Professor Paul Erdős (1913–1996)

An obituary

Professor Paul Erdős (pronounced Airdosh) was born in Hungary on 26 March 1913. It is reported that even at the age of 3, he had invented negative numbers by himself when he set out to answer the question. How much is 250 degrees below 100 degrees? His answer was 150 degrees below zero. A few years later he amused himself by finding out how long does a train take to reach the sun. He had two elder sisters both of whom died of scarlet fever a few days before he was born. So his mother took very good care of him. He never got married and his mother helped him by taking care of his publication list and so on. She travelled with him wherever he went and Erdős was extremely sad after she passed away. His was a life dedicated completely to mathematics. He was so prolific that he had more than 1500 important papers to his credit. He was so versatile that he could contribute significantly to (almost) any mathematical problem. No wonder if, I call his contributions as 'Erdőssian Mathematics'. Some people who found that it was difficult to follow his work, say that he does not do mathematics. For some time I thought that such people invented the concept 'Erdős Number' to make fun of the fact that he writes too many joint papers (the Erdős number of a collaborator with him is 1, that of a collaborator's collaborator is 2 and so on). But later on I found (from A. Ivić) that this concept was invented by Erdős himself (for some purposes). Mathematicians would look upon Erdős as god himself and consider a close association with him as an indication of their merit (close association with him will mean a small Erdős number) and greatness. Erdős' mathematics, so terse and at the same so attractive, is described rightly by Ernst Straus (who worked both with Einstein and Erdős) as follows:

> 'In our century in which mathematics is so strongly dominated by "Theory Doctors" he has remained the prince of problem solvers and absolute monarch of problem posers.'

Looking at the vast number of papers which he has written one cannot help comparing him to L. Euler and looking at the depth and attractiveness of Erdőssian mathematics one cannot deny that he is 'in some sense' the greatest mathematician of this century. Judged from whatever angle I agree with R. Graham that he is certainly amongst the top ten in this century. He had 458 collaborators (that is those with Erdős number 1) and about 4500 with Erdős number 2 and about 100 papers will be published by his collaborators after his death. I am proud to say that I have Erdős number 1 (about which I will say in a later section). On the whole I can say that many subjects will emerge by his contributions.

Some reminiscences

I had heard of Erdős as an extra ordinary mathematician (and I had looked at some of his papers early in my research career) and I was longing to meet him (although I never expected to have that golden opportunity). In 1970–71 I was a neighbour of C. Hooley (at Princeton, Institute for Advanced Study) and he came to know that Erdős will be in New York Academy of Sciences for a few days. He telephoned Erdős and arranged for a meeting with him. He drove me to New York and we were very happy to meet Erdős and his mother. Erdős gave a lunch party to us during which he introduced us to his mother. Thereafter I used to write to him frequently and he promptly replied all my letters. At this point I would like to say that no matter whether you are amateur or a professional he would promptly reply your letters (he wrote about 15 to 20 letters a day). Every time it would be a friendly letter and suggestion of some mathematical problems. Another point about Erdős address: he never used to stay at any place for more than a week and he has travelled practically throughout the world. I always wrote to his Hungarian address. Hungarian Academy used to redirect my letters to his address. The letters would have to travel throughout the world to reach him. What is very amazing is that he remembered what he wrote to people. Each letter to them would contain suggestion of several problems, contributions for collaborative work besides enquiring welfare of family members and friends.

He has visited TIFR twice. On one occasion guest house was (luckily for us)

Professor Paul Erdős.

*This obituary is based on two reports on this topic (namely obituary) one from Brestilov Novak and another from Aleksander Ivić. In addition, it contains something about Professor Erdős which I report from my experience.

not available. We (my family and I) had him as our guest at our residence. We felt very much honoured and we will keep it as a valuable treasure in our minds the memory of his visit to us. He would work till 2 at night drinking frequently a cup of black coffee. His habit was to work almost round the clock. (Also he liked the food in our house.) Another friend in TIFR closely connected with him is N. M. Singhi.

Encouragement to E. Szémerédi

As has been said before, Erdős used to encourage difficult attractive research very much. He used to state such problems and announce prizes for solutions. Many youngsters have received prizes from him. I once spoke to him about the work of H. Maier and he told me that Maier had knocked off several prizes from him. (H. Maier, known for his highly original work on large and small differences between consecutive primes and so on, is at present a professor at ULM in Germany.) Once Erdős had proposed the following problem:

Let $k \geq 3$ be any fixed integer and N a variable integer. Let S_1 be maximal subset of integers from the set $\{1, 2, \ldots, N\}$ with the property that no k integers of S_1 are in arithmetical progression. Let $M = M(k)$ be the number of integers in S_1. Then

$$\lim_{N \to \infty} \frac{M}{N} = 0 .$$

This was solved by E. Szémerédi in an outstanding paper. Erdős proposed him for Fields Medal. But much to the dismay of the world he was not selected for Fields Medal. Erdős was annoyed. But he did not keep quite. In the same gathering of IMU (International Congress of Mathematicians) he gave $2500, a medal and a citation to Szemeredi. (Note also that $2500 exceeds the amount of the Fields Medal.) No individual has struggled all by himself and upheld the cause of deep and attractive mathematics as Paul Erdős.

A mathematician with no home, no job, a great seeker after properties of $1, 2, 3, \ldots$, Erdős would travel from place to place bearing the torch of knowledge and he would propose problems, announce prizes and help people to progress in their work. (At this point one remark: He remembered references to many (important) papers including his own, so much so that he may be known as a mobile encyclopaedia. People may think that he travelled with a huge luggage but his only luggage was a half empty small size suit case.) I mention a few attractive problems of his with prizes.

(1) Let $1 = b_1, b_2, b_3, \ldots$ be any increasing sequence of integers with the property that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty .$$

Then given any fixed integer k there exist $k(\geq 3)$ distinct integers b_ν $(\nu = n_1, n_2, n_3, \ldots, n_k)$ forming an arithmetical progression. (Prize $20,000).

(2) Let p_n denote the nth prime. Then prove or disprove that

$$\lim_{n \to \infty} \frac{p_{n+1} - p_n}{\log n \, \log\log n \, \log\log\log n} > 0 .$$

(Prize $20,000).

He would also increase the prize from time to time if a problem remained unsolved for longer periods.

Some contributions of Erdős

It is practically impossible to touch upon all the ≥ 1500 papers of Erdős. A. Ivić remarked that he would pity the person who prepared the bibliography of all the papers of Erdős. (Suffice to say that all these papers involve considerable skill and originality.) I would single out three discoveries of Erdős.

(a) G. H. Hardy and J. E. Littlewood could prove that

$$\lim_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} < 1$$

(that the above limit is ≤ 1 is a trivial consequence of the prime number theorem which asserts that $p_n(n \log n)^{-1} \to 1$ as $n \to \infty$) only on the assumption of a very difficult *unproved hypothesis* called Generalized Riemann hypothesis. Erdős succeeded in proving this without using any unproved hypothesis. He achieved it by a very ingenious use of Brun's Sieve.

(b) The prime number theorem stated just now was proved (by methods independent of theory of functions of one complex variable) by P. Erdős and A. Selberg independently of each other.

(c) For integers $n(\geq 2)$ let $\omega(n)$ denote the number of distinct prime factors of n (for our purpose it does not matter if the prime factors are counted with multiplicity). Let $f(n)$ be any positive function which $\to \infty$ as $n \to \infty$. Let $F(x)$ denote the number of integers $n \leq x$ with the property

$$|\omega(n) - \log\log n| \leq f(n) \sqrt{\log\log n} .$$

Then G. H. Hardy and S. Ramanujan proved (for the first time after conjecturing it) the result that

$$\frac{F(x)}{x} \to 1 \quad \text{as} \quad x \to \infty .$$

P. Erdős and M. Kac put forth the best of their talents and proved by an ingenious method the following result (which says a lot more about the theorem of G. H. Hardy and S. Ramanujan). Let C_1 and C_2 be any two real constants with $C_1 < C_2$. Let $F(x, C_1, C_2)$ denote the number of integers $\leq x$ with the property

$$C_1 \sqrt{\log\log n} \leq \omega(n) - \log\log n \leq C_2 \sqrt{\log\log n} .$$

Then

$$\frac{F(x, C_1, C_2)}{x} \to \left(\int_{-\infty}^{\infty} e^{-u^2} du \right)^{-1} \left(\int_{C_1}^{C_2} e^{-u^2} du \right).$$

(A passing remark: An Indian who has studied this theorem and worked out some variants of this theorem is A. Krishnaswamy.) Erdős–Kac theorem forms the foundation of a deep aspect of probabilistic number theory.

Encouragement to me and some others

Let u and k be positive integers and let $P(u, k)$ denote the greatest prime factor of $(u+1)(u+2)\ldots(u+k)$. One of the early interests of P. Erdős is what I call Sylvester–Erdős Theorem (Sylvester was the first to prove it and Erdős supplied a simpler proof). It states that

$$\min_{u \geq k} P(u,k) > k.$$

Note that this is a two-variable generalization of Chebyshev's theorem that $P(k,k) > k$ (which is nothing but the well-known Bertrand's postulate). Erdős also proved that

$$\min_{u > k^{3/2}} P(u,k) > \frac{1}{3} k \log k$$

provided k exceeds a large constant (we make from now on this hypothesis about k).

Independently of all this (not knowing this) I considered the special case $k = [u^{\frac{1}{2}}]$ and proved that $P(u,k) > u^{\lambda}$ with $\lambda = \frac{1}{2} + \frac{1}{13}$ by using Selberg Sieve and Vander–Corput sums. Erdős encouraged me to consider the case $u = [k^{\frac{1}{100}}]$ and things of that kind and I was led to prove that (apart from ε's δ's)

$$\min_{u \geq k^{3/2}} P(u,k) > k \log k.$$

However it was R. Tijdeman who sharpened (by Halberstam–Roth method, completely different from mine) the lower bound to $2k \log k$. Using (A. Baker's method on linear forms) myself and T. N. Shorey (improved Tijdeman's result to the fact that 2 can be replaced by any constant) and (using I. M. Vionogradov's methods) M. Jutila could improve it further by combining it with myself T. N. Shorey procedure. The final outcome is

$$\min_{u \geq k^{3/2}} P(u,k) >$$

$$\mu k \log k \log \log k (log \log \log k)^{-1},$$

where $\mu > 0$ is a positive constant (we have the result $p_{n+1} - p_n = O(p_n^{5/8})$ which implied that if $k \leq u \leq k^{3/2}$, $P(u,k) > u$). This is a very good achievement (the best known to-day) due to myself, T. N. Shorey and M. Jutila. Needless to say that Erdős was encouraging all the time at all stages of our work.

Another problem concerns is generalization of Mirsky's result that the number of square free numbers s and such that

$$2n = p + s \; (p - \text{prime})$$

is asymptotic to a constant (> 0) times $n(\log n)^{-1}$. Myself and G. J. Babu considered the number of solutions in integers (with certain property be stated below) satisfying

$$2n = p + s$$

is asymptotic to a constant (> 0) times $n(\log n)^{-1}$. The property is this: Let $b_1, b_2, \ldots$ be away increasing sequence of positive integers (with $b_1 \geq 2$) such that for any two distinct integers i, j we have $(b_i, b_j) = 1$ and

$$\sum_{j=1}^{\infty} (\varphi(b_j))^{-1} < \infty.$$

(The numbers s are assumed to be not divisible by any b_j.) We had to introduce some further conditions and Erdős removed practically all the conditions. The result was a joint paper by myself, G. J. Babu and Erdős. In another joint paper we considered also further relaxation of conditions and obtained the lower bound of the same order instead of an asymptotic formula. All this was possible only because of various letters from Erdős.

I should mention that he has joint papers with T. N. Shorey, R. Balasubramanian and a complete account of these contributions would require a longer article.

Erdős was an honorary fellow of the Hardy–Ramanujan Society besides having many other honours throughout the world. He was on the editorial boards of many journals in the world (*Acta Arithmetica, Journal of Combinatorics, Bulletin of the Calcutta Mathematical Society, Journal of the Indian Mathematical Society*, and so on). His service to mathematics is so great that we feel that the world is not the same without him. In fact there exists in the minds of some of his followers a fact that they belong to the Erdős International Academy of Mathematics, Erdős International University and so on. Professor A. Ivić is one of them. Erdős passed away on 20 September 1996 in Warsza, Poland.

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