Planck's radiation thermodynamics and its consequence: Laue's thermodynamics of interference

*Amar Nath Nigam**

A chronological account of Planck's radiation thermodynamical relations, their consequences as applied to interference by Laue are given from a pedagogic point of view. Planck's approach, Laue's thermodynamics of interference and partial coherence and importance are described in brief. It is found that the Planck's law as well as the mathematical definition of partial coherence could be derived with preference of the latter aspect over the former. An experiment about the visibility has been proposed for the first time.

Radiation thermodynamics was developed mainly by Wien and Planck and later applied to interference by Laue. All the original literature, therefore, exists in German journals. Excellent review articles exists on the work of Wien and Planck in English but Laue's work has been overlooked. The present article aims at giving a chronological account of these works from a pedagogic point of view.

To treat radiation as a thermodynamic system owes its origin to the following facts: a) Radiation exerts pressure which can be calculated from the electromagnetic theory of Maxwell. b) The conception of temperature was extended from material bodies to radiation. This was done by Wiedemann[1] who argued that a sufficiently heated body emits light. This light was ascribed the temperature of the emitter. The luminescent materials emit light at room and even lower temperatures when excited externally by radiation of a suitable frequency. The emitted luminescent radiation was assigned the temperature at which the luminescent material would emit the same kind of light on heating without being decomposed. Thus the temperature of the luminescent light is higher than that of the surrounding medium. c) Wien[2] suggested the extension of the entropy concept to the radiation. If a body heated at temperature T loses energy Q by radiation, the former suffers a loss in entropy $S = Q/T$ and the radiation gains this entropy which like energy gets propagated away. d) Experiments on black body radiation show that the thermal radiation attains a state of equilibrium with the cavity walls, thereafter no further energy exchange occurs. Thus all thermodynamical variables can be assigned to radiation.

Planck's entry

At this stage when enough evidence was available to treat radiation as a thermodynamic system, Max Planck entered the scene. His route to radiation entropy was different. He had a strong conviction that the irreversible radiation processes of a thermal cavity should be accounted for by the electromagnetic theory of light. In his first two introductory papers[3,4] Planck investigated the equations of motion of an electric dipole oscillator in presence of an electromagnetic wave. The oscillator absorbs the wave and re-emits it. The reaction of the re-emitted wave causes radiation damping in the oscillator. This differs from the frictional damping in three major aspects. Radiation damping is independent of the oscillator material, the amplitude of the oscillations is always finite and the principle of energy conservation is satisfied, energy is *not* consumed as in the case of frictional or mechanical damping. This last property convinced Planck[5] that the irreversible radiation processes should be accounted for by the 'conservative interaction' of the radiation damping. To ensure irreversibility of emission, Planck argued as follows. The incident wave excites the oscillator to re-emit energy that propagates away with amplitude proportional to $1/r$. If after a small time interval all the magnetic fields be reversed, the wave will travel back again the same distance r. On rearrival at the oscillator, the amplitude is proportional to $i/2r$. All the phases and amplitudes get altered and it is just impossible to reproduce the original incident wave. The irreversibility is thus established. A characteristic of irreversibility is the establishment of the state of equilibrium. The system cannot by itself return to any of the states it possessed previously.

A spherical cavity (without oscillator) emits thermal waves normal to its surface. These spherical wavefronts converge toward the centre from where they diverge out, get reflected by the walls of the spherical cavity and this process of alternate convergence to and divergence from the centre continues periodically with a time interval $2R/c$, which is the time between two successive equidirectional passages of the same wave through a given point in the space of the cavity. Such waves are called tuned or ordered waves. Now if an oscillator be put at the centre, it re-emits waves that are not tuned or are called disordered. The tuned waves do not satisfy the conditions for the establishment of equilibrium and have to be avoided for explaining irreversibility.

These were the initial steps that led to the idea of 'natural radiation', thus to *entropy of radiation* in electromagnetic terms[6]. Intensity of radiation can be expressed as a Fourier series. Each term is called a partial oscillation and in general has a random phase. Radiation with random phases in its partial oscillations is called 'natural radiation'. This randomness in phases gives rise to entropy of radiation. Thus the untuned or disordered radiation is indeed natural radiation. The existence of radiation entropy is another philosophical thought in Planck's approach, supplemented by his

*Deceased on 13 September 1997.

remark that radiation carries with it entropy along with the electromagnetic energy. The phase randomness in natural radiation is analogous to the position and velocity randomness in gas molecules: they are in a state of molecular chaos.

When natural radiation interacts with a single oscillator, the latter suffers interactions that are random. This gives rise to the entropy of a single oscillator in presence of the natural radiation. This forms an answer to Jeans' objection[7,8]: how can a single oscillator have an entropy? It is not the lone oscillator but the random interactions of the natural radiation with it that give rise to the entropy of the oscillator. How this argument of Planck escaped Jeans' attention is not clear. The expression for the total entropy of the cavity plus the radiation was written as

$$S_{\text{Total}} = \sum S_{\text{oscillators}} + \int s \, \mathrm{d}J. \qquad (1)$$

The second term is the entropy of the radiation field, s is the entropy density. How Planck arrived at his radiation formula from the entropy considerations has been described by the present author[9] earlier partly in this journal and partly elsewhere[10]. His equation for entropy corresponding to his radiation formula was

$$S = \frac{a'}{a}\left\{\left(\frac{U}{a'\nu}+1\right)\ln\left(\frac{U}{a'\nu}+1\right) - \frac{U}{a\nu}\ln\frac{U}{a\nu}\right\}, \qquad (2)$$

where a', a are constants.

We write below the expression for radiation intensity derived by Planck and later used by Laue. However Planck[8] gave a justification for his intensive expression (e is the base of natural logarithm, b a constant, θ a parameter),

$$S_{oscill} = -\frac{U}{a\nu}\ln\frac{U}{e\nu b} = \frac{U}{\theta} \qquad (3)$$

= electromagnetic entropy given to the radiation emitted,

which was one of the bases for deriving eq. (2), where U is given by the classical result,

$$U = \frac{c^2}{\nu^2} R_\nu. \qquad (4)$$

Here R_ν is involved in defining the energy of a linearly polarized beam in the spectral range $\mathrm{d}\nu$ via $R_\nu \, \mathrm{d}\nu \mathrm{d}\sigma \mathrm{d}\Omega$ as the energy emitted by a surface element $\mathrm{d}\sigma$ in time $\mathrm{d}t$ in the solid angle $\mathrm{d}\Omega$ where c is the speed of light.

When radiation traverses through a dielectric, according to Kirchhoff–Clausius law (reference not quoted by Planck[11]), R_ν changes to R/η^2, where η is the refractive index. Consider radiation emitted from a surface element $\mathrm{d}\sigma$ to another one $\mathrm{d}\sigma'$ situated at a distance r. In time $\mathrm{d}t$, in the frequency interval ν to $\nu + \mathrm{d}\nu$, energy flowing is

$$\{2R_\nu \, \mathrm{d}\sigma \, \mathrm{d}\sigma' \mathrm{d}\nu \, \mathrm{d}t\}/r^2, \qquad (5)$$

where $2R_\nu$ is defined as the brightness of the unpolarized radiation of frequency ν. Total intensity is defined by

$$J_\nu = 2R_\nu \int \mathrm{d}\sigma \int \frac{\mathrm{d}\sigma'}{r^2}, \qquad (6)$$

which is written as

$$J_\nu = 2R_\nu F\omega, \qquad (7)$$

where F is the area of the image surface and ω is the solid angle. Now eq. (2) gives the temperature of the radiation

$$\theta = a\nu / \{\ln(ab\nu^3 \omega F/c^2 J_\nu)\} \qquad (8)$$

and for a medium with refractive index n,

$$\theta = a\nu / \{\ln(2b\nu^3 \omega n^2 F/c^2 J_\nu)\}. \qquad (9)$$

The so-calculated temperature θ of the radiation is called the electromagnetic temperature and is maintained as long as the radiation propagated unhindered by absorption or splitting of the beam. The free propagation of radiation even with spreading is a reversible process. If in refraction or reflection there is no loss in energy again, the process is reversible. Any weakening of intensity due to splitting of radiation in two or more directions as in case of diffuse reflection causes a lowering in the temperature of radiation. Therefore interference and diffraction are also irreversible processes. Let us see how Laue contradicted this statement for interference in a Michelson's interferometer where no diffraction effects occur.

Laue's thermodynamics of interference

In thermodynamics we distinguish between reversible and irreversible processes from two points of views:

a) According to Clausius, a process is reversible if it can run both forward and backward. Planck modified this statement by adding the condition: initial and final thermodynamic states must coincide.
b) Entropy in an irreversible process always increases but in a reversible process it either decreases or remains constant.

We have two principles characterizing entropy: a) the principle of increase of entropy and b) additive property of entropy. Using the additive property of entropy, Laue[12] proved that in case of interference the entropy of coherent bundles of rays always decreases. For this purpose he chose the example of interference in a Michelson's interferometer to which equation (2) of Planck (Note 1) can be applied directly.

Laue puts expression (5) in a slightly different form. The energy of a plane polarized radiation bundle with intensity R in vacuum, frequency coming out of a small opening with angle falling on a focal plane with area f at angle θ in time is proportional to

$$tf\omega \ln \theta \, R \, \mathrm{d}\nu. \qquad (10)$$

The eq. (2) assumes the form

$$S = tf\omega \ln \theta k \frac{\nu^2}{c^2}\left[\left(1+\frac{c^2 R}{h\nu^3}\right)\ln\left(1+\frac{c^2R}{h\nu^3}\right) - \frac{c^2R}{h\nu^3}\ln\frac{c^2R}{h\nu^3}\right]\mathrm{d}\nu, \qquad (11)$$

where c is the speed of light, k and h are Boltzmann's and Planck's constants respectively. Let there be two radiation bundles with intensities R_1 and R_2. How does their entropy behave at constant energy assuming the validity of the addition theorem of entropy? In interference, the amplitude sums $R_1 + R_2$ and the differences $R_1 \sim R_2$ are involved. The constancy of energy implies $R_1 + R_2 =$ constant while entropy is a function of $R_1 \sim R_2$ $(\Delta S = \Delta Q/T$, $\Delta Q = R_1 \sim R_2)$. Putting,

$$\frac{c^2 R_1}{h\nu^3} = (a + x),\ \frac{c^2 R_2}{h\nu^3} = (a - x), \qquad (12)$$

we obtain

$$\left.\begin{aligned} x &= \frac{1}{2}\frac{c^2}{h\nu^3}(R_1 \sim R_2) \\ a &= \frac{1}{2}\frac{c^2}{h\nu^3}(R_1 + R_2) \end{aligned}\right\}. \qquad (13)$$

Thus x is proportional to entropy and a to energy. By addition theorem, the entropy of the two bundles is

$$f(x) = (1 + a + x)\ln(1 + a + x) - (a + x)\ln(a + x) + (1 + a - x)\ln(1 + a - x) - (a - x)\ln(a - x). \qquad (14)$$

Differentiation with respect to x gives

$$\left.\begin{aligned} f'(x) &= \ln\frac{1+a+x}{a+x} - \ln\frac{1+a-x}{a-x} \\ f''(x) &= -\frac{2(a^2 + a - x^2)}{[(1+a)^2 - x^2](a^2 - x^2)} \end{aligned}\right\}. \qquad (15)$$

Equating $f'(x)$ to zero gives the range $-a < x < a$ in which $f''(x) < 0$. Thus $f(x)$ has only one maximum which lies at $x = 0$. Entropy proportional to x thus decreases with the difference $R_1 \sim R_2$. The shape of $f(x)$ will be drawn in Figure 4. With the decrease in entropy the temperature of the bundle of rays in absolute units increases as the energy remains constant.

Laue applied these results to a Michelson's interferometer (Figure 1) whose splitting plate P is totally unsilvered and is made out of a material having negligible absorption coefficients. This assumption makes the phase difference between the interfering rays independent of wavelength. The two side mirrors S_1 and S_2 are fully silvered and are perfect reflectors. Let a wave of *unit intensity* be incident on P. The reflected wave incident on S_1 will have intensity r, the transmitted wave incident on S_2 has intensity $1 - r$, where r is the coefficient of reflection of the surface of the plate P. The reflected wave is represented by dashed lines while the transmitted one by continuous. The mirror S_1 reflects the ray with the same intensity r which after transmission by P has the intensity $r(1 - r)$. Likewise the ray transmitted by P has the intensity $1 - r$ and is reflected from S_2 with the same intensity then partially reflected by P with intensity $(1 - r)r$ and transmitted with intensity $(1 - r)^2$. The pair marked (1) travels opposite to the incident wave with phase difference $2(\delta_r - \delta_t)$, δ_r is the phase change on reflection and δ_t is the phase change on transmission through the plate P. The pair marked (2) travels with phase difference zero, each wave having intensity $r(1 - r)$. They interfere to give a maximum of intensity $4r(1 - r)$. The remaining intensity $1 - 4r(1 - r)$ equals that of the pair (1).

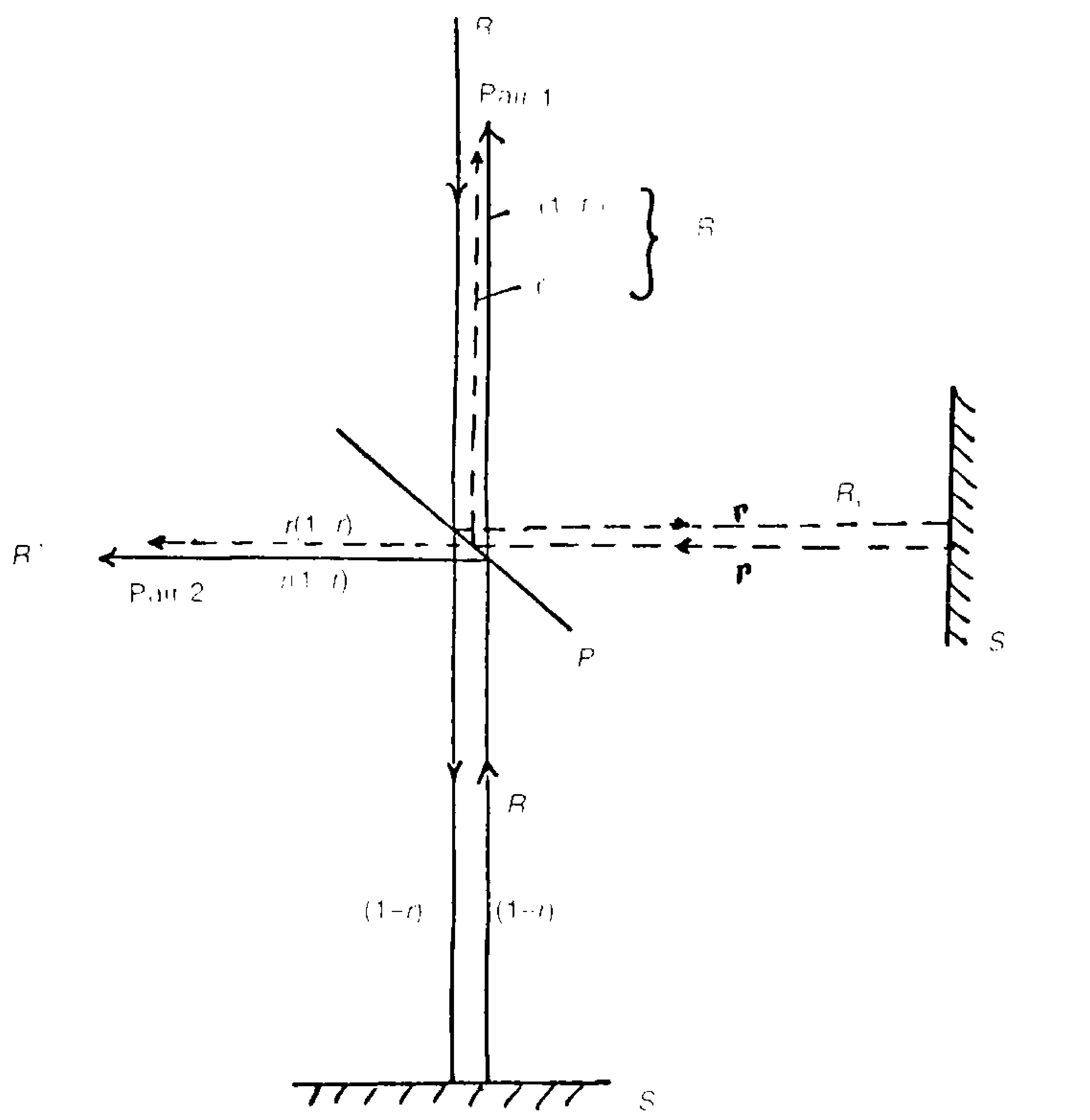

Figure 1. Laue's analysis of interference in Michelson's interferometer.

$$I_{(1)} = (1 - 2r)^2. \qquad (16)$$

Waves of the first pair have intensities r^2 and $(1 - r)^2$, i.e. the amplitudes are r and $1 - r$. The resultant on interference is $1 - 2r$. Their phase difference $2(\delta_r - \delta_t) = \pm\pi$. We now use indices (1) and (2) for first and second reflection and transmission then

$$\left.\begin{aligned} (\delta_r^{(1)} + \delta_t^{(2)}) - (\delta_t^{(1)} + \delta_r^{(2)}) &= 0 \\ (\delta_r^{(1)} + \delta_r^{(2)}) - (\delta_t^{(1)} + \delta_r^{(2)}) &= \pm\pi \end{aligned}\right\}. \qquad (17)$$

These equations hold true for all wavelengths. If the bundle incident at P has intensity R, the two rays that originate by reflection and transmission have intensities.

$$\left.\begin{aligned} R_1 &= rR \\ R_2 &= (1 - r)R \end{aligned}\right\}. \qquad (18)$$

These are reflected normally by S_1 and S_2, return to P and are transformed to

$$\left.\begin{aligned} R_1' &= 4r(1 - r)R \\ R_2' &= (1 - 2r)^2 R \end{aligned}\right\}. \qquad (19)$$

By energy conservation

$$R_1' + R_2' = R_1 + R_2. \qquad (20)$$

Energies of the final and initial states are equal. To see the change in entropy let us put $r < 1/2$ then by eq. (21) $R_2 < R_1$. If $r > 1/4$ by (18) and (19) $R_1' > R_2$ and by (20) $R_2' < R_1$.

$$|R_1' - R_2'| > |R_1 - R_2|. \qquad (21)$$

When $r > 1/2$, $R_1 > R_2$ and for $r < 1/4$, $R_1' > R_1$ and again (21) holds. Thus entropy decreases when

$$\frac{1}{4} < r < \frac{3}{4}, \qquad (22)$$

otherwise it always increases. In the above range of r, interference is a reversible process and is an exception to the general statement that in interference entropy increases.

Next Laue discussed the relation of *Perpetuum mobile* (perpetual motion) of the second kind to the thermodynamics of interference. The continual working of a machine that would create its own energy in violation of the first law of thermodynamics is *perpetuum mobile* of the first kind, while the continual operation of a machine which would work by utilizing energy of only one reservoir in violation with the second law is called *perpetuum mobile* of the second kind[13]. Laue considered two diathermic media (Figure 2) separated by a surface E. S_1, S_2, S_3 and S_4 are perfect black bodies. S_1 sends out rays that on reflection and refraction at f get absorbed by S_2 and S_3 respectively. Likewise rays sent by S_4 are absorbed by S_3 and S_2. The entire system is in thermal equilibrium.

If the rays from S_1 and S_2 interfere, then either S_2 receives more energy from f and S_3 less or vice versa. The thermal equilibrium is then disturbed. The bodies will be heated or cooled without any associated changes. This would violate the impossibility of perpetual motion of the second kind. Equilibrium is attained only by natural radiation in which coherence does not

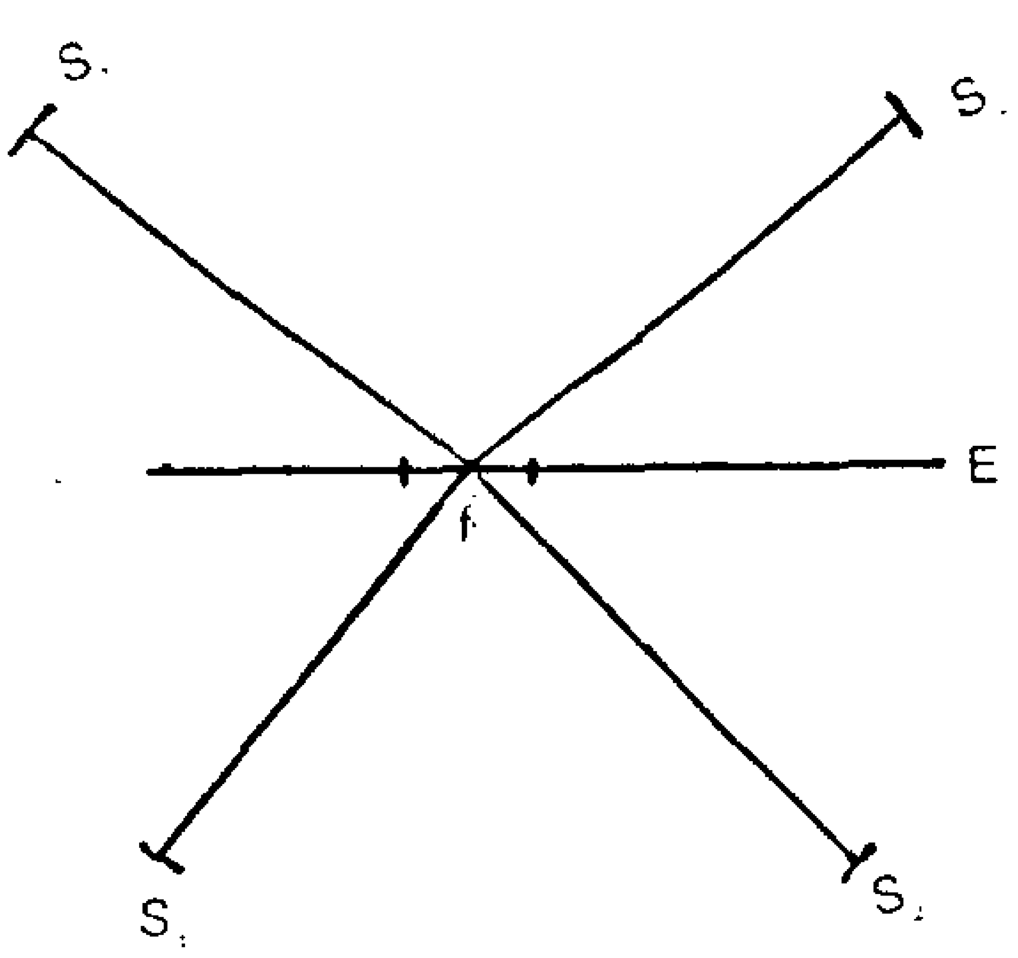

Figure 2. Impossibility of Perpetuum Mobile inside a cavity maintained at constant temperature.

exist and no interference can occur. This forms the basis of the validity of the second law.

Boltzmann's definition of entropy in terms of thermodynamic probability W is

$$S = k \log W. \qquad (23)$$

This explains all the above discussed facts. If a system consists of two subsystems with entropies S_1 and S_2,

$$S_1 = k \log W_1 \text{ and } S_2 = k \log W_2. \qquad (24)$$

By addition theorem, $W = W_1 . W_2$ which implies that the subsystems are totally independent of each other. In interference the two coherent beams are not independent of each other and the addition theorem is no longer applicable.

We are omitting Laue's discussion of certain paradoxes and pass on to his precise definition of partial coherence.

Partial coherence from interference thermodynamics

Michelson defined visibility of interference pattern by the ratio

$$V = (I_{\max} - I_{\min})/(I_{\max} + I_{\min}). \qquad (25)$$

$V = 1$ when $I_{\min} = 0$, i.e. when two coherent rays with equal intensities interfere. $V = 0$ when the rays are incoherent. For intermediate values of V, the interfering rays are said to be partially coherent. No measure of partial coherence was so far defined. There is no perfectly monochromatic source. Every spectral line has a line width. In a Michelson's interferometer if the path difference is increased, there comes a stage when interference ceases to occur. This is interpreted in terms of the line width of the radiation used.

Laue's starting point was the Fourier integral representation of two waves with frequencies ν and ν',

$$f(t) = \int \mathrm{d}\nu F_\nu \cos(2\pi\nu t - \varphi_\nu),$$
$$g(t) = \int \mathrm{d}\nu' G_{\nu'} \cos(2\pi\nu' - \gamma_\nu). \qquad (26)$$

If the sin function is used in place of cos the corresponding quantities are written as f^* and g^*. When the two waves (26) are superposed, their energy is the average

$$(\overline{f+g})^2 = f^2 + g^2 + 2\overline{fg}, \qquad (27)$$

where

$$\overline{fg} = \frac{1}{\tau}\int_t^{t+y} \mathrm{d}t \iint \mathrm{d}\nu \mathrm{d}\nu' F_\nu G_{\nu'}$$
$$\cos(2\pi\nu t - \varphi_\nu)\cos(2\pi\nu' t - \gamma_{\nu'}). \qquad (28)$$

Here τ is the shortest time interval needed for doing a measurement, it is large compared with the periods of oscillations. Integration with respect to time gives

$$\overline{fg} = \frac{1}{2}\iint \mathrm{d}\nu \mathrm{d}\nu' F_\nu G_{\nu'} \left\{\frac{\sin\pi(\nu' - \nu)y}{\pi(\nu' - \nu)}\right\}$$
$$\cos[\pi(\nu' - \nu)(2t + y) - (\gamma_{\nu'} - \varphi_\nu)]. \qquad (29)$$

The term with $\nu' + \nu$ is negligibly small and is neglected. Introducing variables

$$\lambda = \frac{1}{2}(\nu' - \nu), \quad \mu = \frac{1}{2}(\nu' - \nu), \qquad (30)$$

only a narrow strip about the line $\mu = 0$ comes into consideration where

$$\{\sin(2\pi\mu y)\}/2\pi\mu y = 1. \qquad (31)$$

Eq. (29) assumes the form

$$\overline{fg} = \frac{1}{4}\iint \mathrm{d}\lambda \mathrm{d}\mu F_{\lambda-\mu} G_{\lambda+\mu}$$
$$\cos[4\pi\mu t - (\gamma_{\lambda+\mu} - \varphi_{\lambda-\mu})]. \qquad (32)$$

Thus $\overline{fg} = 0$ for both $\nu = \nu'$ and $\nu \neq \nu'$. The phase difference $\gamma_{\lambda+\mu} - \varphi_{\lambda-\mu}$ fluctuates randomly and rapidly with time. For coherence we introduce a constant a by

$$\gamma_\nu - \varphi_\nu = a \qquad (33)$$

and relate

$$G_\nu = \rho F_\nu, \qquad (34)$$

ρ being another constant. The above two equations were used to write

$$g = \rho(f \cos a + f^* \sin a). \qquad (35)$$

Thus

$$\overline{fg} = \rho(\overline{f^2}\cos a + \overline{ff^*}\sin a). \qquad (36)$$

Putting

$$\left.\begin{aligned} g = f^* \text{ i.e. } \rho = 1,\ a = \gamma_\nu - \varphi_\nu = \pi/2 \\ \gamma_{\lambda+\mu} - \varphi_{\lambda-\mu} = \frac{\pi}{2} + (\varphi_{\lambda+\mu} - \varphi_{\lambda-\mu}) \end{aligned}\right\}. \quad (37)$$

Equation (32) due to integration symmetric about $\mu = 0$ gives

$$\overline{ff^*} = \frac{1}{4}\iint \mathrm{d}\lambda \mathrm{d}\mu F_{\lambda+\mu}F_{\lambda-\mu}$$
$$\sin\,[4\pi\mu t - (\varphi_{\lambda+\mu} - \varphi_{\lambda-\mu}) = 0. \quad (38)$$

Thus changing f to f^* implies replacing φ_ν by $\varphi_\nu + \pi/2$. In eq. (29) all functions change sign,

$$\left.\begin{aligned} &\text{a) } (f^*)^* = -f, \\ &\text{b) } \overline{(f+g^*)} = f^* + g^* \end{aligned}\right\}. \quad (39)$$

The following averages were derived:

$$\left.\begin{aligned} &\text{a) } \overline{fg} = \rho\cos a\,\bar{f}^2 \\ &\text{b) } \overline{fg^*} = -\rho\sin a\,\bar{f}^2 \\ &\text{c) } \overline{f^*g^*} = \overline{fg} \\ &\text{d) } \overline{f^{*2}} = \overline{f^2} \\ &\text{e) } \overline{f^*g} = -\overline{fg^*} \end{aligned}\right\}. \quad (40)$$

Rewriting (36) as a completely coherent function

$$g_f = \rho\,(f\cos a + f^*\sin a), \quad (41)$$

we say that g contains a coherent function f. Let g_f' be an incoherent function, then we have a partially coherent function,

$$g = g_f + g_f'. \quad (42)$$

Further,

$$\overline{g^2} = \overline{g_f^2} + \overline{g_f'^2}. \quad (43)$$

Equations (40a), (40b) and (40e) are still valid but $g \to g_f$. Equations (38), (41) and (40d) give

$$\begin{aligned} &\text{(a) } \overline{g_f^2} = \rho^2\bar{f}^2 \\ &\text{(b) } \overline{fg^2} = -\cos^2 a\,\overline{f^2}\,\overline{g_f^2} \\ &\text{(c) } \overline{f^*g^*} = \overline{fg^{*2}} = \sin^2 a\cdot\overline{f^2}\cdot\overline{g^2_f}. \end{aligned} \quad (44)$$

Adding (b) and (c) and rearranging,

$$\frac{\overline{fg^2} + \overline{fg^{*2}}}{\overline{f^2}\,\overline{g^2}} = \frac{\overline{g_f^2}}{\overline{g^2}} = i_{fg}. \quad (45)$$

i_{fg} is called coherence of f in g. The quantity

$$j_{fg} = 1 - i_{fg} \quad (46)$$

was defined as incoherence of f in g. The limiting cases $i_{fg} = 0$ and 1 correspond to absolute incoherence and complete coherence.

Entropy of two partially coherent bundles (Note 3)

If R is the intensity of radiation in vacuum, then in a medium of refractive index n it is R/n^2. This is called specific intensity defined as

$$k = R/n^2. \quad (47)$$

The entropy S of radiation is[14]

$$S = \sigma\,L\,(k) \quad (48)$$

where σ is a constant and in analogy with eq. (15),

$$L(k) = \frac{k\nu^2}{c^2}\left[\left(1 + \frac{c^2k}{h\nu^3}\right)\ln\left(1 + \frac{c^2k}{h\nu^3}\right) - \frac{c^2k}{h\nu^3}\ln\left(\frac{c^2k}{h\nu^3}\right)\right]. \quad (49)$$

Now consider two partially coherent radiation bundles φ and ψ in Figure 3 polarized in plane of incidence fall on the surface of separation of two dielectric media. If k_1 and k_2 be the specific intensities, total energy is $\sigma(k_1 + k_2) =$ constant and

$$k_1 + k_2 = I' \quad (50)$$

is an invariant. At the boundary there originate two new waves,

$$f = \delta\varphi,\ g = -\rho\varphi, \quad (51)$$

where the negative sign of ρ means a phase change of π in reflection. Since $\overline{f^2} + \overline{g^2} = \overline{\varphi^2}$, we have

$$\delta^2 + \rho^2 = 1. \quad (52)$$

If $\varphi = 0$

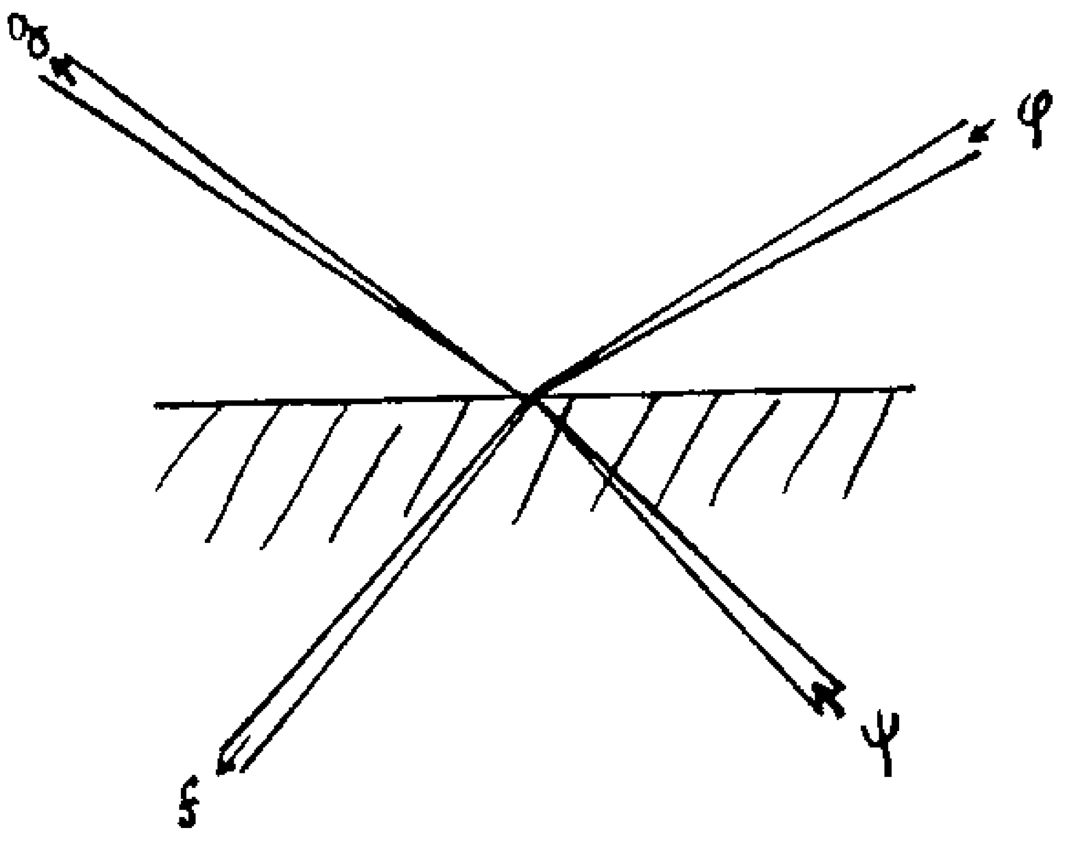

Figure 3. Reflection and refraction of waves at the surface separating two diathermic media.

$$f = \rho\psi,\quad g = \delta\psi. \quad (53)$$

Thus in general

$$\left.\begin{aligned} f = \delta\varphi + \rho\psi \\ g = -\rho\varphi + \delta\psi \end{aligned}\right\}. \quad (54)$$

Since the incoherence j satisfies the equation

$$\begin{aligned} jk_1k_2 &= (1 - i)k_1k_2 \\ &= \overline{f^2g^2} - (\overline{fg^2} + \overline{fg^{*2}}), \end{aligned}$$

using (52) Laue expressed $\overline{f^2}, \overline{g^2}, \overline{fg}$ and $\overline{fg^2}$ in terms of φ and ψ, showing that

$$\begin{aligned} &\overline{f^2}\,\overline{g^2} - (\overline{fg^2} + \overline{fg^{*2}}) \\ &= \overline{\varphi^2}\,\overline{\psi^2} - (\overline{\varphi\psi^2} - \overline{\varphi\psi^{*2}}), \end{aligned} \quad (55)$$

which implies that

$$jk_1k_2 = I'' \quad (56)$$

is a second invariant. For complete incoherence,

$$\left.\begin{aligned} &j = 1,\ k_1 + k_2 = I_1',\ k_1k_2 = I'' \\ &k_1 = \tfrac{1}{2}(I' + \sqrt{I'^2 - 4I''} \\ &K_2 = \tfrac{1}{2}(I' - \sqrt{I'^2 - 4I''} \end{aligned}\right\}. \quad (57)$$

Thus the entropy for two bundles is

$$\begin{aligned} S &= \sigma[L(k_1) + L(k_2)] \\ &= \sigma\left[L\left\{\tfrac{1}{2}I' + \sqrt{I'^2 - 4I''}\right\} + L\left\{\tfrac{1}{2}(I' - \sqrt{I'^2 - 4I''}\right\}\right], \end{aligned}$$

which for $j \neq 1$, becomes

$$S = a\Big[L\Big\{\tfrac{1}{2}(k_1 + k_2) + \sqrt{(k_1 + k_2)^2 - 4jk_1k_2}\Big\} + L\Big\{\tfrac{1}{2}(k_1 + k_2) - \sqrt{(k_1 + k_2)^2 - 4jk_1k_2}\Big\}\Big] \quad (58)$$

and

$$(k_1 + k_2)^2 - 4jk_1k_2 = (k_1 - k_2)^2 + 4ik_1k_2. \quad (59)$$

To discuss this result graphically, Laue put

$$\frac{c^3k_1}{h\nu^3} = a + x, \quad \frac{c^2k_2}{h\nu^3} = a - x \quad (60)$$

eqs (49) and (58) show that S is proportional to

$$\phi(x,i) = f(a + \sqrt{a^2 - (1-i)(a^2 - x^2)}) - f(a - \sqrt{a^2 - (1-i)(a^2 - x^2)}), \quad (61)$$

where f stands for the function

$$f(x) = (1 + x)\ln(1 + x) - x\ln x. \quad (62)$$

The quantity $a = c^2(k_1 + k_2)/h\nu^3$ remains unchanged in reflection and refraction and can be treated as a constant. Laue regarded

$$x = \frac{c^2(k_1 - k_2)}{2h\nu^3}, i, \phi(x,i)$$

as three rectangular coordinates and looked for surfaces that determine (61). Since x appears as x^2 in (61) it is symmetric about the plane $x = 0$. The domain of interest is defined by $0 \le i \le 1$ and $-a \le x \le a$.

For $i = 0$,

$$\phi = F(x + a) + f(a - x).$$

For $x = \pm a$, $\phi = f(2a)$, it rises with decreasing x, till $x = 0$, i.e. for equally intense radiation bundles $\phi = 2f(a)$. This is shown in the upper part of Figure 4. The quantity $\phi - f(2a)$ is in arbitrary units and x is plotted on abscissa. The lines $i = 1$, $x = \pm a$ and $\phi = f(2a)$ enclose entropy surfaces.

Laue calculated

$$\frac{\partial\phi}{\partial i} = \Big\{f'(a + \sqrt{a^2 - (1-i)(a^2 - x^2)}) - f'(a - \sqrt{a^2 - (1-i)(a^2 - x^2)})\Big\}$$

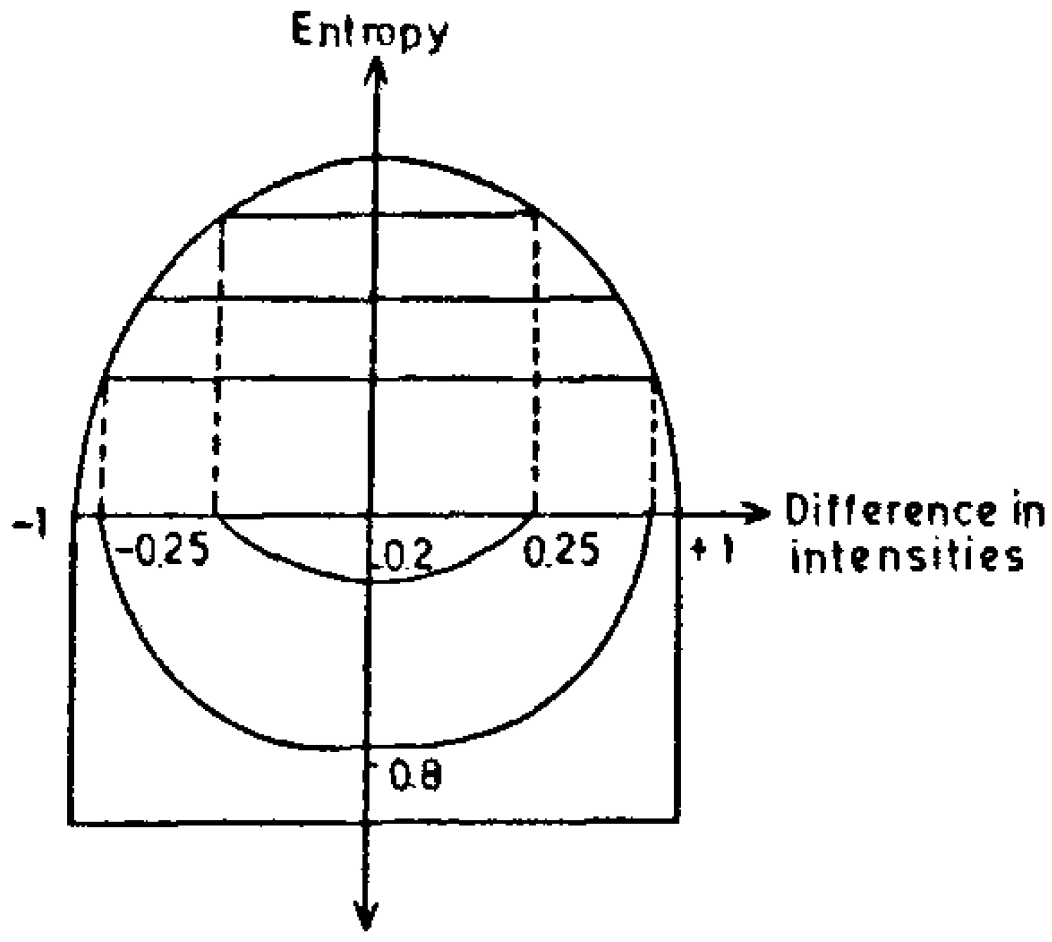

Figure 4. Upper part: variation of entropy in interference with the difference in intensities of two coherent beams. Lower part: surfaces of constant entropy.

$$\times \frac{a^2 - x^2}{\sqrt{a^2 - (1-i)(a^2 - x^2)}}$$

and since

$$f'(x) = \ln\left(1 + \frac{1}{x}\right)$$

decreases with increasing x, the physically meaningful range of curly brackets is negative but its multiplier is positive, thus $\partial\phi/\partial i < 0$ which holds at $x = 0$ and $i = 0$ where

$$\frac{\partial\phi}{\partial i} = \lim_{i\to 0} \frac{f'\{a(1+\sqrt{i}\} - f'\{a(1-\sqrt{i})\}}{\sqrt{i}} = 2af''(a) = \frac{-2}{1+a}$$

with increasing coherence; for constant intensities the entropy of bundles decreases. Along the boundary lines $x = \pm a$, $\partial\phi/\partial i = 0$ but since k_1 and k_2 here are both zero, coherence has no meaning.

The mean conclusions drawn by Laue were:

(1) The entropy surfaces are perpendicular to the three bounding lines. Curves of equal entropy are,

$$(1-i)(a^2 - x^2) = (a^2 - x_0^2), \quad (63)$$

where x_0 is a parameter of the curve. This is shown in the lower part of Figure 4. The differential equation of these curves is:

$$\frac{di}{dx} = -\frac{2x(1-i)}{a^2 - x^2}. \quad (64)$$

(2) At $x = \pm a$, $i = 1$, di/dx is indeterminate; there are singular points. The lines $x = \pm a$, $i = 1$ are lowest level lines. The highest level lines are defined by $x_0^2 = 0$.

(3) For all pairs of rays of equal energy, there are two incoherent radiation bundles with equal intensity having highest entropy; for all level lines, i has a maximum value x_0^2/a_2.

These curves of equal entropy show changes like the coherence of a ray pair in a simultaneous reflection and refraction. All the states which are represented by the points of the same curve can be transformed into each other by a suitable choice of reflection coefficient. For example, two incoherent rays are transformed into two partially coherent ones when their intensities are wholly or partially compensated. The maximum coherence is achieved by complete compensation. If the intensities of incoherent rays are K_1' and K_2' then

$$i_{max} = \frac{x_0^2}{a^2} = \left[\frac{K_1' - K_2'}{K_1' + K_2'}\right]^2. \quad (65)$$

Conversely, every pair of partially coherent ray bundles can be by simultaneous reflection and refraction, transformed into an incoherent pair when the difference between their intensities is increased. Exceptions are completely coherent and completely incoherent bundles.

The last part of this paper of Laue[14] discusses the partial coherence of three radiation bundles. This aspect will not be included here.

Importance of partial coherence in modern optics

Laue used a single frequency in his calculations. However monochromatic a source may be, it has a spectral distribution over a small frequency range. This indeed required a modification in Laue's theory. A rigorous treatment is given by Born and Wolf[15], but a lucid discussion is due to Matveev[16].

With a source having a spectral line with a given intensity distribution (Gaussian or Lorentzian) one calculates out the visibility factor (eq. (25)) in terms of certain integrals. It is found that for small path differences, visibility is better for Gaussian line shape and for larger path differences it is better for Lorentzian line.

Another consequence of the modern development is distinction between temporal and spatial coherence. Every analysis of interference involving division of amplitude uses temporal coherence which involves the spectral width of the frequency used. If the intensity maxima of $\lambda + d\lambda$ falls on the intensity minima of λ, the interference pattern becomes less distinct (more blurred). If θ is the angle between the rays and the interferometer axis,

$$2d \ln\theta = m(\lambda + d\lambda) = (m + 1/2)\lambda,$$
$$m\Delta\lambda = \lambda/2,$$

or

$$2d \ln\theta = \frac{\lambda(\lambda + \Delta\lambda)}{2\Delta\lambda} \approx \frac{\lambda^2}{2\Delta\lambda}$$
$$= \frac{\lambda\nu}{2\Delta\lambda} = cy_c,$$

or

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\nu}{\nu}, \ \Delta\nu\, y_c \approx 1.$$

y_c is called the coherence time and cy_c the coherence length. These considerations were applied to the Fabry–Renot interferometer by Saha[17]. When quadrupole radiation is used, visibility changes[18].

When the division of wavefront is involved, e.g. Frenel's biprism or mirror, the size of the source comes into picture. Spatial coherence is now used. The increase in the size of the source causes decrease in the visibility of the fringes. Different points in the source give rise to different set of fringes mutually displaced with respect to each other. The pattern gets blurred. Path difference between the rays coming from the extreme parts of the source must be less than $\lambda/2$. The best description of spatial coherence is given in terms of the stochastic theory. These details are reserved for a separate review article.

1. Wiedmann, *Ann. Phys.*, 1889, **XXXVII**, 177.
2. Wien, W., *Wied. Annalen*, 1894, **52**, 132.
3. Planck, M., *Berliner Berichte*, 1895, 289.
4. Planck, M., *Berliner Berichte*, 1896, 164.
5. Planck, M., *Berliner Berichte*, 1897, 57.
6. Planck, M., *Berliner Berichte*, 1899, 449.
7. Jeans, J. H., *Nature*, 1905, **72**, 293.
8. Planck, M., *Ann. Phys.*, 1900, **1**, 729.
9. Nigam, A. N., *Curr. Sci.*, 1994, **67**, 127 (see page 128).
10. Nigam, A. N., *Bull. IAPT*, 1993, **10**, 72.
11. Planck, M., *Ann. Phys.*, 1900, **1**, 69.
12. Von Laue, M., *Ann. Phys.*, 1906, **20**, 365.
13. Saha, M. N. and Srivastava, B. N., *A Treatise on Heat*, Indian Press, Allahabad and Calcutta, 1950, pp. 244, 292 and 306.
14. Von Laue, M., *Ann. Phys.*, 1907, **23**, 1.
15. Born, M. and Wolf, E., *Principles of Optics*, Pergamon Press, 4th Edition, Chapter, V, 1970.
16. Matveev, A. N., *Optics*, Mir Publishers, Moscow, 1985, Chapter 5.
17. Saha, M. N., *Phys. Rev.*, 1917, **10**, 782.
18. Wawilov, S. I., *Die Mikrostructure des Lichtes*, Academie Verlag, Berlin, 1954, Part II Chapter III, German Translation of the original Russian by Georg Schultz.

Notes

1. Equation (2) is the stage of bifurcation. If one proceeds according to Planck, one can arrive at his quantum formula of energy distribution – the Planck's law. On the other hand, if one follows Laue's method, one arrives at thermodynamics of interference and mathematical definition of partial coherence. Since the present article deals only with the thermodynamics of radiation, the latter aspect has been preferred over the former.
2. If the variation in visibility with the reflection coefficient r of the plate P of Michelson's interferometer be studied, it would be possible to test Laue's prediction. This can be done in two ways. Either the plate P be subjected to controlled partial silvering accurately varying r or by choosing different dielectric materials that have different reflectivities. To the best of the author's knowledge, such an experiment has never been reported.
3. Laue's derivation is quite lengthy. Only a summarized version highlighting the main points is given here.

ACKNOWLEDGEMENTS. I thank Miss Charlotte Allram of Zentralbibliothek für Physik in Vienna for sending the cost-free xerox copies of the original papers used in writing this article.

AMAR NATH NIGAM

8/20 Arya Nagar,
Kanpur 208 002, India