A Concise Introduction to the Theory of Integration. Daniel W. Stroock. Birkhauser Verlag AG, P.O. Box 133, CH-4010, Basel, Switzerland. 1998. 3rd edn. pp. 272. Price sFr 50/DM 58.

Though the theory of integration is believed to have its roots in the 'method of exhaustion' known to ancient Greeks, it was the work of Newton and Leibnitz that enabled this method to grow into a *systematic tool* for calculating areas and volumes of geometric figures. Slowly this theory became more concerned with purely analytic questions as well, and its applicability extended to physics, differential equations, Fourier analysis and probability. The so-called classical theory of integration culminated in the 'Riemann integral'. In spite of its versatility the theory was not quite 'rounded'; for example, conditions under which limit and integral can be interchanged are fairly stringent; a function dominated by a Riemann integrable function is not necessarily Riemann integrable (in contract to the 'comparison test' for infinite series); and so on.

A more satisfying theory has grown since the pioneering work of Lebesgue about hundred years ago. This theory is more elegant, more complete with powerful convergence theorems.

In Riemann's approach, the interval $[a,b]$, over which integration is performed, is subdivided into a finite number of subintervals by partition $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$; and one takes Riemann sums of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}),$$

where $t_{k-1} \leq x_k \leq t_k$, as 'approximating sequence' to the Riemann integral of the function f. In Lebesgue's approach, the function f is approximated by 'simple functions' f_n taking only finitely many values and $[a,b]$ is subdivided into 'sets' where f_n takes constant values. Thus if f_n takes the values $a_1^{(n)}$, $a_2^{(n)}, \cdots a_{k_n}^{(n)}$ then the 'Lebesgue integral' of f is approximated by

$$\sum_{i=1}^{k_n} a_i^{(n)} \lambda(\{x \in [a,b]: f_n(x) = a_i^{(n)}\})$$

where λ (A) denotes the 'length of the set A'.

Both the approaches may be termed 'obvious' definitions; but chronologically Riemann's approach took precedence, because of obvious tie-up with the topology and order of the real line. Also, it is clear that Lebesgue's approach will involve defining 'length' or 'measure' of sets which are not necessarily intervals, and which can be quite 'wild'. However, thanks to fundamental contributions by Lebesgue, Borel, Caratheodory, Fatou, Fubini, and others an elegant theory has been developed subsuming Riemann integration. Ever since, its influence on functional analysis, partial differential equations, probability theory, harmonic analysis, etc. has been phenomenal. This approach has also spawned a variety of integrals in various contexts, like Bochner integral, Wiener integral, Ito integral, ...

But one might wonder, why should a book be written at the end of 20th century concerning a theory which was developed in the beginning of the century? A reason could be that there are not many books that present the theory of Lebesgue integration and the topics usually covered in an advanced calculus course (like surface area/measure, divergence theorem, etc.) in a unified fashion. It is not an uncommon experience (especially in India) that even those who have learnt the dominated convergence theorem, Radon–Nikodym theorem, etc. in all gory rigorous details in a course on measure theory have learnt divergence theorem, Green's identity, etc. in a semiprecise way in a course on advanced calculus.

This could explain the success of Stroock's remarkable book going into its third edition. Jacobi's result that 'the surface measure is in truth the derivative of the Lebesgue measure across the surface' has been nicely brought out in chapter V of the book under review, resulting in a rigorous treatment of divergence theorem. A technical (some might even say dry) subject like measure theory and integration is made attractive only if its applications to/connections with other branches of mathematics are illustrated. As the author is a distinguished probabilist/analyst who has made seminal contribution to the interface of probability theory with partial differential equations/harmonic analysis/functional analysis, flavour of all these subjects is brought out in the book, especially in chapters V–VII.

The first four chapters deal with basics of Lebesgue measure and Lebesgue integration (construction, Euclidean invariance, measurable functions, monotone and dominated convergence theorems, Fubini's theorem, ...) whereas the last chapter (chapter VIII) gives a smattering of the abstract theory (including Radon–Nikodym theorem, Riesz representation theorem, and existence of infinite family of independent random variables).

Some of the interesting aspects of the book are: Green's identity and its applications to Poisson equation and harmonic functions (chapter V); heat flow semigroup, one-sided stable laws of order 1/2, Poisson semigroup, etc. developed as exercises in the section on convolutions and approximate identities (chapter VI); proving Lebesgue differentiation theorem using Hardy–Littlewood maximal inequality (chapter III); L^2-theory of Fourier transforms in terms of Hermite functions, discussions on wave equation, and uncertainty principle (chapter VII). Topics which are normally in probabilists' domain like uniform integrability, Borel–Cantelli lemma, Dynkin's $\pi - \lambda$ system also find place in the book.

Chapter VII on Fourier analysis and a solution manual are the new features of the third edition; the latter should make the book less formidable.

For a diligent student the book can be highly rewarding, serving as a launching pad for an intensive study of any branch of analysis including probability theory. Of course, the '*diligent student*' need not be one specializing in mathematics; the author mentions that in his course at MIT (which resulted in the present book) students of electrical engineering and economics outnumbered those of mathematics.

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