On the future of Riemann Hypothesis*

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Riemann Hypothesis, Lindelöf Hypothesis, Density Hypothesis and their relations to distribution of prime numbers are discussed. Some achievements of G. Halász and P. Turán are also mentioned.

IN popular language, Riemann Hypothesis (RH) is the same as the convergence (for every $\sigma > \frac{1}{2}$) of

$$M(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}, (s = \sigma + it), \tag{1}$$

where $\mu(1) = 1$, $\mu(n) = 0$ if $n(> 1)$ is divisible by the square of a prime and otherwise $\mu(n) = \pm 1$ according as n has even or odd number of prime factors. The absolute convergence of the series eq. (1) in $\sigma > 1$ is a trivial fact. But it is already a non-trivial fact that eq. (1) converges on $\sigma = 1$. To prove that it converges for some s with $\sigma < 1$ (it is called quasi RH, i.e. qRH) is so hopeless that one may describe it as a problem which may take many centuries to be solved. However, it is not difficult to prove that it does not converge for any s with $\sigma \le \frac{1}{2}$ and this assertion comes out from the fact that there exists some pole say at $s = \beta + i\gamma$ with $\beta \ge \frac{1}{2}$ for the meromorphic continuation of $M(s)$ (of course to prove the existence of such a pole we need the functional equation called FE which we describe later in the article) to the whole plane, which is not difficult to establish. RH (or qRH) is thus equivalent to the convergence for real values of s (i.e. $t = 0$) for the relevant values of s. Thus RH (resp. qRH) is equivalent to

$$\sum_{n=1}^{N} \mu(n) = O(N^{\alpha+\varepsilon}), \text{ for every fixed } \varepsilon > 0, \tag{2}$$

for $\alpha = \frac{1}{2}$ (resp. some $\alpha < 1$). Actually the meromorphic continuation of $M(s)$ to the whole plane is easy to establish since in $\sigma > 1$ we have easily $M(s) = (\zeta(s))^{-1}$ where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \Pi_p (1 - p^{-s})^{-1},$$

the product being over all primes. Now in $\sigma > 0$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} \left(n^{-s} - \int_n^{n+1} \frac{du}{u^s} \right) + \frac{1}{s-1},$$

and repetition of this process arbitrary number of times shows that $\zeta(s) - \frac{1}{s-1}$ is an entire function. To prove that $\zeta(s) \ne 0$ for $\sigma > \alpha$ it suffices by an important famous theorem of Landau (on singularities of Dirichlet series (resp. integrals) with positive coefficients (resp. positive integrands)) to prove that (for some positive integer N_0) there holds,

$$\sum_{n=1}^{N} \mu(n) \ge -N^{\alpha}, \tag{3}$$

for all $N \ge N_0$. But the precise implications of eq. (3) when $\alpha = \alpha(N) < 1$ (say when $\alpha(N) = 1 - (\log N)^{-\eta}$, $0 < \eta < \frac{2}{5}$) are not known even today. If eq. (3) can be proved for some $\eta < \frac{2}{5}$ we can expect a non-trivial improvement of the Vinogradov zero-free region

$$(\sigma \ge 1 - \lambda (\log T)^{-\frac{2}{3}} (\log \log T)^{-\frac{1}{3}}, |t| \le T, T \ge T_0),$$

where $\lambda > 0$ is a certain absolute constant.

The precise implications of

$$\left| \sum_{n=1}^{N} \mu(n) \right| \le N^{\alpha} \text{ (valid for all } N \ge N_0) \tag{4}$$

were investigated by Ramachandra *et al.*[1] when $\alpha = \alpha(N)$ is close to 1. Some important investigations of eq. (3) when α is fixed and independent of N were carried out earlier by Balasubramanian and Ramachandra[2].

As has been remarked already it is not hard to continue $\zeta(s) - \frac{1}{s-1}$ as an entire function. Also it is a simple matter to prove

$$0 = \zeta(-2) = \zeta(-4) = \zeta(-6) = \dots. \tag{5}$$

But to prove that these are the only zeros of $\zeta(s)$ in $\sigma < 0$ we need the functional equation, namely

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(s) = \xi(1-s). \text{ (FE)}$$

Again, it is a simple matter (no FE is necessary) to prove that for all $T \ge T_0(\varepsilon, \delta)$ there exist $\ge T^{1-\varepsilon}$ zeros of $\zeta(s)$ in $\sigma \ge \frac{1}{2} - \delta$, $T \le t \le 2T$ (see Ramachandra[3,4]). But to deduce from this that there are zeros in $\sigma \ge \frac{1}{2}$ we need the FE. Hardy (see Titchmarsh[5]) showed that there are infinitely many zeros on $\sigma = \frac{1}{2}$, using the FE (Riemann had proved earlier that there exist some zeros on $\sigma = \frac{1}{2}$ using the FE (see ref. 5)). In fact Hardy and Littlewood showed (using FE) that there exists a zero $\rho = \frac{1}{2} + i\gamma$ with $T \le \gamma \le T + T^{\frac{1}{4}+\varepsilon}$ for all $T \ge T_0(\varepsilon)$. The constant $\frac{1}{4}$ was reduced to $\frac{1}{6}$ by Balasubramanian[6]. The latest, however, is $\frac{5}{32}$ due to Karatsuba (see ref. 7).

Let θ denote the least upper bound of the real parts of the zeros of $\zeta(s)$. Then it is not hard to prove that (see ref. 3)

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$$\tfrac{1}{2} \le \theta \le 1. \tag{6}$$

RH asserts that $\theta = \frac{1}{2}$. But the most pessimistic side of θ is that we are unable to decide whether the constant θ satisfies $\theta < 1$ or $\theta > \frac{1}{2}$. Vinogradov's zero-free region mentioned earlier is the deepest and the most precious result in the whole of the theory of $\zeta(s)$. Vinogradov deduced his zero-free region from his deep result which is as follows: For all $\delta (0 \le \delta < \frac{1}{100})$ and all $t \ge 10$ there holds

$$|\zeta(1-\delta+it)| \le (t^b \log t)^A, \ (b = \delta^{\frac{3}{2}}), \tag{7}$$

for some absolute constant $A > 0$ (The precise value of A is unimportant). Actually his method gives the upper bound $Bt^{Ab}(\log t)^{\frac{2}{3}}$, where $B > 0$ is an absolute constant (see ref. 5). In 1920s Vinogradov proved that the quantity $\zeta(1+it)(\log t)^{-\frac{2}{3}}$ *is bounded for* $t \ge 2$. *Even today (after* the lapse of nearly 80 years) there is no progress of the type

$$\zeta(1+it)(\log t)^{-\frac{2}{3}} \to 0. \tag{8}$$

This looks highly challenging (because qRH trivially implies that $|\zeta(1+it)| (\log \log t)^{-1}$ is bounded above for $t \ge 100$). Actually if in eq. (7) we can prove that $\frac{3}{2}$ can be replaced by a slightly higher value we cannot only prove eq. (8) but we can also widen the zero-free region of Vinogradov. All these look highly unlikely to be achieved even in the next few decades and perhaps even in the next century.

Let $\rho_n = \beta_n + i\gamma_n$ $(0 < \gamma_1 \le \gamma_2 \le \gamma_3 \le \cdots)$ be all the zeros of $\zeta(s)$ with $\beta_n \ge \frac{1}{2}$. Littlewood showed (see ref. 5) that

$$\gamma_{n+1} - \gamma_n = O((\log \log \log \gamma_n)^{-1}),$$

for all n such that $\gamma \ge 10000$. This was supplemented by the following result of Ramachandra and Sankaranarayanan[8]. Suppose $\zeta(s) \ne 0$ in $\sigma \ge \frac{1}{2} + \frac{c}{\log \log t}$, where $t \ge t_0$ and $c > 0$ is a certain absolute constant. Then for $\gamma_n \ge 10000$ we have

$$\gamma_{n+1} - \gamma_n = O((\log \log \gamma_n)^{-1}).$$

(Actually the two authors proved a slightly more general result of which this is a corollary.) One of the problems is to improve upon this bound for the gaps, assuming RH. Another problem is to prove that all the zeros of $\zeta(s)$ are simple.

I should make it clear that this is not a survey article. (We do not by any means give a complete list of references.) The main emphasis is on exposition of open problems. The interested readers are referred to Titchmarsh[5] or Ivic[7].

From RH it has been deduced that (due to J. E. Littlewood)

$$\zeta(\tfrac{1}{2}+it)t^{-\varepsilon} \to 0, \tag{9}$$

for every fixed $\varepsilon > 0$, as $t \to \infty$ (This is called Lindelöf Hypothesis). The best unconditional ε is any ε with $\varepsilon > \frac{17}{108}$, due to Huxley and Kolesnik[9].

Density Hypothesis (DH): It is eq. (10) (stated below) yet to be proved without assuming LH. Let $N(\sigma, T)$ be the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \ge \sigma (\ge 0)$ and $|\gamma| \le T$. Then Ingham proved for the first time that eq. (9) implies (see ref. 5) that for every fixed $\varepsilon > 0$ there holds

$$N(\sigma, T) \le T^{(2+\varepsilon)(1-\sigma)} (\log T)^{C_1}, (T \ge 10), \tag{10}$$

where $C_1 = C_1(\varepsilon) > 0$ is a constant. It may be mentioned that an interesting consequence of eq. (10) is that if p_n denotes the nth prime, then for all $n = 1, 2, 3, \ldots$ we have

$$p_{n+1} - p_n < C_2 \, p_n^{\frac{1}{2}+\varepsilon}, \tag{11}$$

where $\varepsilon > 0$ is any fixed real number and $C_2 = C_2(\varepsilon) > 0$ depends only on ε. The result $p_{n+1} - p_n < C_2 \, p_n^{1/2} \log p_n$ (where $C_2 > 0$ is an absolute constant) is a consequence of RH. However, it should be mentioned that $p_{n+1} - p_n < C_2 \, p_n^{1/2}$ is beyond the reach of RH at present. Also it should be mentioned that the best known unconditional ε in eq. (11) is $\frac{7}{200}$ according to a recent result of R. C. Baker and G. Harman.

Halász–Turán theorem: LH implies that for every $\varepsilon > 0$ and every $\delta > 0$ we have

$$N(\tfrac{3}{4}+\delta, T) < C_3 T^{\varepsilon}, \ (T \ge 10), \tag{12}$$

where $C_3 = C_3(\varepsilon) > 0$ depends only on ε and δ.

Halász and Turán thought (and even announced) that LH implies eq. (12) with $\frac{1}{2}$ in place of $\frac{3}{4}$. But they did not settle the problem and this is so far a difficult open problem.

I would state two problems which I call Turán's dreams.

(1) Prove Halász–Turán theorem with $\frac{3}{4}$ replaced by $\frac{1}{2}$, and (2) prove (unconditionally) that there exists an absolute constant $A_1 > 0$ for which there holds

$$N(1-\delta, T) < (T^{\delta^2} \log T)^{A_1}, \tag{13}$$

where $T \ge 10$ and $0 < \delta \le \frac{1}{100}$.

Actually Halász and Turán proved (unconditionally) using Vinogradov's deep result eq. (7) that

$$N(1-\delta, T) < (T^{\delta^{\frac{3}{2}}} \log T)^{A_2}, \tag{14}$$

where $0 < \delta < \frac{1}{100}$, $T \ge 10$ and $A_2 > 0$ is an absolute constant.

Inspired by Halász–Turán theorem there is a lot of unconditional progress on DH (and other problems mentioned here) notably by Montgomery who is the leader of this progress. Other contributors are Huxley, Jutila and Ramachandra and others (For all these results see Titchmarsh[5] or Ivic[7]).

Prove or disprove (unconditionally) that

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^6 \, dt < T^{\varepsilon}, \tag{15}$$

for every $\varepsilon > 0$ and all $T \geq T_0(\varepsilon)$.

Note that eq. (15) is a consequence of LH which is a consequence of RH. Trivially, given the truth of eq. (15) with some (positive) exponent in place of 6, we can deduce its truth for all lower (positive real) exponents. So we look for the highest real power in place of 6 known today. This value is 4, due to Hardy and Littlewood (I do not know any proof of their result which avoids FE). The proof eq. (15) with 2 in place of 6 is easy and does not need FE. The following deep result due to Heath-Brown is certainly worthy of mention here (see ref. 5 or ref. 7). There holds

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{12} \, dt < T^{1+\varepsilon}, \tag{16}$$

for every $\varepsilon > 0$ and all $T \geq T_0(\varepsilon)$.

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Role of *Bar* locus in development of legs and antenna in *Drosophila melanogaster*

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The X-linked *Bar* (*B*) mutation of *Drosophila melanogaster*, responsible for the well-known Bar eye phenotype due to over-expression of the BarH1 homeo-domain protein, is shown to enhance the abnormalities in legs and antennae of flies carrying a viable combination of certain *decapentaplegic* (*dpp*) loss of function mutant alleles. It is also shown that the homeo-domain carrying BarH1/BarH2 protein products of the *B* locus are expressed in a characteristic annular pattern in areas of normal larval leg and antennal discs that correspond to the distal regions of adult fly appendages. *dpp*-mutant background partly disrupts the expression pattern of Bar homeo-proteins in these discs and a combination of *B* and *dpp*-mutant alleles disrupts the Bar expression patterns in these imaginal discs much more severely. This is in agreement with the more severe phenotypes of legs and antennae of such flies. We suggest that the homeo-box containing *B* genes function as new members of the proximal distal sector genes and are important for patterning these appendages along their proximo-distal axes.

THE Bar eye mutant phenotype of *Drosophila melanogaster* is associated with a tandem duplication (*Bar* duplication) of the 16A1-7 region of the X chromosome[1], and is characterized by a drastically reduced number of ommatidia in the compound eyes of adult flies[2]. Organization of the *B* locus is complex since it harbours at least two homeo-box containing genes, the *BarH1* and *BarH2*, of which *BarH1* is reported to be over-expressed due to the *Bar* duplication[3,4]. The *decapentaplegic*, *dpp*, gene product is a member of the TGFβ family[5] and has very important roles in morphogenesis in many developmental pathways in *Drosophila*. The gene *dpp* is expressed in the eye discs of third-instar larvae of *Drosophila* in the anteriorly moving morphogenetic furrow and this is responsible for induction of differentiation of the precursor cells into ommatidia[6]. Over-expression of BarH1 homeoprotein in eye discs of *B* mutant larvae is associated with attenuation of *dpp* gene expression in the morphogenetic furrow[7]. As a result, ommatidial precursor cells fail to differentiate and instead, undergo apoptotic death. Consequently, the number of ommatidia in adult eyes of *B* mutant flies is substantially reduced[7]. All other adult structures are

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