Are chaotic particle trajectories fractals?

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It is shown that chaotic particle trajectories in a simple model of two superposed internal inertia-gravity waves may come under the class of natural fractals. The fractal dimension is estimated to lie between 1.2 and 1.5, depending on the strength of the perturbation forcing introduced by the second wave. The significance of the present result is that fractal scaling behaviour of particle paths may arise even before the Eulerian flow becomes turbulent. Implications of the results are discussed from theoretical and experimental viewpoints.

MANDELBROT[1] raises the question whether or not the transition to turbulence in fluids can be associated with the circumstances under which the fluid particle trajectory is a fractal. Richardson[2] was also intrigued about whether it would become necessary to describe the position of an air particle in the atmosphere by something rather like Weierstrass's continuous, *nondifferentiable* function. In the last decade, it was appreciated that enhanced mixing, which is reminiscent of a symptom of Eulerian turbulence, could arise in simple and regular (i.e. non-turbulent in the Eulerian sense) fluid flows purely through dynamical chaos which arises in nonlinear systems[3,4]. This phenomenon is now widely known as 'chaotic advection' or 'Lagrangian turbulence'. We investigate whether chaotic fluid particle trajectories in Lagrangian turbulence exhibit fractal scaling behaviour. Our study is motivated by an experimental indication from oceanography that the Lagrangian trajectories of ocean drifters placed in the surface layer of the Kuroshio extension display a fractal dimension of approximately 1.3, within a range of space and time scales usually attributed to geophysical fluid dynamical turbulence[5].

Let us briefly mention the idea of chaotic advection. Suppose that there is a given velocity field $\mathbf{V}(\mathbf{X}, t)$ in an incompressible fluid at position $\mathbf{X}$ at time t, this solution of the Eulerian equations having been already obtained. The equations for the Lagrangian motion of passive fluid particles in an incompressible fluid are of the form

$$\frac{dX}{dt} = \mathbf{V}(\mathbf{X}, t), \tag{1}$$

where $\nabla \cdot \mathbf{V} = 0$ and $\mathbf{X} \in \Re^3$. In general, $\mathbf{V}$ in eq. (1) is a nonlinear function of $\mathbf{X}$ and t. Our present understanding of the theory of dynamical systems suggests that solutions to equations of the form (1), if effectively defined in $\Re^3$, may typically exhibit chaos which is characterized by exponential divergence of nearby trajectories. This is true even for time-periodic, two-dimensional (2D) flows and spatially-periodic steady three-dimensional (3D) flows.

A fractal is a geometrical object whose shape is irregular and/or fragmented at all scales. Mathematically, a fractal is defined as a set for which the Hausdorff–Besicovitch dimension (or fractal dimension D) strictly exceeds the topological dimension D_T which is always an integer[1]. The value of D for a fractal curve which lies on a 2D surface lies between the value of D_T for the curve (which is one) and the dimension of the surface (which is two). A basic characteristic of fractal curves is their 'scaling' behaviour. Scaling implies that every part of the curve is in some sense a reduced version of the whole curve. We defer the discussion of the method to obtain scaling laws to the next section. At this point, it is also necessary to distinguish between 'mathematical' fractals as opposed to 'natural' fractals. In the study of *mathematical fractals*, the scaling properties are assumed to hold good on the entire spectrum of time or space scales. In the case of natural fractals, which are encountered in nature and are of primary concern in this study, small- and large-scale constraints usually confine the scaling properties to a finite range of scales.

The model system used in this study is an idealized flow induced by two superposed, vertically trapped, internal inertia-gravity waves (IGWs) in the atmosphere which was shown in an earlier paper by Joseph[6] to exhibit chaotic particle paths. In that study, linearized solutions for vertically trapped, horizontally propagating IGWs were obtained for a continuously stratified, incompressible, inviscid, isentropic and rotating fluid, with uniform buoyancy frequency (see ref. 7 for explanations). The basic state was one of a motionless fluid in hydrostatic equilibrium. The velocity field corresponding to a single IGW was found to be incapable of producing chaotic particle paths. However, when two IGWs with different wave numbers were superposed, numerical evidence suggested that the resulting velocity field was sufficient to produce chaotic particle paths in some regions of the flow domain. Let us denote the superposed velocity field in the form

$$\mathbf{V}(\mathbf{X}, t) = \mathbf{V}_1(\mathbf{X}, t) + \varepsilon \mathbf{V}_2(\mathbf{X}, t), \tag{2}$$

where the subscripts 1 and 2 correspond to the first and second wave, respectively, and ε is a parameter denoting the strength of the perturbation forcing introduced by the second wave. After making a Galilean-type transformation to reduce the propagating waves to rest, the advection equations governing the fluid particle trajectories are of the form

$$\frac{dx}{dt} = (A_1 \cos x + B_1 \cos y + C_1 \sin y)\cos z - D_1,$$

$$\frac{dy}{dt} = (A_2 \cos y + B_2 \cos x + C_2 \sin x)\cos z - D_2, \quad (3)$$

$$\frac{dz}{dt} = (A_3 \sin x + B_3 \sin y)\sin z.$$

For the expressions of the subscripted coefficients A, B, C, D, and values of parameters, typical of the Earth's atmosphere, used in this study, the reader may refer to ref. 6. The first and second waves have periods nearly 3 h and 7 h, respectively.

For a given initial condition (x_0, y_0, z_0) at $t = t_0$, we solve eq. (3) to get the particle's position for any $t > t_0$ which constitute the trajectory of the particle parameterized in time. Since the wave is trapped in the vertical direction, our primary concern is with the two time-series corresponding to the x and y coordinates of the particle's position. A common procedure to determine the fractal scaling behaviour is based on considering the average absolute displacements in x and y directions, $\langle \Delta x(k\Delta t)\rangle = \langle |x(t + k\Delta t)) - x(t)|\rangle$ and $\langle \Delta y(k\Delta t)\rangle = \langle |y(t + k\Delta t) - y(t)|\rangle$ where $\langle\ \rangle$ denotes an appropriate time-average for each k, the vertical bars indicate absolute values, and the delay time $k\Delta t$ is an integer multiple of the sampling time Δt (ref. 5). If

$$\langle \Delta x(k\Delta t)\rangle = k^{H_x} \langle \Delta x(\Delta t)\rangle,$$

$$\langle \Delta y(k\Delta t)\rangle = k^{H_y} \langle \Delta y(\Delta t)\rangle, \quad (4)$$

where the equality is in the sense of distributions, then the time-series of particle's positions are scaling functions. Then, $x(t)$ and $y(t)$ are said to be self-affine, simple (or mono) fractal signals with scaling exponents H_x and H_y in x and y directions, respectively. For this case, the graphs of $\langle \Delta x(k\Delta t)\rangle$ and $\langle \Delta y(k\Delta t)\rangle$ versus k fall on straight lines on a double logarithmic plot. The slope of this line gives the value of the scaling exponent H. For self-similar, isotropic monofractal curves, the scaling exponents are the same (i.e. $H_x = H_y$). For self-similar monofractals on a 2D surface, the fractal dimension is given by

$$D = \min[1/H, 2]. \quad (5)$$

Hence for $0 < H < 1$, the particle trajectory is a simple fractal curve such that $1 < D \leq 2$. Pure Brownian motion corresponds to $H = 1/2$. For any other value of H, the motion is called fractional Brownian motion (fBm)[1]. The trace of an fBm is a nonstationary process and the degree of nonstationarity depends on the value of H (ref. 8). As H increases, the dependence of the position at a given instant on the position in the past also increases.

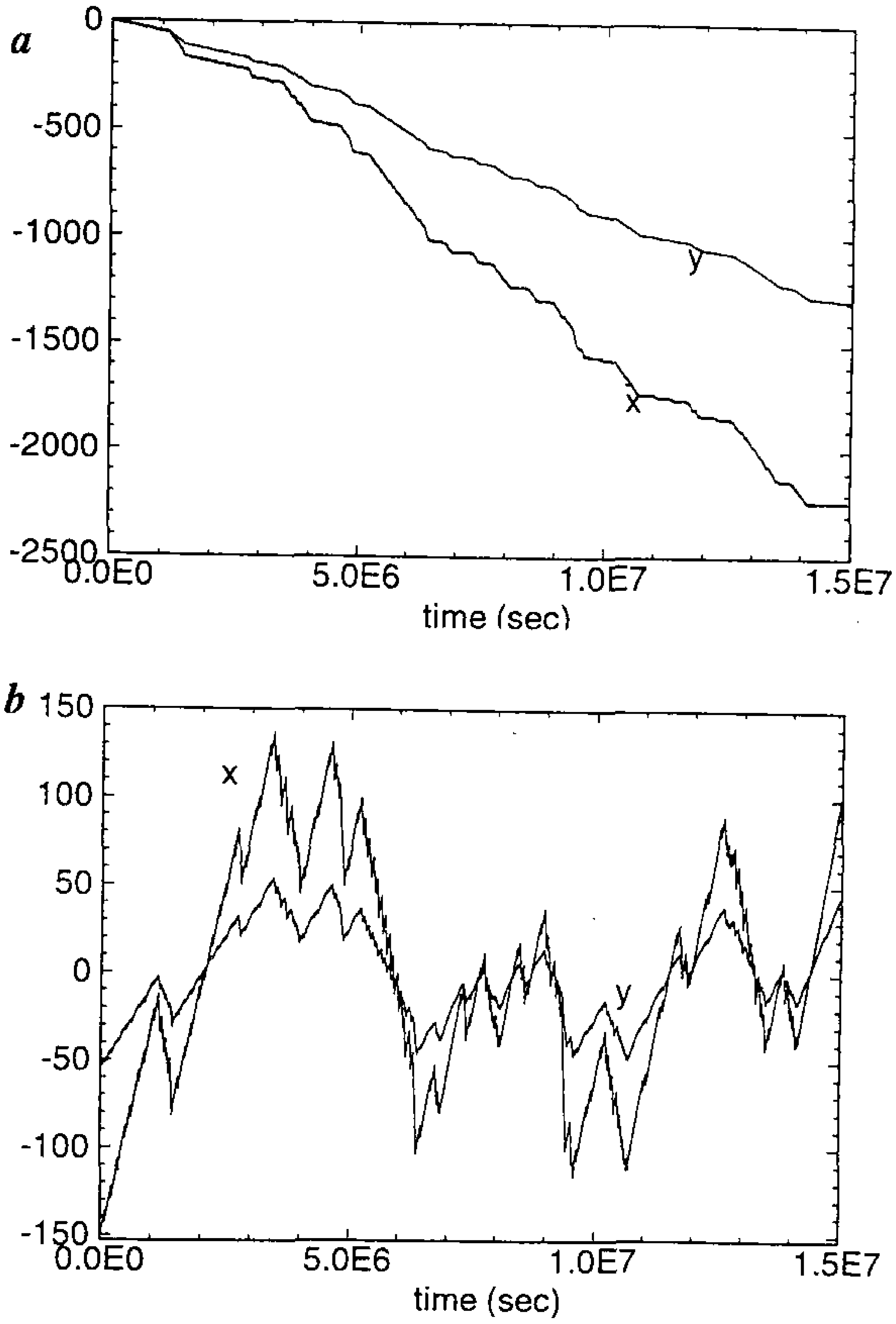

Figure 1. Time-series of particle's x and y coordinates, when $\varepsilon = 0.5$. ***a***, the original time-series, ***b***, the detrended time-series.

In addition, the departure from the origin increases. Thus, in effect the exploration of a given area decreases, and the particle leaves the area sooner and the trace of the motion becomes smoother.

First, we will present the results for a typical value of $\varepsilon (= 0.5)$, which was found to exhibit chaotic particle paths in ref. 6. The sampling time-interval (Δt) for the trajectory is 2000 s. Figure 1 *a* shows the x and y coordinates of particle's position as function of time, obtained by solving eq. (1), with initial position at (0, 0, 0.2), using a Runge–Kutta method[9]. The motion of the particle has been described in ref. 6 as a combination of trapped helical and untrapped undulatory motions. Our analysis is applied on the detrended time-series of particle's position which is presented in Figure 1 *b*. The removal of trend implies that we are studying the fluid particle motions about the rectilinear trajectory generated by a constant mean flow. The scaling behaviour, in both x and y directions, is evident from Figure 2 *a* for a range of k from about 2 to 200, which corresponds to a time ranging from about 1 h to 4.6 days. The scaling behaviour is strikingly similar in both x and y. This may

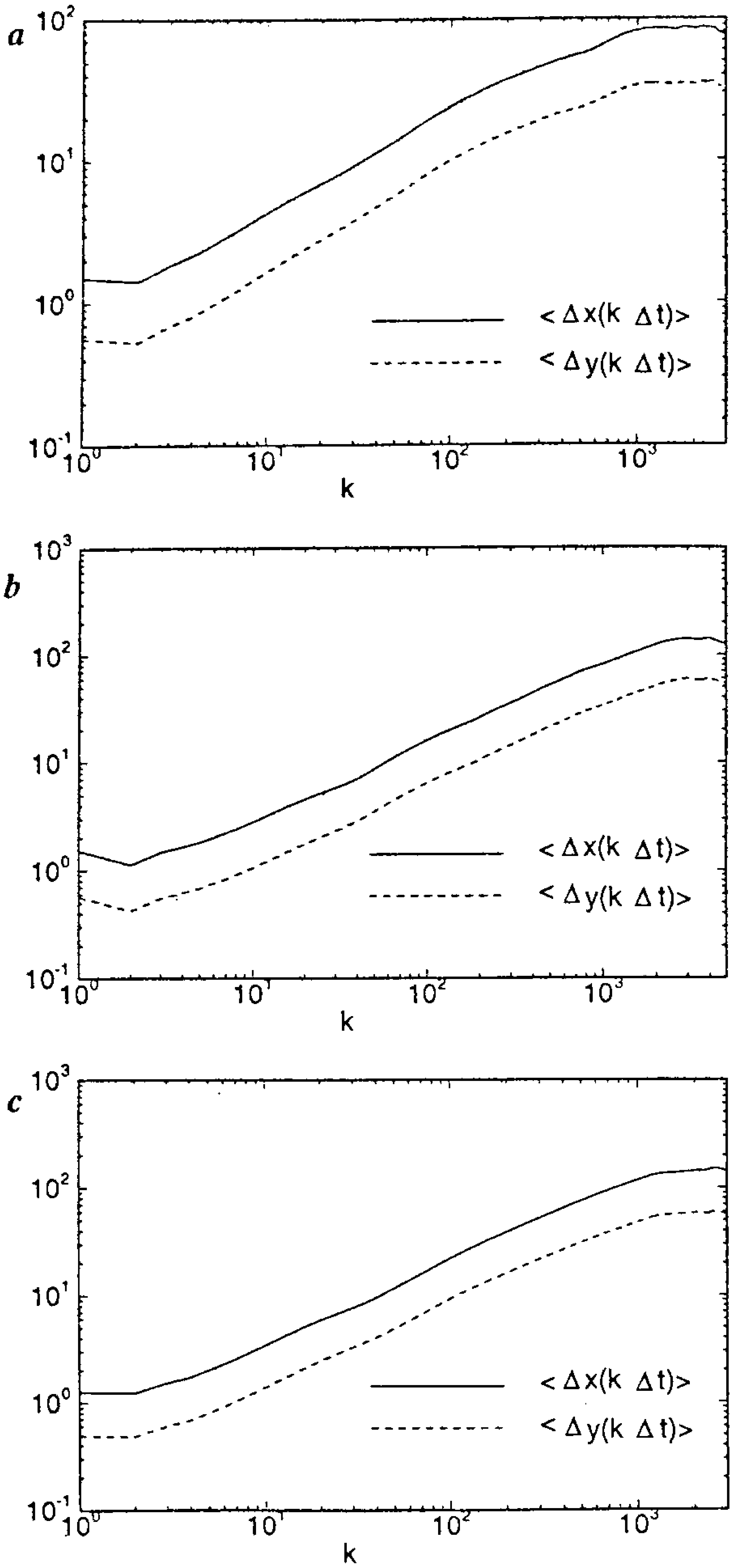

Figure 2. Plot of $\langle \Delta x(k\Delta t) \rangle$ and $\langle \Delta y(k\Delta t) \rangle$ versus k in double logarithmic axes, for (*a*) $\varepsilon = 0.5$; (*b*) $\varepsilon = 0.3$; (*c*) $\varepsilon = 0.7$.

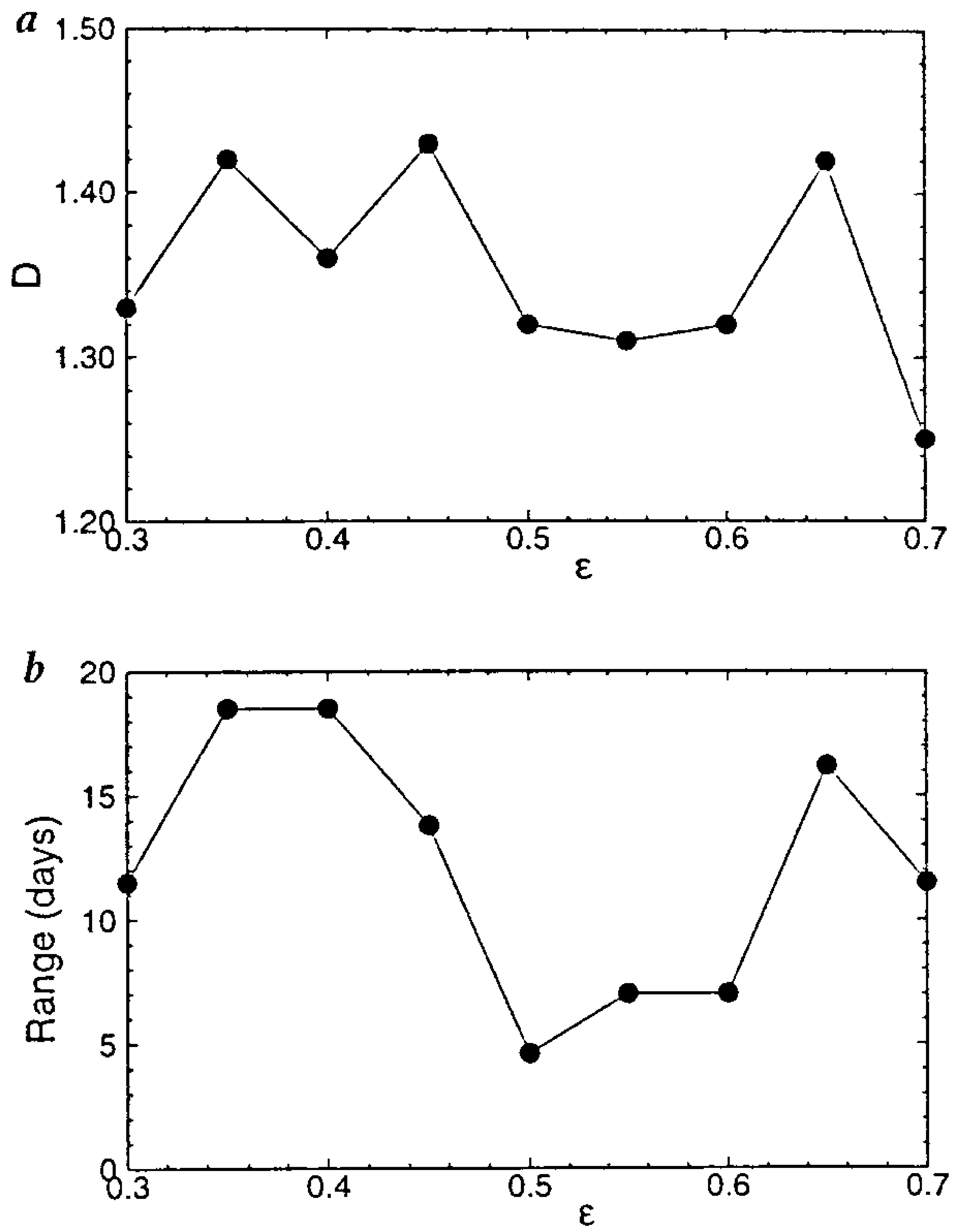

Figure 3. *a*, Plot of the fractal dimension (D) versus the perturbation forcing parameter (ε). *b*, Plot of scaling range as a function of the perturbation forcing parameter (ε).

be reflecting the fact that the anisotropy introduced by the rotation of the Earth is not strong enough to induce significant differences in the scaling behaviour for the zonal and meridional directions for the present model system. The values of the scaling exponents obtained are $H_x = 0.746 \pm 0.006$ and $H_y = 0.779 \pm 0.005$, where the indicated uncertainties are the statistical errors of the least square fits to the slopes estimated as the least-squares 95% confidence limits on the individual values of the slope. The average of the scaling exponents in x and y direction is approximately 0.76 which gives the fractal dimension (D) as 1.32 from eq. (5). This result indicates that the motion of the particle corresponds to an fBm and the trajectory samples only a portion of the x–y plane that is accessible to it.

Next, we examine whether any quantitative change occurs if the value of ε is varied. Results of a similar analysis, when $\varepsilon = 0.3$ and $\varepsilon = 0.7$, are presented in Figure 2 *b*, *c*, respectively. A linear slope is evident, in both cases, for k ranging from about 2 to 500 (which in real time is from about 1 h to 12 days). The values of H_x and H_y, when $\varepsilon = 0.3$, are 0.748 ± 0.003 and 0.760 ± 0.003, respectively. An average value of H about 0.75 gives the value of D to be 1.33. For the case when $\varepsilon = 0.7$, we obtain the values of H_x and H_y as 0.801 ± 0.003 and 0.798 ± 0.003, respectively. Now, the average of the scaling exponents is 0.80 which results in a fractal dimension of 1.25. The variation of D for different values of ε is presented in Figure 3 *a*. For the initial condition chosen in this study, the trajectory becomes chaotic only when ε is increased to about 0.3. We notice that there is no monotonic dependence of D on ε. The values of D lie between 1.2 and 1.5. In Figure 3 *b*, the range over which the scaling behaviour holds is shown as a function of ε. The lower cut-off is at 1 h for all the cases. The scaling range shows a threshold minimum for intermediate values of ε. A deterministic explanation for the variation of D and the scaling range might exist, but, it is not clear to us at the moment. The interesting aspect of this work is the indication that fractal scaling behaviour of fluid particle trajectories may occur even before the Eulerian flow becomes turbulent. Wave motions are ubiquitous in many branches of fluid dynamics. Even though our work is a case study of IGWs, it is likely that similar results could be found in different flow scenarios if the flow fields are topologically equivalent.

Be that it may, we need to address the theoretical and experimental implications of fractal scaling behaviour of chaotic trajectories in fluid flows. In contrast to the general practice in the study of nonlinear dynamical systems, where the emphasis is on a higher dimensional phase space (of Fourier modes), the present results are obtained in the physical space of spatial variables. From an experimental viewpoint, the determination of D in the phase space is extremely difficult, whereas it is more definitive in the physical space. The fundamental issue here is whether given the fractal scaling behaviour can we infer about the dynamical processes responsible for the generation of this fractal behaviour? Unfortunately, there is no clear answer to this question because by restricting attention to the physical space rather than the phase space, it is usually not possible to deduce dynamical models uniquely.

In the context of dissipative systems, the fractal dimension of chaotic trajectories may be interpreted as an estimate of the dimension of the chaotic attractor to which the trajectory asymptotically approach and a small value of D indicates that the underlying dynamics is that of a low-order dynamical system. But, for incompressible fluid flows such as the model considered in this study, there cannot be any attractors and the above interpretation is not generally valid. However, it may be noted that the present model velocity field is horizontally divergent and this may contribute to the fractal nature of trajectories observed on the x–y plane. Bennetin *et al.*[10] has pointed out that an apparent fractal dimension which is smaller than the available phase space dimension can occur in Hamiltonian systems as a transient phenomenon. It is likely that this transient process also contributes to the value of D observed in the present study. Again, a small value of D for an experimentally observed trajectory does not, by itself, unambiguously identify that the underlying dynamics (Eulerian) is that of a low-order system. For example, a nonlinear Hamiltonian model simulating 2D turbulence (thus, includes a large number of wave-components) can also result in small values of D (nearly 4/3) as shown by Osborne and Caponio[11]. Given an experimentally observed trajectory, following a method described by Brown and Smith[12], it is possible to obtain the fraction of energy contained on average in the n largest modes as a function of n from the spectrum of the trajectory covariance matrix. The application of this method to submerged float trajectories in the North Atlantic Ocean suggested that the underlying dynamics is that of a low-order system. However, differences between float, 2-mode, 32-mode, and random-walk simulated trajectory results were found to be very small. Therefore, what one could say is that simple dynamical models could be constructed in such a way that their outcome is statistically the same, up to some level of approximation, as the measured results. The present situation is similar to a problem encountered in the study of turbulence. There is experimental evidence to suggest that several aspects of fully-developed turbulence (such as interfaces, constant-property surfaces, dissipative structures, etc.) are fractals[13,14]. It is not clear how, given the fractal dimensions of several of its facets, one can solve the inverse problem of reconstructing the turbulent flow itself. The fact that certain experimental data exhibit fractal scaling properties over a finite range of scales is an important result. It is likely that the basic dynamical processes active in the flow are responsible for the scaling range. It is a major challenge for future theoretical and experimental studies to relate the fractal properties to the actual dynamical processes and to explore the origin of the finiteness of the scaling range.

The mathematical notion of fractal dimension in physical space is useful in the context of certain practically important physical processes such as mixing and transport in fluid flows. Mixing in real flows is governed by an interplay between advective transport by the velocity field and molecular diffusion. For short times of practical interest, advection usually dominates the transport process. Transport in fluid flows is often characterized by the variance of the displacement of a particle distribution given by

$$\sigma^2(t) = \lim_{t\to\infty} \langle (x(t) - \langle x(t) \rangle)^2 \rangle \sim t^\gamma.$$

Enhanced normal (or Brownian) diffusion ($\gamma = 1$) results from particle trajectories which, like the Brownian motion, can be treated as a random walk, but with significantly enhanced transport rates. Random walk prediction with a finite second moment gives normal diffusive transport by the Central Limit Theorem. However, transport in several flows with both persistent vortices and coherent jets is reported to be 'anomalous' (with $\gamma > 1$) in experiments[15,16] as well as in simple 2D models of chaotic advection[17-20]. Anomalous diffusion results from a competition between sticking events inside or near persistent vortices and long distance flight events in the jet regions. Lévy random walk models, characterized by power law probability distribution functions for flight lengths and times, lead to transport process which is superdiffusive with $\gamma > 1$ (refs 21–23). Lévy flights have the interesting property that the mean square displacement per step diverges and the set of sites visited by the walk is a fractal[24]. Hence, experimental observation of fractal particle trajectories gives an indication that the transport process may be of the anomalous type. It is worthwhile to note here that weakly turbulent flow in rotating annulus experiments by Weeks *et al.*[16] resulted in normal diffusion. Pasmanter[17] also showed that it is only in the limit that the chaotic region occupies all the space, transport is of the Brownian type. Then, chaotic paths can cover all the available space and D would approach 2. Moreover, it is reported in ref. 16 that theoretical prediction by I. Mezic and S. Wiggins (unpublished) for turbulent flows is also normal diffusion (at time scales large compared to the eddy turnover times). If the above results are generally valid, the experimental observations of fractal trajectories have a wider implication for our understanding about transport phenomena in fluid flows. A fractal dimension less than 2 may indicate that transport is mainly due to advective chaos in a non-turbulent flow field in which regular and chaotic regions coexist to produce sticking and flight events. In contrast, a fractal dimension of 2 (typical of Brownian motion) in a planar flow may suggest that the flow is either uniformly chaotic or is approaching a turbulent regime. The above scenario, in which advective chaos plays the role of a transitional regime, also provides a smooth variation for D from 1 (in laminar regime) to 2 (in turbulent regime) as plausibly demanded by continuity requirements.

A close analogy to the above heuristic arguments holds good for the model system considered in this study as well. In ref. 6, it was shown that chaotic trajectories undergo a combination of trapped helical motions in the vicinity of regular toroidal region and long distance jumps in an open undulatory flow region. By releasing clouds of particles in the chaotic region, transport was found to be anomalous with γ ranging from about 1.6 to 2.7, depending on the value of the perturbation forcing parameter ε and the initial position of the cloud release. However, a direct comparison with theoretical predictions is premature at the moment. For this, Lévy random walks need to be generalized for higher dimensional embedding space taking into account the anisotropy introduced by stratification and rotation which are avenues for future work. It is anticipated that the study of Lévy walks will become more and more important in the coming years for a better understanding of transport phenomena in geophysical flows.

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