Table 2. Effect of different concentrations of sodium bicarbonate on *in vivo* germination of *G. densiflorum*. Each set consists of 25 encapsulated PLBs and had 10 replicates

Concentration of sodium bicarbonate (mg l^{-1})	Concentration of bavastin (mg l^{-1})	Regeneration percentage ± SE
0	4	6.00 ± 0.30
5	4	12.00 ± 1.08
10	4	20.4** ± 1.69
15	4	24.2** ± 1.28
20	4	28.40** ± 1.04
25	4	22.6* ± 0.60
30	4	20.20** ± 0.74
40	4	18.6*** ± 0.78

Germination percentage followed by asterisks in each treatment within the same column is significantly different from control (artificial seeds without $NaHCO_3$), using Student's *t* test at *5% level; **1% level and ***0.1% level.

supplementation of the encapsulating matrix with suitable food preservative and fungicide. It was interesting to note that besides preventing desiccation, the food preservative sodium bicarbonate alone was effective in checking contamination. However, in the long run, presence of a fungicide appears to be crucial to resist contamination. Development of the protocol for artificial seed production in *G. densiflorum* suggests that the lengthy and empirical process of hardening could be avoided for transplantation of *in vitro*-grown plantlets from laboratory condition to natural conditions (Table 2). Moreover, the development of the protocol for the endangered orchid *G. densiflorum* may be an useful addition to the *in vivo* germination and regeneration of plantlets for storage and transplantation of precious and costly hybrid orchids as well as for conservation of endangered germplasms. The judicious and intelligent coupling of artificial seed technology with that of microcomputer in achieving automated encapsulation and regeneration of plantlets would tremendously increase the efficiency of encapsulation and production of homogeneous and high quality artificial seeds, and will thus revolutionize the current concept of commercial micropropagation method by the beginning of twenty-first century.

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Received 10 August 1998; revised accepted 29 January 1999

On the use of symbolic computations in geosciences

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Demonstration of Computer Algebra Systems (CAS) in general, with a special reference to earth sciences, is given. Mathematical manipulations for solving complicated geophysical problems are carried out on a computer by a symbolic computational system, Maple V. Such intelligent systems can be utilized to solve large size scientific and engineering problems, which will result in enormous savings of time and manual resources.

MATHEMATICAL modelling of physical systems is a powerful tool to solve scientific and engineering problems. Although, analytical solutions to any problem are much more accurate, but scientists frequently have to resort to numerical techniques to solve for realistic models since analytical solutions for such models require gigantic human efforts and are time consuming. Even for numerical methods, algebraic formulas have to be obtained before any numerical technique is used. Symbolic manipulations of such complicated mathematical expressions are considered to be a daunting task by many practising scientists. With the advent of symbolic computation packages, also known as Computer Algebra Systems (CAS), the burden on practising engineers and scientists have become easier in handling large algebraic formulations.

Following is an account of a couple of problems in geophysics, which are quite time consuming if solved manually. For this purpose, we have used a popular symbolic computation program, Maple V, developed at the University of Waterloo, Canada. This is one of the first attempts at demonstrating the ability of such algebraic manipulation programs as an aid to solve complex real-world problems. The resulting enormous time saving can be utilized in comprehension of the results.

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Hopefully, many of the unsolved scientific and engineering problems, which were not earlier attempted because of their huge nature, would be easier to solve now by utilizing the potential of the symbolic computational packages.

The concept of computer algebra systems (CAS) is not new. In fact, Newton[1] laid its foundation in *Arithmatica Universalis* where he systematically discussed the methods of manipulating universal mathematical expressions (i.e. formulas containing symbolic indeterminates), and algorithms for solving equations build around these expressions. But it came into practice only in 1953, when Kahrimanian[2] and Nolan[3] published their theses, separately, on *Analytical Differentiation on a Digital Computer*. After the development of LISP, several CAS were written in it. In late 60s and 70s, multipurpose algebraic packages were written for personal computers. In 1980, Geddes and Gonnet[4] developed **Maple** at the University of Waterloo, Canada, which was one of the most efficient programs developed for symbolic, numeric and graphical manipulations. The latest version is **Maple V** which has come a long way from its predecessors.

Maple V has already been used in solving real-world problems. Pratibha[5] and Corless *et al.*[6] demonstrated how, through efficient programming, **Maple V** can handle very large algebraic expressions by emulating a human mathematician. Pratibha and Jeffrey[7] studied the behaviour of electrorheological fluids by replacing the numerical solution by a more accurate series approximation. A large number of terms (upto order of 50) in the series expansion were handled and all the algebraic manipulations were carried out on a computer with the help of **Maple V**.

Systems like **Maple V** can be utilized in several areas of mathematics, for example differential and integral calculus, matrix manipulations, analysis of complex variables, calculations of eigen functions, roots of polynomials, solution of linear equations, etc. the list and the possibility of its applications in sciences and engineering are endless.

Earthscientists have now begun to realize that simple assumptions about earth's structure are not sufficient to interpret the data observed during the geophysical surveys. One has to resort to complicated earth models to explain the anomalies. Large complicated mathematical expressions result due to the model complexity. Sometimes it becomes very difficult to handle such large expressions manually and therefore one leaves the problem out of frustration.

Anticlines and synclines are some of the important geophysical models used in gravity data interpretation. These two models assume significance in the context of sedimentary basins and are important from the point of view of petroleum exploration. The variation in the value of the acceleration due to gravity, termed as gravity anomaly, is observed on the earth's surface, and interpreted to infer about the rock-types present beneath the surface and their structure (shape, extent etc.). The gravity anomaly observed at the surface largely depends on the structure and the variation in the density of the rocks present underneath. In sedimentary basins, it is often necessary to assume a continuous change in the density contrast as the depth changes. Analytical formulae can be readily derived for uniform density contrast. The problem is much more complex when one has to try for nonuniform density contrast, even for simple structures such as anticlines and synclines. Rao and Raju[8] provide an example of such a problem. They carried out gravity inversion of anticlinal and synclinal structures with nonuniform density contrast. In particular, they considered a hyperbolic increase of density with depth, represented as:

$$\Delta\rho(z) = \frac{\Delta\rho_0\beta^2}{(z+\beta)^2},$$

where $\Delta\rho(z)$ is the density contrast at depth z, $\Delta\rho_0$ is the density contrast at the earth's surface, and β is a constant with units of length. An anticlinal structure (Figure 1 *a*) with depths to the top, Z_1 and bottom, Z_2, and with an angle of inclination, α, would produce a gravity anomaly at the surface, which is to be evaluated by considering a cross-sectional area $du.dz$ in the body of the anticline and integrating its gravity effect between proper limits as given by Rao and Raju[8]:

$$\Delta g(x) = 2\gamma\Delta\rho_0\beta^2 \int_{z=Z_1}^{Z_2}\int_{u=-(z-Z_1)\cot(\alpha)}^{(z-Z_1)\cot(\alpha)} \frac{z}{(z+\beta)^2((u-x)^2+z^2)}\,du\,dz.$$

Similar integration results for the gravity effect of a synclinal structure (Figure 2 *a*) are:

$$\Delta g(x) = 2\gamma\Delta\rho_0\beta^2 \int_{z=Z_1}^{Z_2}\int_{u=(z-Z_2)\cot(\alpha)}^{-(z-Z_2)\cot(\alpha)} \frac{z}{(z+\beta)^2((u-x)^2+z^2)}\,du\,dz.$$

Analytical evaluation of these integrals and their derivatives with respect to the unknown parameters Z_1, Z_2, and α are required before an optimization scheme can be implemented to match the anomaly with the computed values. The evaluation of the integrals and the derivatives are quite time consuming and highly error prone, if evaluated manually. A more detailed discussion on the accuracy of these derivatives is submitted elsewhere[9].

Rao and Raju[8] have manually derived analytical expressions for the anomaly. The formulae derived have already been given by Rao and Raju[8]. The derivation of

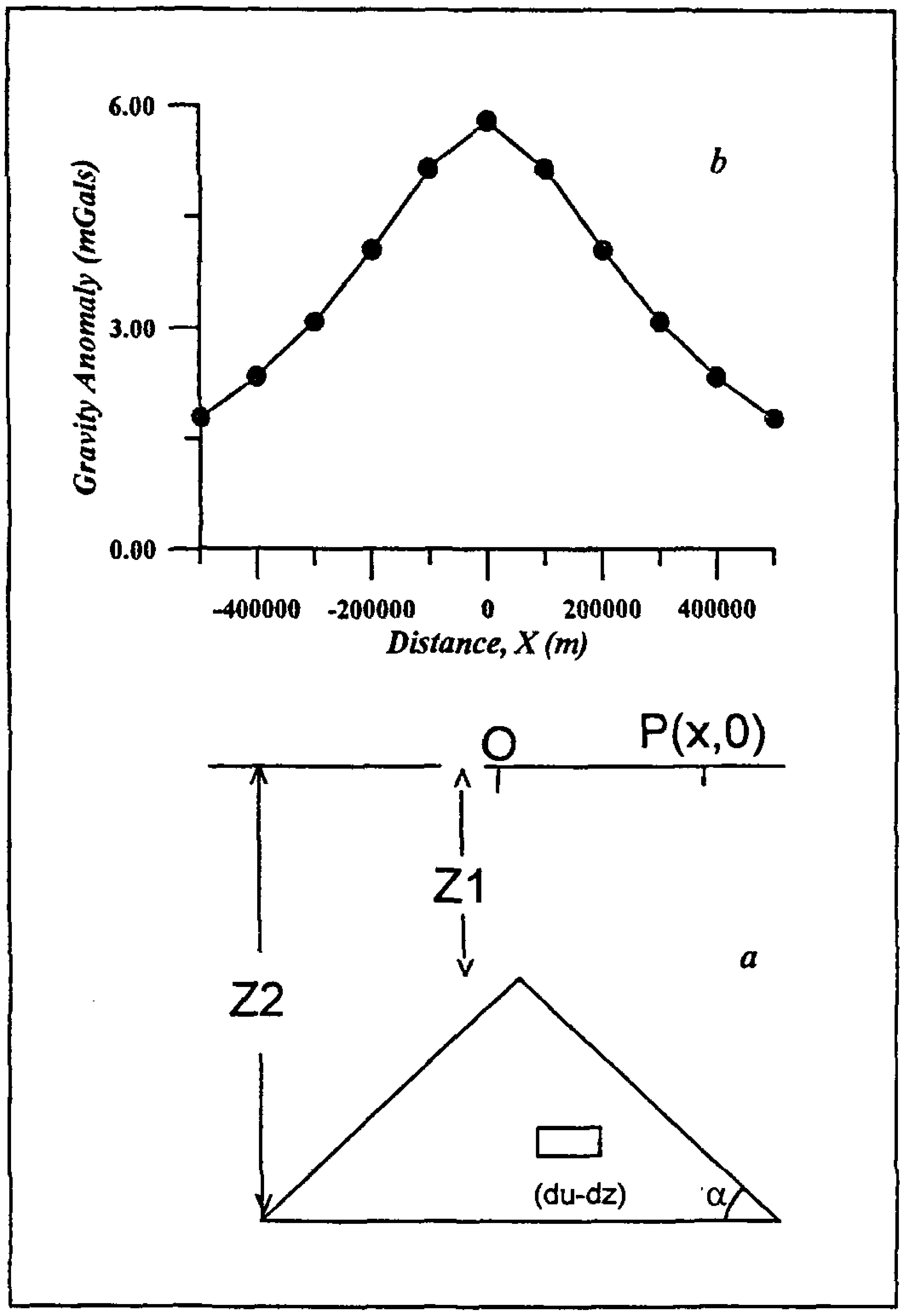

Figure 1. Gravity anomaly, computed using Maple V, over a symmetric anticlinal structure. Model parameters are chosen from Rao and Raju[8].

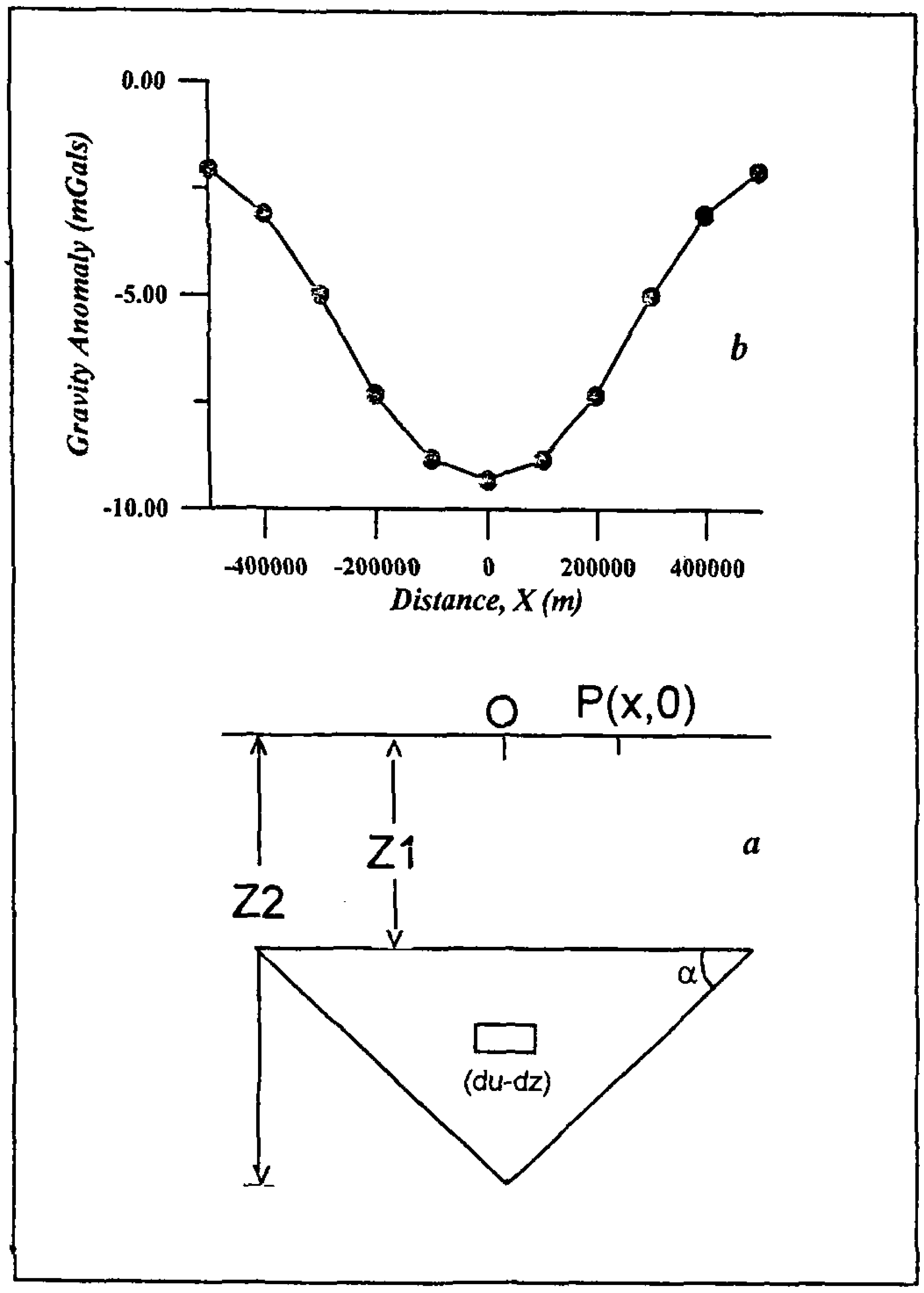

Figure 2. Gravity anomaly, computed using Maple V, over a symmetric synclinal structure. Model parameters are chosen from Rao and Raju[8].

these expressions must have taken a good deal of their time. We have derived these expressions with the help of **Maple V**. The derivation was achieved in a very short time (of the order of 100 s) using a small computer program. Apart from symbolic manipulations, **Maple V** has enormous numeric capabilities as well. The values of gravity anomaly for anticlinal model given in Rao and Raju[8] with parameters ($Z_1 = 1$ km, $Z_2 = 5$ km, and $\alpha = 45°$) have been computed using the expressions obtained by our program. The values of ρ_0 and β are assumed to be 2.5 g/cm^3 and 2.5 km, respectively. The gravity anomaly thus obtained is shown in Figure 1 *b*. The parameters chosen for the synclinal model are $Z_1 = 1$ km, $Z_2 = 6$ km, and $\alpha = 60°$, and the anomaly is shown in Figure 2 *b*.

The evaluation of the derivatives of the anomaly with respect to their parameters are also required in order to apply any inversion scheme. This evaluation can either be carried out numerically or analytically; the analytical expressions undoubtedly being more accurate than their numerical counterparts. Rao and Raju[8] have given some formulae for these derivatives which are not correct because the size of the problem makes these untractable manually, and therefore errorprone. Correct form of these derivatives have been easily derived with **Maple V** (ref. 9). Expression for one of the derivatives, $(\partial \Delta g/\partial Z_1)$, for the anticlinal model (Figure 1 *a*) is provided here for demonstration, which is:

$$\frac{\partial \Delta g}{\partial Z_1} = 2\gamma \Delta\rho_0 \beta^2 \left\{ \frac{Z_2 \cot\alpha}{(Z_2+\beta)R_2^2} + \frac{Z_2 \cot\alpha}{(Z_2+\beta)R_3^2} + \right.$$

$$\frac{A}{G_1}\left[\frac{Z_1}{R_1^2} - \frac{1}{Z_1+\beta} + \frac{F_1 \cot\alpha}{R_3^2}\right] +$$

$$\frac{B}{G_2}\left[\frac{Z_1}{R_1^2} - \frac{1}{Z_1+\beta} + \frac{F_2 \cot\alpha}{R_2^2}\right] +$$

$$\frac{H_1}{2G_1}\left[\frac{E_9}{E_1} - \frac{E_{10}}{E_2}\right] - \frac{H_2}{2G_2}\left[\frac{E_{11}}{E_3} - \frac{E_{12}}{E_4}\right] +$$

$$\left. T_4 \cdot T_{16} + T_5 \cdot T_{17} + T_8 \cdot T_{18} - T_9 \cdot T_{19} \right\},$$

where,

$$A = Z_1\cot\alpha - x, \qquad B = Z_1\cot\alpha + x,$$

$$G_1 = A^2 + 2A\beta\cot\alpha + \beta^2\operatorname{cosec}^2\alpha$$

$$G_2 = B^2 + 2B\beta\cot\alpha + \beta^2\operatorname{cosec}^2\alpha,$$

$$H_1 = 2\beta\operatorname{cosec}^2\alpha + 2A\cot\alpha,$$

$$H_2 = 2\beta\operatorname{cosec}^2\alpha + 2B\cot\alpha,$$

$$R_1 = (Z_1^2 + x^2)^{1/2}, \qquad R_2 = (Z_2^2 + F_2^2)^{1/2},$$

$$R_3 = (Z_2^2 + F_1^2)^{1/2}, \qquad P_1 = Z_2 + F_1\cot\alpha,$$

$$Q_1 = Z_1 + x\cot\alpha \qquad R_{11} = Z_1 - x\cot\alpha,$$

$$S_1 = Z_2 - F_2\cot\alpha, \qquad F_1 = x + (Z_2 - Z_1)\cot\alpha,$$

$$F_2 = x - (Z_2 - Z_1)\cot\alpha,$$

and

$$E_1 = A^2 + P_1^2, \qquad E_2 = A^2 + Q_1^2,$$

$$E_3 = B^2 + R_{11}^2, \qquad E_4 = B^2 + S_1^2,$$

$$E_9 = -A\cot^2\alpha - P_1\cot\alpha, \qquad E_{10} = A - Q_1\cot\alpha,$$

$$E_{11} = B - R_{11}\cot\alpha, \qquad E_{12} = -B\cot^2\alpha - S_1\cot\alpha,$$

$$T_4 = \ln(E_{13}) + \ln(E_{14}), \qquad T_5 = \ln(E_{13}) + \ln(E_{15}),$$

$$T_8 = \tan^{-1}(P_1/A) - \tan^{-1}(Q_1/A),$$

$$T_9 = \tan^{-1}(R_{11}/B) - \tan^{-1}(S_1/B),$$

$$T_{16} = (G_1\cot\alpha - AE_{16})/G_1^2,$$

$$T_{17} = (G_2\cot\alpha - BE_{17})/G_2^2,$$

$$T_{18} = (2G_1\cot^2\alpha - H_1E_{16})/2G_1^2,$$

$$T_{19} = (2G_2\cot^2\alpha - H_2E_{17})/2G_2^2,$$

with

$$E_{13} = (Z_2 + \beta)/(Z_1 + \beta), \quad E_{14} = R_1/R_3, \quad E_{15} = R_1/R_2,$$

$$E_{16} = 2A\cot\alpha + 2\beta\cot^2\alpha, \qquad E_{17} = 2B\cot\alpha + 2\beta\cot^2\alpha.$$

The handling of such large expressions through **Maple V** demonstrates that its use may not be limited to problems in earth sciences alone. All branches of sciences and engineering can utilize its potential equality.

Another example to establish **Maple V** as a powerful tool for scientific problems is chosen from earthquake seismology. A double couple source function is the most generalized static model of an earthquake source. Several years ago, one of us, Kamal[10], calculated the strong ground motion on a homogeneous earth for a vertical line double couple earthquake source, situated at depth. We have rederived the results with the help of **Maple V.**

In the following section is a brief statement of the problem, which is that a homogeneous, isotropic perfectly elastic medium occupies the region $-\infty \leq x \leq \infty$, $z \geq 0$. The xy plane is chosen as the surface of the medium. The positive z direction is along increasing depth. The elastic field is generated by a line source equivalent to a shear dislocation buried in the half space at depth h extending parallel to y-axis from $-\infty$ to ∞, so that the problem is two-dimensional. The compressional wave velocity and shear wave velocity in half space are α and β, respectively. The density of the medium is ρ. The system is causal, i.e. till time $t = 0$, the medium is at rest everywhere. The general form of the potentials (ϕ and ψ) in frequency-wavenumber domain, which satisfies the wave equations are:

$$\phi^- = a_1e^{(i\omega t - ikx + z\nu_p)} + a_2e^{(i\omega t - ikx - z\nu_p)}, \ z < h$$

$$\psi^- = b_1e^{(i\omega t - ikx + z\nu_s)} + b_2e^{(i\omega t - ikx - z\nu_s)}, \ z < h$$

$$\phi^+ = Ae^{(i\omega t - ikx + z\nu_s)}, \ z > h$$

$$\psi^+ = Be^{(i\omega t - ikx + z\nu_s)}, \ z > h.$$

where,

$$\nu_p = \sqrt{\left(k^2 - \frac{\omega^2}{\alpha^2}\right)}, \qquad \nu_s = \sqrt{\left(k^2 - \frac{\omega^2}{\beta^2}\right)},$$

and − and + superscripts refer to the quantifies above and below the fictitious boundary at $z = h$ respectively.

In order to find the *formal solution* to the problem, one has to solve a system of six linear equations to determine a_1, a_2, b_1, b_2, A and B. This is obtained by applying six boundary conditions. The surface of the medium is stress free, the tangential stresses (τ_{xz}) as well as the normal stresses (τ_{zz}) should vanish at the boundary. The source is introduced through stress discontinuity at $z = h$, but the displacement components u and w are continuous at $z = h$.

$$\tau_{zz}^-|_{z=0} = 0, \qquad \tau_{xz}^-|_{z=0} = 0.$$

$$\tau_{zz}^+ - \tau_{zz}^-|_{z=h} = -F_ze^{i\omega t}, \qquad \tau_{xz}^+ - \tau_{xz}^-|_{z=h} = -F_xe^{i\omega t}.$$

$$u^+ - u^-|_{z=h} = 0, \qquad w^+ - w^-|_{z=h} = 0.$$

This results in six linear equations of moderate size, because the earth model is simple enough. This was done manually by Kamal[10], which took him a few days to solve for the unknowns a_1, a_2, b_1, b_2, A and B and equally good time to check the results for their correctness. We programmed these equations in **Maple V** in terms of a_1, a_2, b_1, b_2, A and B by using the definitions of τ_{zz}, τ_{xz}, u and w. Then a one-line command in **Maple V** would solve the system of equations to give complicated expressions for the constants a_1, a_2, b_1, b_2, A and B. The entire exercise took only 55 s on a computer and there was no need to check it for accuracy. The solution obtained is given as:

$$a_1 = \frac{e^{-\nu_p h}}{2\rho\omega^2}\left[\frac{ik}{\nu_p}F_x - F_z\right], \quad b_1 = -\frac{e^{-\nu_s h}}{2\rho\omega^2}\left[F_x + \frac{ik}{\nu_s}F_z\right]$$

$$a_2 = \frac{1}{\rho\omega^2}\left[\frac{-2ik\left(2k^2 - \frac{\omega^2}{\beta^2}\right)e^{-\nu_p h}(F_x\nu_s + ikF_z)}{F_-} - \frac{F_+ e^{\nu_p h}(\nu_p F_z - ikF_x)}{2F_-\nu_p}\right]$$

$$b_2 = \frac{1}{\rho\omega^2}\left[\frac{2ik\left(2k^2 - \frac{\omega^2}{\beta^2}\right)e^{-\nu_p h}(F_z\nu_p - ikF_x)}{F_-} - \frac{F_+ e^{\nu_s h}(\nu_s F_x + ikF_x)}{2F_-\nu_s}\right]$$

where

$$F_- = 4k^2\nu_p\nu_s - \left(2k^2 - \frac{\omega^2}{\beta^2}\right)^2, \text{ and}$$

$$F_+ = 4k^2\nu_p\nu_s + \left(2k^2 - \frac{\omega^2}{\beta^2}\right)^2.$$

Expressions for A and B are not provided here because they are only useful when one is interested in the expressions at depth. For the surface seismograms ($z = 0$), above results will provide the potentials ϕ and ψ for a single line force. One can easily extend this to a solution for a double couple also, which is a more realistic earthquake source. The above treatment with the use of **Maple V** is very easy to handle and makes the life of a seismologist much easier.

Thus, Computer Algebra Systems, such as **Maple V**, are very useful in solving complicated scientific problems, if utilized properly. We have demonstrated its use in a couple of geophysical problems. However, these problems are otherwise also solvable, but involve a larger time and human effort.

Our first effort in applying the symbolic manipulations to geophysics should encourage scientists to explore other possibilities in this field as well. Scientists and engineers in other fields should also be encouraged to solve mathematical problems involving complex models. If the problems can be solved analytically (using the computer algebra systems), the results will be more accurate than their numerical counterparts and will provide fresh insight into the solutions.

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ACKNOWLEDGEMENTS. We thank the Council of Scientific and Industrial Research, New Delhi, for partial support during this study.

Received 26 October 1998; revised accepted 29 January 1999