Liquid drops in rise against gravity through a viscous medium: Drag force by the method of dimensions and comparison with liquid drops in fall under gravity

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We have presented previously the results observed in the study of liquid drops of Reynolds number (20–305), Eotvos number (2×10^{-2} to 105×10^{-2}) and Morton number (4.75×10^{-12} to 3.75×10^{-7}), in free fall without exhibiting oscillation through an immiscible liquid column where the liquid in the column is lighter than the liquid drops which tend to fall (downward motion) under the influence of gravity from the top of the liquid column. Suppose the column liquid is heavier, then, obviously, the motion of the liquid drop is completely curtailed and the drop floats. However, the motion of the drop may be achieved by its rise (upward motion) from the bottom of the liquid column. The rise of the drop occurs against (the influence of) gravity. The motion of the drop in rise against gravity may be thought or assumed to be opposite to that of the motion of the drop in fall under gravity. Therefore, we have devoted attention here to study the drops in rise against gravity through an immiscible liquid column to understand whether the expressions satisfying the drop in fall under gravity also hold good for the drop in rise against gravity or not and to determine the proportionality constant K of the drag force expression (eq. (1)) which suits the drop in rise against gravity and another constant S (eq. (6)) just as observed for drops in fall under gravity. In the present study, the drops in rise against gravity considered are in the range of Reynolds number, R_e (25–495), Eotvos number, E_t [(2×10^{-2}) to (70×10^{-2})] and Morton number, M_o [(2.06×10^{-12}) to (7.13×10^{-10})]. The important observations made in the present study are that (i) the value of the proportionality constant of the drag force expression for drop in rise against gravity differs from the value of the proportionality constant of the drag force expression for drop in fall under gravity and (ii) the value of the constant S in the case of drops in rise against gravity differs from the value of the constant S in the case of the drops in fall under gravity[1]. For drops in rise against gravity, here, the predicted value for K and S are respectively 4.6082 (≈ 4.6) and 0.3176 (≈ 0.32) $m^{-1} s^2$; for drops in fall under gravity, they were predicted as K = 3.6094 (≈ 3.6) and $S = 0.2483$ (≈ 0.25) $m^{-1} s^2$.

RECENTLY Srinivasan and Satyanarayana[1] have presented the results observed in the study of liquid drops in the range of Reynolds number, R_e (20–305) ($R_e = \sigma u D/\eta$), Eotvos number E_t (2×10^{-2} to 105×10^{-2}) ($E_t = g\Delta\rho D^2/\gamma$), and Morton number, M_o (4.75×10^{-12} to 3.75×10^{-7}) ($M_o = g\eta^4\Delta\rho/\sigma^2\gamma^3$), in free fall without exhibiting oscillation through an immiscible liquid column (column liquid is lighter than the drop). $\Delta\rho$ is the absolute value of density difference, $|\rho-\sigma|$. In their study[1], they obtained a fresh expression for the drag force by the method of dimensions for the drop in fall from the top of the liquid column under the influence of gravity. When the liquid drop is lighter, the motion of the drop occurs by rise from the bottom of the column against gravity. Since the motion of the drop in rise against gravity may be thought to be opposite to that of the motion of the drop in fall under gravity, an attempt has been made in this paper to study the rise of liquid drop against gravity without exhibiting oscillation through an immiscible liquid column by the method of dimensions and to arrive at results to know whether they are in line or not with the observations that were already made in the case of liquid drops falling without exhibiting oscillation under the influence of gravity[1] through an immiscible liquid column. Studies on the rise of liquid drops through an immiscible liquid column have been carried out by several authors[2–28]. Literature review shows that none has attempted to deal with the problem of rise of liquid drops with only six variables F, D, u, η, ρ and σ in the dimensional analysis of the drag force F (F is the drag force acting on the drop in rise; D is the diameter of the drop; u is the terminal velocity attained by the drop; η is the viscosity of the liquid in the column; ρ is the density of the liquid drop; and σ is the density of the liquid in the column). Therefore these six variables F, D, u, η, ρ, σ alone are considered to arrive at an expression for the drag force (eq. (1)) acting upon the drop in rise against gravity without exhibiting oscillation through an immiscible liquid column by the method of dimensions. Since Srinivasan and Satyanarayana[1] have employed only these six variables to arrive at a fresh expression for the drag force acting on the drop in fall under gravity without exhibiting oscillation through an immiscible liquid column, an attempt has also been made to know whether the expressions (eq. (1)–(10)) given by them by the method of dimensions for drops in fall under gravity hold good for the drops here in rise, against gravity or not. Drops of Reynolds number, R_e (25–495), Eotvos number, E_t [from (2×10^{-2}) to (70×10^{-2})] and Morton number, M_o [from (2.06×10^{-12}) to (7.13×10^{-10})], which rise without oscillations against gravity through an immiscible liquid column are dealt with in the present study. The experimental data points predict 4.6082 (≈ 4.6) for the constant K of the drag force expression (eq. (1)) which suits drop in rise against gravity. The predicted value of K is

3.6094 (≈ 3.6) for drops[1] in fall under gravity. For drops in rise against gravity, the experimental data points here predict the value for the constant S as 0.3176 (≈ 0.32) $m^{-1}\ s^2$ which, for drops in fall under gravity has been predicted as 0.2483 (≈ 0.25) $m^{-1}\ s^2$ (ref. 1).

The experiments show that for drops (sufficiently small) in rise against gravity in the range of Reynolds number, R_e (from 25 to 495), Eotvos number, E_t [from (2×10^{-2}) to (70×10^{-2})] and Morton number, M_o [from (2.06×10^{-12}) to (7.13×10^{-10})] without exhibiting oscillation through an immiscible liquid column, the expressions suggested by Srinivasan and Satyanarayana[1] for drops in fall under gravity without exhibiting oscillation through an immiscible liquid column, hold good, viz.,

$$F = \mathrm{K}D^2u^2\sigma(\eta/\sigma uD)^{1/2}(\rho/\sigma)^{1/2},$$

or (1)

$$F = \mathrm{K}D^{3/2}u^{3/2}\eta^{1/2}\rho^{1/2}.$$

The method of arriving at these expressions has already been dealt with in detail[1]. At the terminal velocity, for drops in rise against gravity without exhibiting oscillation through an immiscible liquid column, the buoyant force $(4\pi r^3\sigma g/3)$ of the liquid is equal to the sum of the weight of the drop, $(4\pi r^3\rho g/3)$ and the drag force F. That is,

$$4\pi r^3\sigma g/3 = (4\pi r^3\rho g/3) + F,$$

$$\therefore F = 4\pi r^3(\sigma-\rho)g/3. \quad (2)$$

Eq. (2) in eq. (1) gives

$$\mathrm{K} = \sqrt{2}\pi g(r/u)^{3/2}(\sigma-\rho)/3\eta^{1/2}\rho^{1/2}, \quad (3)$$

$$u = (2\pi^2g^2r^3(\sigma-\rho)^2/9\eta\mathrm{K}^2\rho)^{1/3}, \quad (4)$$

$$\sqrt{2}\pi gS/3\mathrm{K} = 1, \quad (5)$$

where

$$S = [(r/u)^{3/2}(\sigma-\rho)]/[\eta^{1/2}\rho^{1/2}], \quad (6)$$

and $g = 9.8\ m\ s^{-2}$, the acceleration due to gravity.

Eq. (6) (just as in the case of falling drop) is of vital importance in the sense that it may be used to determine the density ρ of the liquids which are lighter than the immiscible liquid column when conventional methods fail[1]. Eq. (6) may be written as a quadratic equation from which

$$\rho = \{2\sigma + \lambda + [\lambda(\lambda + 4\sigma)]^{1/2}\}/2, \quad (7)$$

where

$$\lambda = S^2\eta/(r/u)^3. \quad (8)$$

S has been found to be approximately a constant for all systems studied here (see Table 3), the mean value of which is 0.3176 $m^{-1}\ s^2$.

Experimentally-observed data points predict a value for the constants S and K. The interval within which the predicted value of S and K holds good have to be ascertained for acceptability. Since an uncertainty analysis will help in establishing the uncertainty or error limits, which reveal the interval within which the predicted value of S and K holds good for acceptability, an attempt has been made here to adopt statistical method or technique[29–40], which will enable to estimate and bring out the confidence limits (interval) within which the experimentally predicted value for S and K is an acceptable value of the parameters. Let $(x_1, \hat{y}_1), (x_2, \hat{y}_2), \ldots (x_n, \hat{y}_n)$ be the n pairs of observations on the variables (observables) x and $\hat{y}$. Let the relationship between x and $\hat{y}$ be of the form

$$\hat{Y} = \hat{A} + \hat{B}X. \quad (9)$$

Let the line of regression (best-fit)[29,30] of $\hat{Y}$ on X be

$$Y = A + BX. \quad (10)$$

Let the error estimate or residual[29,30] as given by the line of best-fit (eq. (10)) be e, that is,

$$e = \hat{Y} - Y, \quad (11)$$

where Y is the estimated value as given by the line of best-fit (eq. (10)) for a given value of X. A (ref. 29) and B (ref. 29) are being estimated from

$$A = (\Sigma\hat{Y} - B\,\Sigma X)/n, \quad (12)$$

where

$$B = \{\Sigma X\hat{Y} - (\Sigma X\,\Sigma\hat{Y})/n\}/\{\Sigma X^2 - (\Sigma X)^2/n\}. \quad (13)$$

The expressions[29] giving $(1-\alpha)$ per cent confidence limits for the regression parameters $\hat{A}$ and $\hat{B}$ are respectively,

$$\hat{A} = A \pm S_a t_{\alpha,\, n-2}, \quad (14)$$

$$\hat{B} = B \pm S_b t_{\alpha,\, n-2}, \quad (15)$$

where $t_{\alpha,\, n-2}$ is the table value for the two-tailed test at α level of significance and for $(n-2)$ degrees of freedom. S_a and S_b are the standard error of A and B respectively, and they are expressed as[29]

$$S_a = \{S_e^2\{(1/n) + [(\Sigma X/n)^2/[\Sigma X^2 - ((\Sigma X)^2/n)]]\}\}^{1/2}, \quad (16)$$

$$S_b = \{S_e^2/\{\Sigma X^2 - [(\Sigma X)^2/n]\}\}^{1/2}. \quad (17)$$

S_e is the mean square error which is expressed as[29]

$$S_e = \{\{[\Sigma\hat{Y}^2 - ((\Sigma\hat{Y})^2/n)] - B[\Sigma X\hat{Y} - ((\Sigma X\,\Sigma\hat{Y})/n)]\}/(n-2)\}^{1/2}. \quad (18)$$

In the present study, eq. (6) may be written as

$$\rho^{1/2}/(\sigma-\rho) = 0 + (1/S)[(r/u)^{3/2}/\eta^{1/2}]. \quad (19)$$

This equation is of the form of eq. (9) and we get $\hat{Y} = \rho^{1/2}/(\sigma-\rho)$; and $X = (r/u)^{3/2}/\eta^{1/2}$. These in eqs 12, 13, 16, 17 and 18, give the estimated value of A, the estimated value of B [$=(1/S)_{\text{estimated}}$], S_a, S_b and S_e respectively. Eqs (14) and (15) give $(1-\alpha)$ percent confidence limits for regression parameters $\hat{A}$ and $\hat{B}$ respectively for $t_{\alpha, n-2}$ table value for the two tailed t-test at α level of significance of $(n-2)$ degrees of freedom. It is obvious from eq. (15),

Highest confidence limit of

$$\hat{B}\ (=1/S) = B + S_b\, t_{\alpha, n-2}, \quad (20)$$

Least confidence limit of

$$\hat{B}\ (=1/S) = B - S_b\, t_{\alpha, n-2}, \quad (21)$$

Confidence interval

$$\hat{B} \text{ is } (B - S_b\, t_{\alpha, n-2}) \text{ to } (B + S_b\, t_{\alpha, n-2}), \quad (22)$$

Least confidence limit of

$$S = \{1/[B + S_b\, t_{\alpha, n-2}]\}, \quad (23)$$

Highest confidence limit of

$$S = \{1/[B - S_b\, t_{\alpha, n-2}]\}, \quad (24)$$

Confidence interval of S is

$$[1/(B + S_b\, t_{\alpha, n-2})] \text{ to } [1/(B - S_b\, t_{\alpha, n-2})]. \quad (25)$$

Since by eq. (5), $\mathbf{K} = \sqrt{2\pi g S/3}$,

Least confidence limit of

$$\mathbf{K} = (\sqrt{2\pi g/3})/[B + S_b\, t_{\alpha, n-2}], \quad (26)$$

Highest confidence limit of

$$\mathbf{K} = (\sqrt{2\pi g/3})/[B - S_b\, t_{\alpha, n-2}], \quad (27)$$

Confidence interval of

$$\mathbf{K} = (\sqrt{2\pi g/3})/[B + S_b\, t_{\alpha, n-2}] \text{ to } (\sqrt{2\pi g/3})/[B - S_b\, t_{\alpha, n-2}], \quad (28)$$

within these limits (interval) the experimentally predicted value for S and $\mathbf{K}$ is an acceptable value of the parameters S and $\mathbf{K}$ respectively.

A long graduated cylinder (diameter 5 cm) filled with the experimental liquid was used. Test liquid drops of known volume were gently injected at the bottom of the liquid column using a graduated Hamilton Precision micro-syringe. For injection, a small side tube attached to the graduated cylinder at the bottom and sealed with a septum was used. The rising drops and the liquids in the column were immiscible. Terminal velocity 'u' was determined by observing the time 't' required by the liquid drop of radius 'r' to cover the distance 'd' between two graduations on the column. The drops studied here while rising are ellipsoidal in shape. If 'V' is the volume of the drop of equivalent radius 'r', then,

$$r = (3V/4\pi)^{1/3}.$$

The thirteen systems studied are given in Table 1. Density, viscosity and interfacial tension given in Table 1 were determined by the specific gravity bottle method, Ostwald viscometer and method of drops, respectively. As seen for drops in fall under gravity[1], here also, in the case of drops in rise against gravity, (r/u) is approximately constant (Table 2). Three data points each for seven liquid-drop pairs are alone given in Table 2. For the other liquid-drop pairs given in Table 1, the experimental results obtained show that the same (r/u, approximately constant) holds good.

The value of S has been found to be approximately constant (Table 3) for all systems, just as observed for

Table 1. The systems and density, ratio of density, viscosity, interfacial tension and ratio of radius to terminal velocity values

Liquid drop	Column liquid	ρ (kgm^{-3})	σ (kgm^{-3})	$\sigma-\rho$ (kgm^{-3})	ρ/σ	Column liquid viscosity η (Nsm^{-2})	Interfacial tension γ ($\times 10^{-3}\ Nm^{-1}$)	r/u (s)
Xylene	Water	857.95	1000.00	142.05	0.85795	0.001000	12.5	0.0160
Benzene	Water	870.78	1000.00	129.22	0.87078	0.001000	69.2	0.0174
Kerosene	Water	797.34	1000.00	202.66	0.79734	0.001000	43.8	0.0125
Turpentine	Water	860.03	1000.00	139.97	0.86003	0.001000	41.5	0.0165
Toluene	Water	860.89	1000.00	139.11	0.86089	0.001000	39.7	0.0165
Iso-amyl acetate	Water	882.15	1000.00	117.85	0.88215	0.001000	29.2	0.0185
Hexane	Water	665.37	1000.00	334.63	0.66537	0.001000	17.4	0.0085
Petroleum ether	Water	667.91	1000.00	332.09	0.66791	0.001000	24.3	0.0085
Cyclohexane	Water	775.04	1000.00	224.96	0.77504	0.001000	20.4	0.0116
Soap oil	Water	857.15	1000.00	142.85	0.85715	0.001000	34.2	0.0160
Heptane	Water	720.37	1000.00	279.63	0.72037	0.001000	38.7	0.0098
Water	Chlorobenzene	1000.00	1097.99	97.99	0.91076	0.000710	46.1	0.0196
Water	Bromobenzene	1000.00	1492.21	492.21	0.67015	0.000850	70.2	0.0071

r = radius of the liquid drop; u = terminal velocity of the drop; ρ = density of the liquid drop; σ = density of the liquid in the column; η = viscosity of the liquid in the column; γ = interfacial tension between the liquid drop and the liquid in the column.

Table 2. Experimental data for liquid-drop pairs

Liquid – drop pair	Volume 'V' of the drop (μl)	Radius 'r' of the drop ($\times 10^{-4}$ m)	Distance 'd' travelled ($\times 10^{-2}$ m)	Time 't' taken (s)	Observed terminal velocity 'u' ($\times 10^{-2}$ ms^{-1})	r/u (s)	Reynolds number R_e	Eotvos number E_t ($\times 10^{-2}$)
Xylene	0.5	4.9237	40	13.0	3.0769	0.0160	30.30	10.80
in								
water	1.0	6.2035	40	10.3	3.8835	0.0160	48.18	17.14
Morton no. = 7.1275×10^{-10}	2.0	7.8159	40	8.2	4.8780	0.0160	76.25	27.21
Benzene	0.5	4.9237	40	14.1	2.8369	0.0174	27.94	1.77
in								
water	1.0	6.2035	40	11.2	3.5714	0.0174	44.31	2.82
Morton no. = 3.8215×10^{-12}	2.0	7.8159	40	8.9	4.4944	0.0174	70.26	4.47
Turpentine	0.5	4.9237	40	13.4	2.9851	0.0165	29.40	3.21
in								
water	1.0	6.2035	40	10.6	3.7736	0.0164	46.82	5.09
Morton no. = 1.9192×10^{-11}	2.0	7.8159	20	4.2	4.7619	0.0164	74.44	8.08
Iso-amyl acetate	0.5	4.9237	40	15.0	2.6667	0.0185	26.26	3.84
in								
water	1.0	6.2035	40	11.9	3.3613	0.0185	41.70	6.09
Morton no. = 4.6388×10^{-11}	2.0	7.8159	40	9.5	4.2105	0.0186	65.82	9.66
Hexane	0.5	4.9237	40	6.9	5.7971	0.0085	57.09	18.28
in								
water	1.0	6.2035	40	5.5	7.2727	0.0085	90.23	29.01
Morton no. = 6.2251×10^{-10}	2.0	7.8159	40	4.4	9.0909	0.0086	142.11	46.05
Cyclohexane	0.5	4.9237	40	9.4	4.2553	0.0116	41.90	10.48
in								
water	1.0	6.2035	40	7.5	5.3333	0.0116	66.17	16.64
Morton no. = 2.5968×10^{-10}	2.0	7 .8159	40	5.9	6.7797	0.0115	105.98	26.41
Water	0.5	4.9237	40	5.8	6.8966	0.0071	119.22	6.66
in								
bromobenzene	1.0	6.2035	40	4.6	8.6957	0.0071	189.40	10.58
Morton no. = 3.2687×10^{-12}	2.0	7.8159	40	3.7	10.8108	0.0072	296.67	16.79

Table 3. The value of $(r/u)^{3/2}/\eta^{1/2}$ and $\rho^{1/2}/(\sigma-\rho)$ and the constant **K** obtained by using eq. (3)

Liquid drop	Column liquid	$X = (r/u)^{3/2}/\eta^{1/2}$	$\hat{Y} = \rho^{1/2}/(\sigma-\rho)$	S^+	K*	Y^{++}	e^{+++}
Xylene	Water	0.0640	0.2062	0.3104	4.5046	0.2019	0.00425
Benzene	Water	0.0726	0.2284	0.3178	4.6128	0.2291	−0.00070
Kerosene	Water	0.0442	0.1393	0.3172	4.6034	0:1394	−0.00003
Turpentine	Water	0.0670	0.2095	0.3199	4.6427	0.2115	−0.00198
Toluene	Water	0.0670	0.2109	0.3178	4.6119	0.2115	−0.00058
Iso-amyl acetate	Water	0.0796	0.2520	0.3157	4.5823	0.2512	0.00087
Hexane	Water	0.0248	0.0771	0.3215	4.6659	0.0780	−0.00093
Petroleum ether	Water	0.0248	0.0778	0.3184	4.6216	0.0780	−0.00019
Cyclohexane	Water	0.0395	0.1238	0.3192	4.6334	0.1245	−0.00080
Soap oil	Water	0.0640	0.2050	0.3123	4.5321	0.2019	0.00300
Heptane	Water	0.0307	0.0960	0.3196	4.6389	0.0966	−0.00067
Water	Chlorobenzene	0.1030	0.3227	0.3191	4.6313	0.3251	−0.00241
Water	Bromobenzene	0.0205	0.0642	0.3194	4.6355	0.0645	−0.00030
			Mean	0.3176	4.6090		

*The constant of eq. (3); ^+S of eq. (6); $^{++}$By eq. (10) $Y = A + BX$
By eq. (12) using data points given in Table 4, $A = -0.0003$
By eq. (13) using data points given in Table 4, $B = 3.1601$
$^{+++}$By eq. (11) $e = \hat{Y} - Y$
From Figure 1, $S = 0.3175$ m^{-1} s^2; **K** = 4.6074
Estimated value of **K** = $\sqrt{2\pi g}/3B = 4.5920$; $g = 9.8$ m s^{-2}; (r/u), ρ, σ, and η from Table 1.

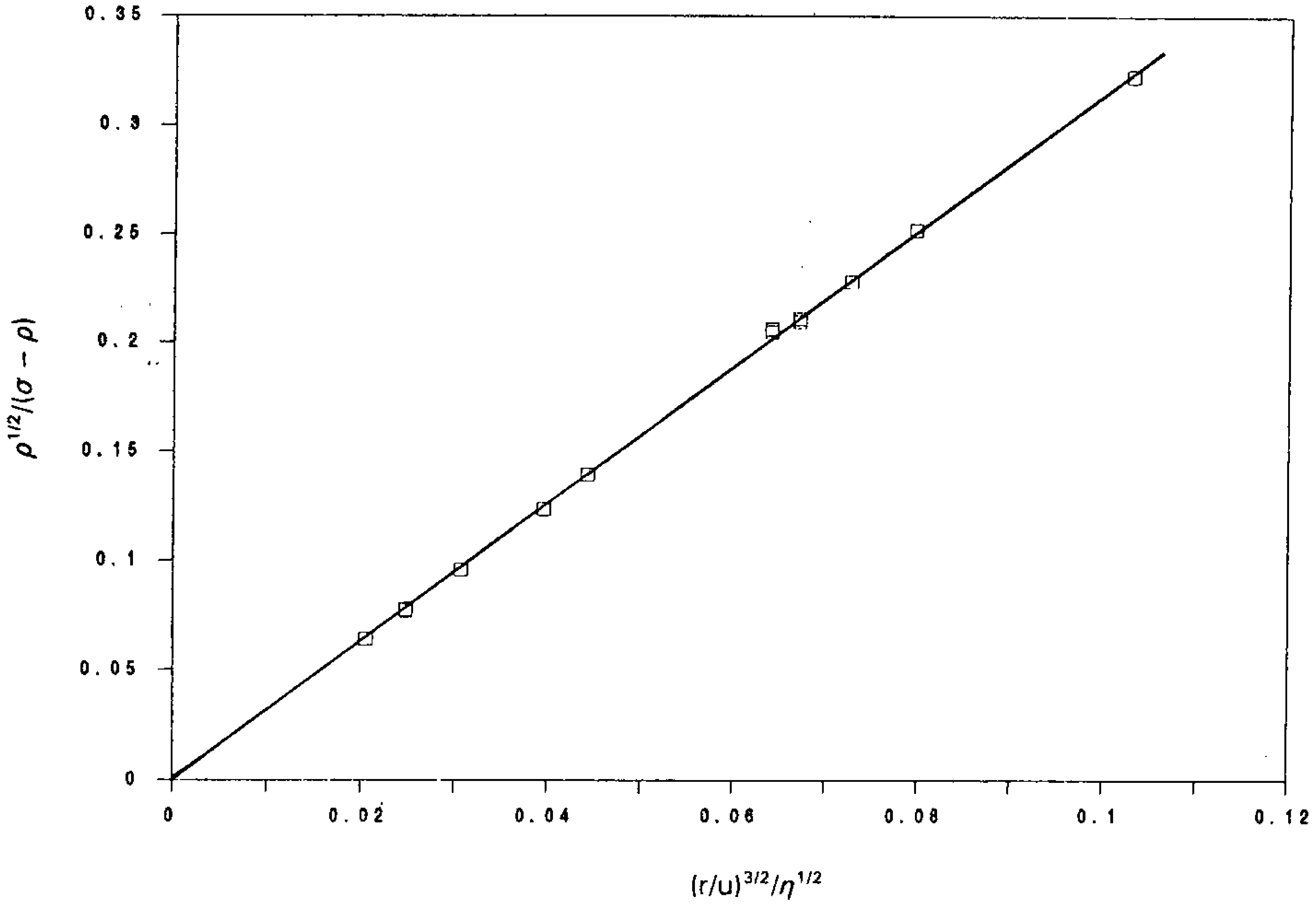

Figure 1. Plot of data given in Table 3.

drops in fall under gravity[1]. Experimental data points given in the fourth and fifth columns of Table 3 in eq. (6) give S of which $S_{mean} = 0.3176$ m^{-1} s^2 (Table 3). Comparison with drops in fall under gravity[1] (where the mean value of S is 0.2483 m^{-1} s^2) clearly shows that the value of S remains positive for drops in rise against gravity as observed for drops in fall under gravity[1]. The expression revealing the relation between S and **K**, here eq. (5), is the same as eq. (7) given by Srinivasan and Satyanarayana[1] and also for expressions ρ and λ.

Comparison of eqs (2)–(4), (6) and with those of expressions, F, **K**, u and S given by Srinivasan and Satyanarayana[1], shows that they are of similar form except for the fact that ρ and σ have just interchanged their positions.

S (eq. (6)) involves two terms $(r/u)^{3/2}/\eta^{1/2}$ and $\rho^{1/2}/(\rho-\sigma)$. The values of these terms are given in Table 3. A graph (Figure 1) is drawn with $(r/u)^{3/2}/\eta^{1/2}$ along the X-axis and $\rho^{1/2}/(\sigma-\rho)$ along the Y-axis. Referring to eq. (3), it may be seen that the slope of the line in the graph is $\sqrt{2\pi g}/3\mathbf{K}$ which works out to be 3.1500. The slope value leads to $\mathbf{K} = 4.6074$. This slope value, by eq. (5), i.e., $\sqrt{2\pi g S}/3\mathbf{K} = 1$ gives $S = 0.3175$ m^{-1} s^2.

Experimental values of $(r/u)^{3/2}/\eta^{1/2}$ and $\rho^{1/2}/(\sigma-\rho)$ given in Table 3, in eq. (3), give the value of **K** of which $\mathbf{K}_{mean} = 4.6090$ (Table 3). Eq. (5) may also be used to determine **K** and $\mathbf{K}_{mean}$.

While attempting to predict values for S and **K**, it becomes necessary to carry out uncertainty analysis which will help in establishing with experimental data points, the uncertainty or error limits of S and **K**. Therefore, as a point of interest, error analysis has also been carried out here, with experimental data points to estimate confidence limits (interval) of S and **K**, and to find out the error estimate 'e'.

With $n = 13$ and using the data given in Table 3, viz. $X = (r/u)^{3/2}/\eta^{1/2}$ and $\hat{Y} = \rho^{1/2}/(\sigma-\rho)$, or data points given in Table 4 in eqs (12), (13), (16), (17) and (18) give respectively

$A = -0.0003$,

$B = 3.1601$,

$S_a = 0.0013$,

$S_b = 0.0220$,

$S_e = 0.0019$.

$1/B$ gives the estimated value of S which is 0.3164. This value is nearly equal to the observed value of S given in the 6th column of Table 3 with negligible percentage of error. Likewise, $\sqrt{2\pi g}/3B$ gives the estimated value of **K** which is 4.5920. This value differs with negligible percentage of error with the observed values of **K** given in the 7th column of Table 3. The error estimate or residual 'e' is given in Table 3. From Table 3, it may be seen that 'e' is negligibly small.

Now, for $t_{\alpha,\, n-2}$, the table value of t for $n-2$ (= 11) degrees of freedom and α (5, 2, 1 and 0.1) per cent level of significance[29] $t_{\alpha,\, 11}$ are respectively 2.201, 2.718, 3.106 and 4.437 for the two-tailed t-test. For these values, with value $B = 3.1601$ and $S_b = 0.0220$, eqs 20–28 give the significant values related to the confidence limits (interval) of $\hat{B}$, S and **K** are as shown in Table 5.

Table 3 gives mean observed value of $S = 0.3176\ \mathrm{m^{-1}\ s^2}$; $\mathbf{K} = 4.6090$. Figure 1 gives $S = 0.3175\ \mathrm{m^{-1}\ s^2}$ and $\mathbf{K} = 4.6074$. The error analysis giving significant values regarding confidence limits of S and $\mathbf{K}$ are given in Table 5. The values of S and $\mathbf{K}$ agree quite reasonably.

The estimated value, $0.3164\ \mathrm{m^{-1}\ s^2}$, the significant values of the enumerated confidence limits (interval) of the parameter S (Table 5) indicate that (i) $S = 0.3176\ \mathrm{m^{-1}\ s^2}$ (Table 3, the mean observed value experimentally predicted for the parameter S) is acceptable and (ii) by virtue of the importance of eq. (6), (the value of the parameter $S = 0.3176\ \mathrm{m^{-1}\ s^2}$) it is possible to get fairly accurate density values by eqs (7) and (8) for liquids which are lighter than the immiscible liquids in the column and available in small quantities just as observed in the case of falling drops[1].

The values of $\mathbf{K}$ determined from the graph (Figure 1) and that determined from eq. (3) (Table 3) are 4.6074 and 4.6090 respectively and the mean of both (observed) is 4.6082. That is, 4.6082 is the value experimentally predicted for the parameter $\mathbf{K}$. Looking at the estimated value, 4.5920, the significant values of the enumerated confidence limits (interval) of the parameter $\mathbf{K}$ (Table 5), it appears that (i) the value 4.6082, experi-

Table 4. Data prepared for error analysis

Liquid drop	$X = (r/u)^{3/2}/\eta^{1/2}$	$\hat{Y} = \rho^{1/2}/(\sigma - \rho)$	X^2	$\hat{Y}^2$	$X\hat{Y}$
Xylene	0.0640	0.2062	0.0041	0.0425	0.0132
Benzene	0.0726	0.2284	0.0053	0.0521	0.0166
Kerosene	0.0442	0.1393	0.0020	0.0194	0.0062
Turpentine	0.0670	0.2095	0.0045	0.0439	0.0140
Toluene	0.0670	0.2109	0.0045	0.0445	0.0141
Iso-amyl acetate	0.0796	0.2520	0.0063	0.0635	0.0201
Hexane	0.0248	0.0771	0.0006	0.0059	0.0019
Petroleum ether	0.0248	0.0778	0.0006	0.0061	0.0019
Cyclohexane	0.0395	0.1238	0.0016	0.0153	0.0049
Soap oil	0.0640	0.2050	0.0041	0.0420	0.0131
Heptane	0.0307	0.0960	0.0009	0.0092	0.0029
Water	0.1030	0.3227	0.0106	0.1041	0.0332
Water	0.0205	0.0642	0.0004	0.0041	0.0013
	$\Sigma X = 0.7016$	$\Sigma \hat{Y} = 2.2129$	$\Sigma X^2 = 0.0455$	$\Sigma \hat{Y}^2 = 0.4528$	$\Sigma X\hat{Y} = 0.1435$

$X = (r/u)^{3/2}/\eta^{1/2}$ from Table 3; $\hat{Y} = \rho^{1/2}/(\sigma - \rho)$ from Table 3.

Table 5. Error analysis – Enumerated significant values (confidence limits – interval) of $\hat{B}$, S and $\mathbf{K}$

$t_{\alpha,\,n-2}$	Confidence limits of $\hat{B}$: Least (eq. (21))	Confidence limits of $\hat{B}$: Highest (eq. (20))	Confidence interval of $\hat{B}$ (eq. (22))	Confidence limits of S: Least (eq. (23))	Confidence limits of S: Highest (eq. (24))	Confidence interval of S (eq. (25))	Confidence limits of $\mathbf{K}$: Least (eq. (26))	Confidence limits of $\mathbf{K}$: Highest (eq. (27))	Confidence interval of $\mathbf{K}$ (eq. (28))
$t_{0.05,\,11}$ 2.201*	3.1117 (3.11)	3.2085 (3.21)	0.0965 (0.097)	0.3117 (0.312)	0.3213 (0.321)	0.0096 (0.010)	4.5238 (4.52)	4.6632 (4.66)	0.1394 (0.139)
$t_{0.02,\,11}$ 2.718*	3.1003 (3.10)	3.2199 (3.22)	0.1196 (0.120)	0.3106 (0.311)	0.3225 (0.323)	0.0119 (0.012)	4.5079 (4.51)	4.6806 (4.68)	0.1727 (0.173)
$t_{0.01,\,11}$ 3.106*	3.0918 (3.09)	3.2284 (3.22)	0.1366 (0.137)	0.3097 (0.310)	0.3234 (0.323)	0.0137 (0.014)	4.4948 (4.49)	4.6942 (4.69)	0.1994 (0.199)
$t_{0.001,\,11}$ 4.437*	3.0625 (3.06)	3.2577 (3.26)	0.1952 (0.195)	0.3070 (0.307)	0.3265 (0.327)	0.0195 (0.020)	4.4551 (4.46)	4.7391 (4.74)	0.2840 (0.284)

***Significant value of $t_{\alpha,\,n-2}$ from Table (ref. 29).**

Estimated value of $B = 3.1601$ (By eq. (13) using data points given in Table 4); Estimated value of $S = 1/B = 0.3164$; Estimated value of $\mathbf{K} = \sqrt{2\pi g/3B} = 4.5920$; $g = 9.8\ \mathrm{m\ s^{-2}}$

Using data points given in Table 4, by eq. (16), $S_a = -0.0013$ (= 0.001); By eq. (17), $S_b = 0.02220$ (0.022); by eq. (18), $S_c = 0.0019$ (= 0.002)

At $t_{0.05,\,11}$, $\hat{B} = 3.1601 \pm 0.0484$, $S = 0.3165 \pm 0.0048$, $\mathbf{K} = 4.5935 \pm 0.0697$
(3.16 ± 0.048) (0.317 ± 0.005) (4.59 ± 0.070)

At $t_{0.02,\,11}$, $\hat{B} = 3.1601 \pm 0.0598$, $S = 0.3166 \pm 0.0060$, $\mathbf{K} = 4.5943 \pm 0.0864$
(3.16 ± 0.060) (0.317 ± 0.006) (4.59 ± 0.086)

At $t_{0.01,\,11}$, $\hat{B} = 3.1606 \pm 0.0688$, $S = 0.3165 \pm 0.0068$, $\mathbf{K} = 4.5945 \pm 0.0999$
(3.16 ± 0.069) (0.317 ± 0.007) (4.60 ± 0.100)

At $t_{0.001,\,11}$, $\hat{B} = 3.1601 \pm 0.0976$, $S = 0.3168 \pm 0.0098$, $\mathbf{K} = 4.5971 \pm 0.1420$
(3.16 ± 0.098) (0.317 ± 0.010) (4.60 ± 0.142)

The mean observed value experimentally predicted (i) for $S = 0.3176$ (= 0.318) $\mathrm{m^{-1}\ s^2}$ (Table 3, Mean value $0.3176\ \mathrm{m^{-1} s^2}$; From Figure 1, $0.3175\ \mathrm{m^{-1}\ s^2}$); (ii) for $\mathbf{K} = 4.6082$ (= 4.61) (Table 3, Mean value 4.6090; From Figure 1, 4.6074).

In the brackets, the values are given in three significant figures for easy perusal of the error limits.

mentally predicted for the parameter **K** is acceptable (ii) and hence, for the motion of a liquid drop in rise or rising upward against gravity through an immiscible liquid column without exhibiting oscillation of fairly high Reynolds number, R_e (from 25 to 495), Eotvos number, E_t [from (2×10^{-2}) to (70×10^{-2})] and Morton number, M_o [from (2.06×10^{-12}) to (7.13×10^{-10})], the experimentally predicted value 4.6082 for the constant **K** in the drag force expression eq. (1) is acceptable.

If only three significant figures (given in brackets in Table 5) for the parameters S and **K** are considered, it may be seen that the error limits lie on either side of the value of $S = 0.317\ m^{-1}\ s^2$ and for the value of **K** = 4.59 or 4.60, the mean of which is 4.60. The experimentally observed value for S and **K** when expressed up to three significant figures, become $0.318\ m^{-1}\ s^2$ and 4.61 respectively. That is, they predict in two significant figures $0.32\ m^{-1}\ s^2$ for S and 4.6 for **K**. In other words, in the drag force expression eq. (1), the value of **K** is 4.6.

The motion of the drop in rise against gravity may be thought to be just opposite to that of the motion of a drop in fall under gravity. The results for the drops considered in rise against gravity show that the proportionality constant **K** in the drag force expression (eq. (1)) is 4.6082 and the mean value of the constant S (eq. (6)) is $0.3176\ m^{-1}\ s^2$. In the case of falling drop, for drops in fall under gravity[1], the value of **K** and S are respectively 3.6094 and $0.2483\ m^{-1}\ s^2$. A comparison shows that values obtained for **K** and S for drops in rise against gravity differ from those values obtained for drop in fall under gravity[1] and furthermore, **K** and S remain positive for both viz., drop in rise against gravity and drop in fall under gravity[1].

Eq. (4) with **K** = 4.6082 may be used to predict the terminal velocity (just as seen for drop in fall under gravity[1]) of the drop rising against gravity without exhibiting oscillation if r, η, σ and ρ are known. Using eqs (7) and (8) with the mean experimental value of $S = 0.3176\ m^{-1}\ s^2$ (Table 3), if r, u, η and σ are known, one may determine (as seen for drop in fall under gravity[1]) density ρ of the drops for which density cannot be determined by the capillary tube method where weighing is a problem or by any other conventional method[1].

Error analysis (confidence limits/interval) indicates that (i) getting fairly accurate value for density ρ with $S = 0.3176\ m^{-1}\ s^2$ is possible and (ii) for the fresh drag force expression eq. (1), the experimentally predicted value 4.6082 for the constant **K** is acceptable.

If only two significant figures are considered, the observed value, the estimated value and the central value about which error limits lie, show that $S = 0.32\ m^{-1}\ s^2$ and **K** = 4.6.

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