Resonance of Ramanujan's Mathematics. Vols. I & II. R. P. Agarwal. New Age International (P) Ltd. 4835/24, Darya Ganj, New Delhi 110 002, 1996. Price: Rs 450 for each volume.

This two-volume work concerns some of the results of Srinivasa Ramanujan (1887–1920), the renowned Indian mathematician, recorded in his famous *Notebooks*, which continue to be a perennial source of inspiration for generations of mathematicians. Ramanujan recorded 3254 results or theorems in three notebooks, from 1903 to 1914.

Volume I has five chapters. Chapter 1 concerns certain summation and asymptotic formulae for generalized (ordinary and basic) hypergeometric series. Chapter 2 is on Rogers–Ramanujan identities. Chapters 3–5 deal with certain definite integrals of interest to Ramanujan.

In Chapter 1, the author presents a major part of Ramanujan's systematic work on ordinary hypergeometric series, an independent rediscovery of the work of others – like the ${}_7F_6(1)$ summation formula of Dougall (1907) and particular cases of this sum, such as, Pfaff (1797) – Saalschütz (1890) theorem for the ${}_3F_2(1)$ and the Gauss and Kummer summation theorems for the ${}_2F_1(1)$ and the ${}_2F_1(-1)$, respectively, and Dixon's summation theorem for a well-poised ${}_3F_2(1)$. Ramanujan's entries in his *Notebooks* concerning the deep results on the asymptotic relations for finite number of terms of a variety of ordinary hypergeometric series, which center around the behaviour of the logarithmic solutions of the hypergeometric differential equation are presented.

Chapter 2 deals with the derivations of the celebrated Rogers–Ramanujan identities, their combinatorial interpretations, their generalizations and their applications in statistical mechanics. The identities were independently found by Rogers (1894) and Ramanujan (1913). There are several proofs of these identities now due to I. Schur (1917), G. N. Watson (1929); combinatorial interpretations due to P. A. MacMahon (1916) and analytic generalizations due to W. N. Bailey (1947, 1948, 1951) and L. J. Slater (1951, 1952). B. Gordon (1961) defined an infinite family of combinatorial identities generalizing the two Rogers–Ramanujan identities, paving the way for a whole host of combinatorial theorems associated with these identities. More recently, using only properties of arithmetic, A. Garsia and S. Milne (1981), converted a proof due to Schur into a combinatorial proof and discovered in the process a basic combinatorial fact, now named as the Garsia–Milne involution principle. As Hardy has stated: 'None of these proofs can be called both "simple" and "straightforward", since the simplest are essentially verifications and no doubt it would be unreasonable to expect a really easy proof.' These identities arise naturally in R. J. Baxter's solution of the two-dimensional Hard Hexagon Model in Statistical Mechanics (1980). This led G. E. Andrews, P. J. Forrester and R. J. Baxter to study the eight-vertex solid-on-solid (SOS) model and to further generalize Rogers–Ramanujan type identities. All these interesting mathematical developments have been very well described in this cogently written chapter.

The next three chapters of Volume I deal with some of the integrals studied by Ramanujan. Definite integrals associated with Fourier transforms and self-reciprocal functions given in Ramanujan's 'lost' *Notebook*, are the contents of Chapter 3. Agarwal has shown how some of Ramanujan's results on integrals in his *Notebooks* can be established with the help of other entries of his, *that every given entry in the Notebooks by him has a purpose, although sometimes a particular entry may look out of context at the first instance. ...We have advocated elsewhere also that it may be interesting many a times to try to prove 'unproven' entry of Ramanujan with the help of some other entry given by him earlier or later on.*

Chapter 4 is focused on Ramanujan's Master Theorem for the evaluation of definite integrals, viz.:

$$\int_0^\infty x^{n-1}F(x)\mathrm{d}x = \Gamma(n)\phi(-n),$$

where

$$F(x) = \sum_{k=0}^{\infty} \frac{\phi(k)(-x)^k}{k!}$$

in some neighbourhood of $x = 0$ and for n not necessarily a positive integer. The three quarterly reports submitted by Ramanujan to the Madras University, as a requirement stipulated by the University for awarding him the First Research Scholarship in Mathematics (even though he had no formal degree or qualification), contain various applications of this theorem to the *evaluation of a huge variety of integrals* and expansion formulae. The appropriate conditions of convergence for its validity have been provided by Hardy in his lectures contained in *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work* (Cambridge Univ. Press, 1940). The q-extension of the Master Theorem by Jackson (1951) and its extension to two variables by R. P. Agarwal (1974) are presented. The fundamental gamma and beta functions of mathematical analysis have been ingeniously extended by Ramanujan and these are related to the 'incomplete' gamma and beta functions (in which the upper limit of the integrals is finite and not ∞). The q-extensions of these works by several mathematicians including J. Thomae (1879), F. H. Jackson (1904), R. Askey (1978), Andrews and Askey (1981), A. Verma and V. K. Jain (1992) are presented.

Chapter 5 is devoted to the study of integrals of the type

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\mathrm{d}t$$

for particular values of a, b, c, d, evaluated by Ramanujan. Mordell (1933) classified these according to the values of the parameters and evaluated them in terms of Jacobi's theta and other related functions, using the method of complex contour integration. Agarwal presents the general method of evaluation of different standard forms of this integral and relates them to those studied by Ramanujan, who used transform calculus to evaluate his integrals. Integrals associated with Riemann's zeta functions and elliptic modular relations are not discussed, as pointed out by the author himself. The chapter concludes with the mention of Ramanujan's formal deduction of a definite integral with fractional derivatives, based on his Master theorem, which agrees with the classical definition of Liouville.

In Volume II, Agarwal presents a critical appraisal of Ramanujan's extensive and intriguing work on elliptic functions. Results pertain to theta functions, partial theta functions, 'mock' theta functions (discovered by Ramanujan during the last year of his life on his return to India), as well as Lambert series and their relationship with elliptic functions, mock theta functions and allied functions. While theta functions and partial theta functions are covered in chapter 1 of this volume and Lambert series and related functions are dealt with in chapter 5, the major part of this volume is concerned with the 'mock' theta function results of Ramanujan, presented in three extensive chapters (2, 3 and 4), contained in the 'lost' *Notebook*, discovered by George E. Andrews in 1976.

After giving the definitions of the four Jacobi theta functions and the 'false' theta functions (viz. theta functions with the 'wrong' signs for the terms) defined by L. J. Rogers, Agarwal introduces the Ramanujan notation for the theta function in terms of the symmetric function:

$$f(a,b) = \sum_{k=-\infty}^{\infty}(ab)^{k(k-1)/2}(a^k + b^k), |ab| < 1.$$

In terms of this symmetric function, Ramanujan defines:

$$f(qe^{2iz}, qe^{-2iz}) = \theta_3(z, q),$$

where

$$q = e^{i\pi\tau} \text{ and } |q| < 1.$$

Several general properties of the function $f(a, b)$ are then presented. Jacobi's triple product identity and other such identities due to D. Hickerson (1988) and G. E. Andrews and D. Hickerson (1991) are derived explicitly, due to their importance to simplify calculations in later chapters on 'mock' theta functions. A number of theta function expansions in the 'lost' *Notebook* pertain to 'partial' theta functions. Some of these are stated and G. E. Andrews (1981) derived a general basic hypergeometric identity and showed the results of Ramanujan as special cases of it.

Ramanujan defined four third order mock theta functions and to this set three more were added by Watson. Watson used the transformations of basic hypergeometric series for obtaining new definitions for all the seven functions. Agarwal himself has contributed significantly to the theory of mock theta functions by showing that most of the general identities of Andrews (referred above) belong to a very general class of basic hypergeometric transformations. His results are presented and those of Andrews deduced from them. Furthermore, Agarwal defines the third order mock theta functions in terms of $a\ {}_2\Phi_1$ basic hypergeometric series and deduces many of the mock theta function properties from the well-known properties of the ${}_2\Phi_1$. Ramanujan gave ten mock theta functions of order five, in two groups of five each, and three functions of order seven. He asserted that the members of each of the two groups of mock theta functions of order five are related amongst those belonging to the same group only, while the three mock theta functions of order seven are not related to each other. The works of Andrews, Agarwal and M. Gupta on the mock theta functions of order five are presented in detail. A. Gupta and Agarwal showed that mock theta functions of order five and seven are defined through basic hypergeometric series of type ${}_3\Phi_2$ and ${}_4\Phi_3$. They have also made attempts to develop the theory of the above types of basic hypergeometric series to obtain new transformations and definitions for certain mock theta functions of orders five and seven. It is pointed out that the absence of a general transformation theory for the ${}_3\Phi_2\ (a, b, c; e, f; z)$ series [except when the argument z is q (the base parameter itself) or ef/abc], is the main reason why there exists no general transformation theory for mock theta functions of order five.

Observations of Agarwal on the relationship between the mock theta functions and the basic hypergeometric series lead him to define: *the order of a mock theta function as* $(2r + 1)$ *if it is expressible in terms of a* ${}_{r+1}\Phi_r$ *series, on a single base* q^k, $k \leq r + 1$. *There may be in the definition of the mock theta function an additive term with* ${}_{r+1}\Phi_r$ *consisting of* θ*-products, which do not affect the order.* While this is an interesting observation by itself, it is simplistic, to say the least. For, mystery still surrounds the incomplete work of Ramanujan in this area, contributing to the belief that Ramanujan was working on a general theory of mock theta functions, whose preliminary results are the ones in the 'lost' *Notebook* of his and that he was snatched away by fate before he could reveal his grand scheme!

This two-volume work of Agarwal is a deep study of some of the work of Ramanujan on ordinary and basic hypergeometric series, on definite integrals and theta, partial theta and mock theta functions. It is very original in parts and the author has consciously tried to have a minimum overlap with the extensive five-part exposition in detail of all the results in the *Notebooks* of Ramanujan by Bruce C. Berndt. (The author's reference to these Parts I to V as the *Berndt Notebooks* is unfortunate, especially in view of the significance of the Ramanujan *Notebooks* and the 'lost' *Notebook*, which are the source materials for both Berndt and Agarwal.) This work supplements the comprehensive work of Berndt and presents, in as coherent a way as is possible, the work of Agarwal and his school of students and coworkers, in particular, in the proper perspective of the evolution of the subject dealt with.

Some of the notations are in every chapter and this could have been avoided. The absence of an index at the end of the book will prove to be a handicap.

It is a valuable supplement for the research scholar and for mathematicians who wish to understand the 'how' and 'why' of what Ramanujan did. This book is a must for all mathematics departmental libraries.

K. SRINIVASA RAO

The Institute of Mathematical Sciences, Chennai 600 113, India