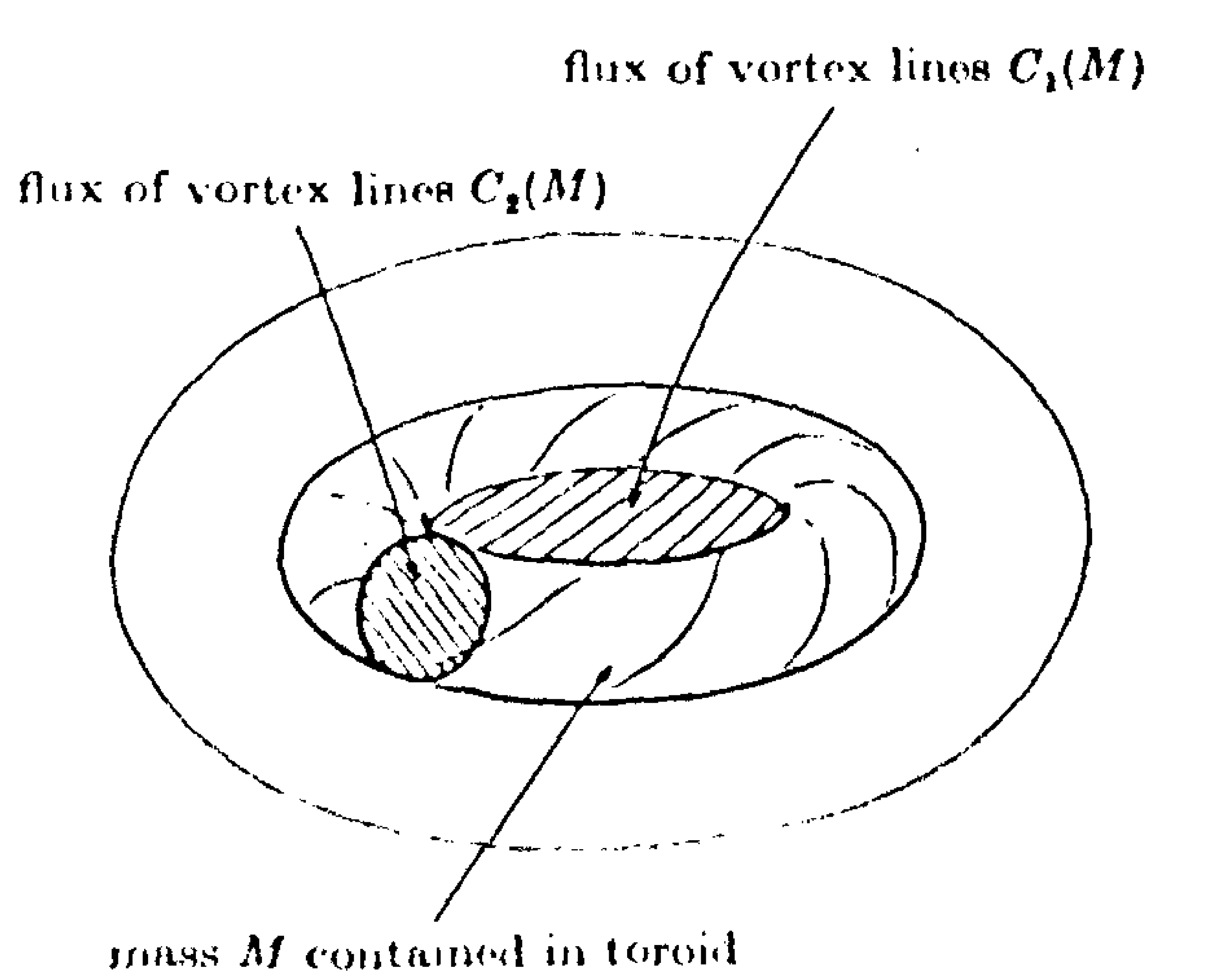

Figure 4. Vortex lines wound on a torus of constant load showing the two surfaces across which the vortex fluxes C_1 and C_2 are measured.

When the surfaces of constant load have the topology of nested cylinders, then if two flows have the same $M(\lambda)$ or $M(C)$ they are isocirculational. The proof consists of showing that there is a mass preserving mapping such that the circulations around all paths are preserved (see ref. 43, p. 187). More complicated nesting of cylinders (Figure 3) were also considered there. When the vortex lines lie on toroids some of which do not intersect the surface of the fluid, these considerations need further generalizations also given there. When the vortex lines are purely toroidal as in Hill's spherical vortex, they close on themselves. In such a case we may define a second load as the closed loop integral $\lambda_2 = \oint \rho/(\omega \cdot \hat{\ell})\mathrm{d}\ell$ along a vortex line and we can again define the mass within the toroid of load λ_2 and the circulation, C_2, around the toroid by the short way. However, the general case is not like that because the vortex lines normally from quasi-ergodic area-filling curves on the toroid that do not close on themselves (except for special toroids). Around such a toroid containing mass M there will be two circulations (Figure 4) $C_1(M)$ the circulation along a path that goes around the toroid once the long way without going around the short way, and $C_2(M)$ the circulation along a path that goes once around by the short way without going around the long way. Then the first and second loads are given by

$$-\left(\frac{\mathrm{d}C_1}{\mathrm{d}M}\right)^{-1} = \lambda_1 \text{ and } \left(\frac{\mathrm{d}C_2}{\mathrm{d}M}\right)^{-1} = \lambda_2 .$$

The minus sign occurs because M is not the mass within 'cylinders' defined earlier. With this sign the two definitions agree if the toroids intersect the surface and become 'cylinders'. The helicity invariant[44,45] for such a case is

$$\mathcal{H} = \int \mathbf{u} \cdot \omega \mathrm{d}^3 x = \int C_1 \frac{\mathrm{d}C_2}{\mathrm{d}M} \mathrm{d}M . \tag{3.6}$$

Again one may prove that, if the surfaces of constant load have the same topology for any two flows and if the functions $C_1(M)$ calculated for each flow are the same and likewise the two $C_2(M)$ are equal, then there exists an isocirculational mass-preserving mapping of one into the other, so the two flows are isocirculational. Thus conservation of $C_1(M)$ and $C_2(M)$ yields *first class* invariants which contain the information of the state of circulation of the fluid in an invariant manner. We have therefore succeeded in replacing Kelvin's second class invariants by first class ones.

On the general flows attributable to vortex lines

For any steady flow we may either consider the vortex lines as fixed in space or as moving with the fluid. Both those 'flows' are mass preserving and both preserve the flux of vorticity through any surface. What is the most general flow that we could attribute to vortex lines if we ask that the 'flow' of vortex lines be mass and flux preserving? Let $\mathbf{v}$ be the fluid velocity and $\mathbf{v}+\mathbf{w}$ be the velocity we attribute to the vortex lines. Then

$$\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0 \qquad (4.1)$$

and

$$\partial\rho/\partial t + \mathrm{div}[(\rho(\mathbf{v}+\mathbf{w}))] = 0 \qquad (4.2)$$

and hence

$$\mathrm{div}(\rho\mathbf{w}) = 0 \qquad (4.3)$$

Similarly,

$$\partial\omega/\partial t + Curl\,(\omega \times \mathbf{v}) = 0 \qquad (4.4)$$

and

$$\partial\omega/\partial t + Curl[\omega \times (\mathbf{v}+\mathbf{w})] = 0 \qquad (4.5)$$

hence

$$Curl\,(\omega \times \mathbf{w}) = 0 \qquad (4.6)$$

Now the arguments defining load and metage in §3 only require that the motion be both mass preserving and vortical flux preserving; thus they apply just as powerfully to $\mathbf{v}+\mathbf{w}$ as they do to $\mathbf{v}$! So

$$\frac{\partial\lambda}{\partial t} + \mathbf{v}\cdot\nabla\lambda = 0 = \frac{\partial\lambda}{\partial t} + (\mathbf{v}+\mathbf{w})\cdot\nabla\lambda \qquad (4.7)$$

so

$$\mathbf{w}\cdot\nabla\lambda = 0 \qquad (4.8)$$

and similarly for metage

$$\mathbf{w}\cdot\nabla\mu = 0. \qquad (4.9)$$

Hence

$$\rho\mathbf{w} = A\nabla\lambda \times \nabla\mu\,. \qquad (4.10)$$

Also since $\omega\cdot\nabla\lambda = 0$ we have by equation (3.4)

$$\omega\times\mathbf{w} = \frac{A}{\rho}(\omega\cdot\nabla\mu)\nabla\lambda = A\nabla\lambda \qquad (4.11)$$

so by equation (4.6) A must be a function of λ only. Thus the most general mass preserving and flux preserving flow that can be attributed to vortex lines is the flow of the fluid $\mathbf{v}(\mathbf{x}, \mathbf{t})$ supplemented by the addition of a flow $\mathbf{w}$ along the intersections of the surfaces of constant load and constant metage of the form

$$\mathbf{w} = \frac{A(\lambda)}{\rho}\nabla\lambda\times\nabla\mu. \qquad (4.12)$$

This covers the case in which the vortex lines intersect the fluid's surface so that μ is defined.

When the vortex lines are wound on toroids with circulations around both the short and the long way, it is necessary to define $\nabla\mu$ somewhat differently. The toroids themselves can be found directly from the flow as the surfaces on which the vortex lines are densely wound. If we label each toroid by the mass inside it M and let the circulation around it by the long way be $C_1(M)$ and by the short way $C_2(M)$, then $dC_1/dM = -\lambda_1^{-1}$ and $dC_2/dM = \lambda_2^{-1}$. By imagining the toroid to be cut and twisted so that the vortex lines become untwisted, one sees that one twist generates a flux of vorticity around the short way equal to the flux around the tube between M and $M + dM$ by the long way. This implies that the pitch of the vortex lines on the toroid is $-dC_1/dC_2 = \lambda_2/\lambda_1$. This pitch is the average ratio of the rotation turned by a vortex line around the z-axis per rotation around the toroid by the short way. It is the generalization to toroids of the idea of the pitch of a screw when the screw is directed around the 'circle' made by the line toroid about which the others are nested.

We still need a definition of metage when the vortex lines are wound on toroids. Consider first an equilibrium situation so $\mathbf{v}$ itself obeys equations (4.1) and (4.4) without the $\partial/\partial t$ terms. Since load is conserved $\mathbf{v}$ lies in the surfaces of constant λ_1 as does ω. Hence $\mathbf{w} = \mathbf{v} + A_1(\lambda_1)\omega/\rho$ satisfy both equations (4.3) and (4.6) for any A. In general the lines of $\mathbf{w}$ will spiral around the toroids just as those of $\mathbf{v}$ and ω do but by choosing $A_1 = A_0(\lambda_1)$ appropriately on each toroid, we can in general make sure that a line of $\mathbf{w}_0$ closes after just one turn around the toroid by the long way with no turns around by the short way*. Since in a steady flow we can consider a vortex line as either fixed or moving with velocity $\mathbf{w}_0$ all points on the line will give the same value of the load. Since $\mathrm{div}(\rho\mathbf{w}_0) = 0$ and $\rho \neq 0$ the lines of $\mathbf{w}_0$ cannot cross and, since they lie on the surfaces of constant λ, the closure of one implies the closure of all the others on that toroid. Since $\mathbf{w}$ preserves both the vortical flux and mass, it preserves load so $\int_P^{P'} \rho/\omega\cdot\hat{\ell}d\ell = \lambda_1$ where P and P' are consecutive intersections of a vortex line with a line of $\mathbf{w}_0$. Now define $\mu = \int_P^Q \rho/\omega\cdot\hat{\ell}d\ell$ along a vortex line starting from the same line of $\mathbf{w}_0$ that we chose originally but now proceeding to an arbitrary point Q before the second crossing at P'. By our original argument the mass on an infinitesimal vortex tube between Q and P will be preserved by the motion $\mathbf{w}_0$ as will the strength of the tube so μ will be preserved by the motion. So μ defined in this way will be preserved, $D_0\mu.Dt = 0$. Although μ has an arbitrary loop of $\mathbf{w}_0$ as its starting point, it is eliminated in $\nabla\mu \times \nabla\lambda_1$ so that is uniquely defined. μ is multivalued but in steps of λ_1. Since $\mathbf{w}_0$ lies in the surfaces of constant λ_1 and constant μ we may write

*There are exceptions, e.g. flows with $\omega \propto \mathbf{u}$ but we shall not give the detailed prescriptions for such special cases here. When $\lambda_1 \equiv 0$ it is necessary to switch the roles of λ_1 and λ_2.

$$\rho \mathbf{w}_0 = B \nabla \lambda_1 \times \nabla \mu$$

then $\omega \times \mathbf{w}_0 = B\nabla\lambda_1$ and since the curl of this has to be zero B has to be a function of λ_1 only. Thus

$$\mathbf{w}_0 = \rho^{-1} B(\lambda_1) \nabla \lambda_1 \times \nabla \mu.$$

Now let us look for the most general flow of vortex lines in our steady flow. Clearly $\mathbf{w}_0$ is one possibility and $\rho^{-1}\omega$ is another since div $\omega = 0$ and $\omega \times \omega = 0$. A general vector field lying on the toroids may be written

$$\mathbf{w} = A_2 \mathbf{w}_0 + B_1 \rho^{-1} \omega.$$

Any flow of vortex lines must preserve λ_1 and so must lie on the toroids so this general form applies

$$\omega \times w = A_2 B(\lambda_1) \nabla \lambda_1$$

and since the curl of this must be zero, we deduce that A_2 must be a function of λ_1 it follows that

$$\text{div}(\rho \mathbf{w}) = \text{div}(B_1 \omega) = 0$$

and so $\omega \cdot \nabla B_1 = 0$. Now for a general toroid ω lies densely so this equation implies that B_1 takes the same value densely all over the toroid. If B_1 is to be continuous this implies B_1 is a function of λ_1. So the general flow of vortex lines is given by $\mathbf{w} = A_3(\lambda_1)\rho^{-1}\nabla\lambda_1 \times \nabla\mu + B_1(\lambda_1)\rho^{-1}\omega$, but that is only for steady flow. [One may wonder here what all the detail is about since any λ dependent combination of $\mathbf{v}$ and $\rho^{-1}\omega$ would have done for the steady case and that is equivalent to what we have finally deduced.] The advantage of introducing μ becomes obvious when we consider the non-steady problem. Pick any point P on a toroid on which ω is densely wound. Starting at P integrate $\rho/\omega \cdot \hat{\ell}\, d\ell$ along the vortex line through P until the integral reaches λ_1, the first load of that toroid. Choose the end point P' as the new starting point and repeat the process so generating a sequence of points P, P', P'', etc. In the steady case these points lie on a curve of constant μ which we may choose to call $\mu = 0$. Now instead of choosing one point P start from a curve that cuts every toroid in such a point. The process may be repeated to generate the surface $\mu = 0$. Any surface of constant μ can then be generated starting from a point on $\mu = 0$ and integrating less far. These constructions work in the steady case and are not invalidated by the time dependence. Finally we deduce that the most general flow we may attribute to vortex lines that are densely wound on toroids are $\mathbf{v} + \mathbf{w}$ where

$$\rho \mathbf{w} = A_3(\lambda_1) \nabla \lambda_1 \times \nabla \mu + B(\lambda_1) \omega.$$

This is the natural generalization of the result (4.12) obtained for lines that hit the surface of the fluid. Of course on those lines that do hit the surface $B(\lambda_1)$ has to be zero.

In principle there are many more complicated cases. The toroids could themselves be knotted etc. but provided they remain toroids the same construction works.

A fundamental theorem in fluid mechanics

Of all barotropic flows with a given state of circulation and a given total angular momentum those with *minimum*† energy are *stable* steady states in some (in general rotating) axes. Those with stationary energy are also steady states but are often unstable. All steady states may be obtained as states of stationary energy in this manner even if they are only steady when viewed from rotating axes. We follow the proof of the appendix to LBK which generalizes Arnold's work[46] to three-dimensional compressible fluid.

Here we are concerned with flows that are close to flows of stationary energy. Let $\xi(\mathbf{r}, t)$ be the small field of displacements that maps points of one flow into those of another so that $\mathbf{r}' = \mathbf{r} + \xi(\mathbf{r}, \mathbf{t})$. The condition that the flow ρ, $\mathbf{u}$ is isocirculational to the flow ρ', $\mathbf{u}'$ is that for all closed paths, I', within the fluid,

$$0 = \Delta \oint_{r} \mathbf{u} \cdot ds = \oint_{r'} \mathbf{u}'(\mathbf{r}+\xi, \mathbf{t}) d(\mathbf{r}+\xi) - \oint \mathbf{u}(\mathbf{r}) \cdot d\mathbf{r}.$$

Therefore

$$0 = \oint_{r} [\mathbf{u}'(\mathbf{r}+\xi) = -\mathbf{u}(\mathbf{r}) + \mathbf{u}'(\mathbf{r}+\xi) \cdot \partial\xi/\partial\mathbf{r}] \cdot d\mathbf{r}$$

$$= \int_{r} [\Delta\mathbf{u} + (\mathbf{u}+\Delta\mathbf{u}) \cdot \partial\xi/\partial\mathbf{r}] \cdot d\mathbf{r}.$$

Hence the condition for isocirculationarity is exactly

$$\Delta\mathbf{u} + (\mathbf{u} + \Delta\mathbf{u}) \cdot \partial\xi/\partial\mathbf{r} = \nabla\delta\gamma^*. \tag{5.1}$$

Hence the second order in ξ (notice that $\delta\gamma^*$ is first order)

$$\Delta\mathbf{u} = \nabla\delta\gamma^* - \mathbf{u}\cdot\partial\xi/\partial\mathbf{r} - (\nabla\delta\gamma^* - \mathbf{u}\cdot\partial\xi/\partial\mathbf{r})\cdot\partial\xi/\partial\mathbf{r} + 0(\xi^3).$$

Now

$$\xi \times \text{curl}\mathbf{u} = (\nabla\mathbf{u})\cdot\xi - (\xi\cdot\nabla)\mathbf{u}$$

$$= \nabla(\mathbf{u}\cdot\xi) - (\nabla\xi)\cdot\mathbf{u} - (\xi\cdot\nabla)\mathbf{u}.$$

Hence

$$\Delta\mathbf{u} = \nabla\delta\gamma + \xi \times \omega + (\xi \cdot \nabla)\mathbf{u} + 0(\xi^2), \tag{5.2}$$

where

$$\delta\gamma = \delta\gamma^* - \mathbf{u} \cdot \xi.$$

Notice that for each chosen ξ and $\delta\gamma$ there exists an isocirculational flow $\delta\mathbf{u}$; so at each time ξ and $\delta\gamma$ may be varied independently. The difference in the kinetic energies of the two flows is

†If the only displacements that do not lower the energy are uniform displacements and uniform rotations accompanied by displacements that do not change the density and velocity fields, then the energy will be considered as at a minimum.

$$\Delta T=\Delta\int \tfrac{1}{2}\mathbf{u}^2 dm=\int \tfrac{1}{2}(\mathbf{u}+\Delta\mathbf{u})^2-\tfrac{1}{2}\mathbf{u}^2 dm$$
$$=\int \mathbf{u}\cdot\Delta\mathbf{u}+\tfrac{1}{2}(\Delta\mathbf{u})^2 dm. \tag{5.3}$$

The change in gravitational potential energy is

$$\Delta V_g=-\int\Delta\psi_{ext}\,dm$$
$$-\tfrac{1}{2}G\iint\left(\frac{1}{|\mathbf{r}+\xi-\mathbf{r}'-\xi'|}-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dm\,dm'. \tag{5.4}$$

To first order the double integrand is

$$(\xi\cdot\nabla)\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)+(\xi'\cdot\nabla')\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)+0(\xi^2),$$

and so writing

$\psi_i=G\int|\mathbf{r}-\mathbf{r}'|^{-1}dm'$, $\psi=\psi_i+\psi_{ext}$

and reversing $\mathbf{r}$ and $\mathbf{r}'$ where necessary we have

$$\Delta V_g=-\int\xi\cdot\nabla\psi dm+0(\xi^2). \tag{5.5}$$

Turning now to the internal energy, the work done in squeezing down unit mass from infinite dilation is

$$-\int p\,dv=-\int p\,d\left(\frac{1}{\rho}\right)=-p/\rho+\int\frac{dp}{\rho}.$$

Thus the internal energy is

$$U=-\int p\,d^3r+\iint\frac{dp}{\rho}dm;$$

hence

$$\Delta U=-\int\delta p\,d^3r+\int\frac{\delta p}{\rho}dm+\int\left[(\xi\cdot\nabla)\int\frac{\delta p}{\rho}\right]dm+0(\xi^2). \tag{5.6}$$

The first two terms above cancel and so collecting first-order terms only, and writing $V=V_s+U$

$$\Delta V=-\int\xi\cdot\nabla\left(\psi-\int\frac{dp}{\rho}\right)dm=-\int\delta\rho\left(\psi-\int\frac{dp}{\rho}\right)d^3r \tag{5.7}$$

where we have used $\delta\rho=-\mathrm{div}\,(\rho\xi)$ and integrated by parts at the last equality. We wish to minimize the total energy subject to fixed total angular momentum

$$0=\Delta J=\int[(\mathbf{r}+\xi)\times(\mathbf{u}+\Delta\mathbf{u})-\mathbf{r}\times\mathbf{u})dm$$
$$=\int[\mathbf{r}\times\Delta\mathbf{u}+\xi\times(\mathbf{u}+\Delta\mathbf{u})]dm. \tag{5.8}$$

We now use Lagrange's method and minimize $E-\Omega\cdot\mathbf{J}$ where the components of Ω are undetermined multipliers that allow us to vary $\delta\gamma$ and ξ as though they were unrestricted, and $E=T+V$. From equations (5.5) to (5.8),

$$\Delta E=\Omega\cdot\Delta\mathbf{J}$$
$$=\int\left\{(\mathbf{u}-\Omega\times\mathbf{r})\cdot\Delta\mathbf{u}+\xi\cdot\left[\Omega\times\mathbf{u}-\nabla\left(\psi-\int\frac{dp}{\rho}\right)\right]\right\}dm+0(\xi^2).$$

If we define $\mathbf{v}=\mathbf{u}-\Omega\times\mathbf{r}$ and use expression (5.2) for $\Delta\mathbf{u}$

$$\Delta E=\Omega\cdot\Delta\mathbf{J}=\int\rho\mathbf{v}\cdot\nabla\delta\gamma d^3r$$
$$+\int\left\{-\xi\cdot(\mathbf{v}\times\omega)+\mathbf{v}\cdot(\xi\cdot\nabla)(\mathbf{v}+\Omega\times\mathbf{r})\right.$$
$$\left.+\xi\cdot\left[(\Omega\times\mathbf{u})-\nabla\left(\psi-\int\frac{dp}{\rho}\right)\right]\right\}dm+0(\xi^2)$$

which may be cast in the form

$$\Delta E=\Omega\cdot\Delta\mathbf{J}=\int\delta\gamma\rho\mathbf{v}\cdot d\mathbf{S}-\int\delta\gamma\,\mathrm{div}(\rho\mathbf{v})d^3r$$

$$+\int\xi\cdot\left\{-\mathbf{v}\times\omega+\nabla\left[\tfrac{1}{2}\mathbf{v}^2+\int\frac{dp}{\rho}-\psi-\tfrac{1}{2}(\Omega\times\mathbf{r})^2\right]\right\}dm+0(\xi^2).$$

At a point of stationary energy for fixed angular momentum we require the right-hand side to be zero for all variations of ξ and $\delta\gamma$; hence

$\rho\mathbf{v}.d\mathbf{S}=0$ on the fluid boundary,

$\mathrm{div}(\rho\mathbf{v})=0$ in the fluid interior

and

$$-\mathbf{v}\times\omega=-\nabla\left[\tfrac{1}{2}\mathbf{v}^2+\int\frac{dp}{\rho}-\psi-\tfrac{1}{2}(\Omega\times\mathbf{r})^2\right].$$

These we recognize as the equations of steady motion of a barotropic fluid referred to axes that rotate at a rate Ω.

Energy principles

To obtain a nonlinear energy principle such as that given by K. Schwarzschild for the special case of uniform rotation one must discover a neat way of ensuring that all the flows considered have a given circulation state $C(M)$. A way of doing this was given in LBK but Katz *et al.*[47], Yahalom *et al.*[48] find another method and I suspect the very best way of doing this is still to be discovered. All of this work can be generalized to non-barotropic fluids. There it is only the circulations on the surfaces of constant specific entropy s that are conserved and Ertel's[49] invariant $e=\rho^{-1}\omega\cdot\nabla s$ comes into its own. The state of circulation of the fluid is specified by the function $M(e,s)$ that gives the total mass with both specific entropy $>s$ and Ertel's invariant $>e$. As far as I know no-one has given a general method of determining the velocity of the flow in terms of $M(e,s)$ but Katz and Lynden-Bell[50] have written down the Lagrangian for such flows and shown that the equations of inviscid fluids follow. Finding variables in terms of which all the flows considered automatically have the given $M(e,s)$ gives a great advantage as degrees of freedom that cannot be excited are automatically eliminated. Finding the best of such variables remains an important challenge for the future (see Ipser and Lindblom[51]).

In concentrating on the physics of vorticity I have not done full justice to the beautiful idea of Joseph Katz. He looks at the Lagrangian of the fluid – sees that it is invariant under a large class of displacements that leave the density and velocity unchanged. The momenta conjugate to such displacements are thus conserved quantities. He uses the theory of continuous (Lie) groups to show that this leads to Ertel's invariant and eventually to load and metage for the case of barotropes. There are close analogues here to the gauge fields such as electromagnetism but that would take us too far from the theme of this paper. An account of the technique applied to fluids can be found in KLB[52].

Although the equations of fluid mechanics in the presence of viscosity are not Lagrangian, nevertheless the viscous forces are linear in the gradients of **u** and lead to quadratic dissipation. It should therefore be possible to find a Rayleigh dissipative function to put the equations in quasi Lagrangian form. Mobbs[53] has looked into variational principles for dissipative flows. The methods of Glansdorf and Prigogine[54] should also apply but they are not always simple to use.

Retrospect

Chandrasekhar's emphasis on style and beauty led him to an elegant mathematical approach to fluid mechanics while his great enthusiasm inspired even those of us who sought a more physical approach. The path he set us on led to some of the most fundamental aspects of fluid mechanics, many of which have not yet found their final and most elegant form of expression. Chandra's challenge lives on!

1. Chandrasekhar, S., *Ellipsoidal Figures of Equilibrium*, Yale U. Press, New Haven, 1969.
2. Lebovitz, N. R., *Astrophys. J.*, 1961, **134**, 500.
3. Chandrasekhar, S. and Lebovitz, N. R., *Astrophys. J.*, 1962, **135**, 238 and **136**, 1032 and 1037.
4. Chandrasekhar, S. and Lebovitz, N. R., *Astrophys. J.*, 1963, **137**, 1142, 1162.
5. Roberts, P. H., *Astrophys. J.*, 1962, **136**, 1108.
6. Eggen, O. J., Lynden-Bell, D. and Sandage, A. R., *Astrophys. J.*, 1962, **136**, 748.
7. Lynden-Bell, D., *Proc. Camb. Phil. Soc.*, 1962, **50**, 709.
8. Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, 1961, pp. 577–587.
9. Chandrasekhar, S., *Astrophys. J.*, 1962, **136**, 1048.
10. Chandrasekhar, S., *Astrophys. J.*, 1963 **137**, 1185.
11. Chandrasekhar, S., *Astrophys. J.*, 1964 **139**, 664.
12. Chandrasekhar, S., *Astrophys. J.*, 1965, **141**, 1043.
13. Chandrasekhar, S., *Astrophys. J.*, 1965, **142**, 890.
14. Chandrasekhar, S., *Astrophys. J.*, 1966, **145**, 842.
15. Chandrasekhar, S., *Astrophys. J.*, 1968, **152**, 293.
16. Lynden-Bell, D., *Astrophys. J.*, 1964, **139**, 1195.
17. Lin, C. C., Mestel, L. and Shu, F. H., *Astrophys. J.*, 1965, **142**, 1431.
18. Lynden-Bell, D., *Observatory*, 1970, **99**, 89.
19. Lynden-Bell, D., *Nature*, 1969, **223**, 690.
20. Zel'dovich, Ya. B., *Astron. Astrophys.*, 1970, **5**, 84.
21. Lynden-Bell, D., *Monthly Not. R. Astron. Soc.*, 1965, **129**, 299.
22. Kerr, F. J., *Monthly Not. R. Astron. Soc.*, 1962, **123**, 327.
23. Lebovitz, N. R., *Astrophys. J.*, 1966, **145**, 878.
24. Lebovitz, N. R., *Ann. Rev. A&A*, 1967, **5**, 465.
25. Lebovitz, N. R. and Lifshitz, A., *Astrophys. J.*, 1996, **458**, 699.
26. Fujimoto, M., *Astrophys. J.*, 1968, **152**, 523.
27. Rosensteel, G. and Huy, Y. T., *Astrophys. J.*, 1991, **366**, 30.
28. Lynden-Bell, D., *Astrophys. J.*, 1965, **142**, 1648.
29. Cristodoulou, D. M., Schlossman, I. and Tohline, J. E., *Astrophys. J.*, 1995, **443**, 551, 563.
30. Cristodoulou, D. M., Kazanas, D., Schlossman, I. and Tohline, J. E., *Astrophys. J.*, 1995, 446, 472, 485, 500, 510.
31. Lamb, H., *Hydrodynamics*, Cambridge University Press, 1895, pp. 139–142.
32. Jeans, J. H., *Problems of Cosmology and Stellar Dynamics*, Cambridge University Press, 1915.
33. Lynden-Bell, D., Thesis, Cambridge University Library, 1960.
34. Crampin, D. J. and Hoyle, F., *Astrophys. J.*, 1964, **140**, 99.
35. Mestel, L., *Monthly Not. R. Astron. Soc.*, 1963, **126**, 553.
36. Clement, M. J., *Astrophys. J.*, 1964, **140**, 1045.
37. Lynden-Bell, D. and Ostriker, J. P., *Monthly Not. R. Astron. Soc.*, 1967, **136**, 293.
38. Schutz, B. F. and Sorkin, R., *Ann. Phys.*, 1977, **107**, 1.
39. Friedman, J. L. and Schutz, B. F., *Astrophys. J.*, 1989a, **221**, 937.
40. Friedman, J. L. and Schutz, B. F., *Astrophys. J.*, 1989b, **222**, 281.
41. Bardeen, J. M., Friedman, J. L., Schutz, B. F. and Sorkin, R., *Astrophys. J.*, 1977, **217**, L73.
42. Hunter, C., *Astrophys. J.*, 1977, **213**, 497.
43. Lynden-Bell, D. and Katz, J., *Proc. Roy. Soc.*, 1981, **A378**, 179 (LBK).
44. Moffat, H. K., *J. Fluid. Mech.*, 1969, **35**, 117.
45. Moreau, J. J., *C. R. Acad. Sci., Paris*, 1961, **252**, 2810.
46. Arnold, V. I., *J. Mech.*, 1966, **5**, 26.
47. Katz, J., Inagaki, S. and Yahalom, A., *Pub. Astr. Soc. Japan*, 1993, **45**, 424.
48. Yahalom, A., Katz, J. and Inagaki, S., *Monthly Not. R. Astron. Soc.*, 1994, **268**, 506.
49. Ertel, H., *Meteorol. Zeit.*, 1942, **59**, 277.
50. Katz, J. and Lynden-Bell, D., *Proc. R. Soc.*, 1982, **A381**, 263.
51. Ipser, J. R. and Lindblom, L., *Astrophys. J.*, 1991, **379**, 285.
52. Katz, J. and Lynden-Bell, D., *Geophys. Astrophys. Fluid Dynam.*, 1985, **33**, 1 (KLB).
53. Mobbs, S. D., *Proc. R. Soc. London*, 1982, **A305**, 1.
54. Glansdorf, P. and Prigogine, I., *Thermodynamic Theory of Structure Stability and Fluctuations*, Wiley Interscience, London, 1971.

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