On the Stroock–Varadhan theory of diffusions

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The Stroock–Varadhan theory of diffusion (to be more specific, refs 1–4) had earned for its authors the 1996 Steele Prize for seminal contribution to research. To get a better appreciation of the theory, let us recount briefly how probabilists viewed a diffusion process prior to the theory.

It is known that the phenomenon of d-dimensional diffusion can be described in terms of a Fokker–Planck equation:

$$\frac{\partial u}{\partial t}(t,z) = L_z^* u(t,z)$$

$$\equiv \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial z_i \partial z_j}(a_{ij}(t,z)u(t,z))$$

$$-\sum_{i=1}^{d} \frac{\partial}{\partial z_i}(b_i(t,z)u(t,z)), \qquad (1)$$

where $a_{ij}(\ldots)$, $b_i(\ldots)$ are respectively infinitesimal dispersion and drift parameters satisfying suitable nondegeneracy and regularity conditions. Let $(s, x; t, z) \mapsto p(s, x; t, z)$, $0 \le s < t$, $x, z \in \mathbb{R}^d$ denote the fundamental solution of the above equation; (that is, $u(t,z) = \int_{\mathbb{R}^d} f(x)\, p(s, x; t, z)\, dx$ solves the above Fokker–Planck equation for $t > s$, $z \in \mathbb{R}^d$ with initial value $u(s, z) = f(z)$ for any bounded continuous function f on $\mathbb{R}^d$). Then $p(s, x; t, z)\, dz$ can be interpreted as the probability of finding the diffusing particle in a dz neighbourhood of z at time t if the particle had started from x at time s. When $a_{ij}(\ldots) \equiv \delta_{ij}$, $b_k \equiv 0$ then the above Fokker–Planck equation is the heat (or diffusion) equation and the corresponding diffusion phenomenon is the standard Brownian motion (with variance parameter 1 in each coordinate) in a homogeneous isotropic medium. As $p(s, x; t, z)$ can be regarded as a 'transition probability density function' which is also well-behaved, using probabilistic techniques it can be shown that a well-defined continuous Markov process can be associated with $p(\cdot, \cdot; \cdot, \cdot)$; such a process is called a diffusion process corresponding to the Fokker–Planck equation (eq. (1)). This is the classical approach (also called the 'semigroup approach') to diffusion processes pioneered by Kolmogorov and Feller. However this depends heavily on the theory of partial differential equations for the existence of a fundamental solution p, which in turn depends on certain regularity of a_{ij}, b_i and nondegeneracy of $((a_{ij}(\ldots)))$.

In the meantime Paul Levy had suggested that a diffusion process could be represented as a stochastic differential equation:

$$dZ(t) = \sigma(t, Z(t))\, dB(t) + b(t, Z(t))\, dt,$$

that is,

$$dZ_i(t) = \sum_j \sigma_{ij}(t, Z(t))\, dB_j(t)$$

$$+ b_i(t, Z(t))\, dt, \qquad (2)$$

where $dB(t)$ can be taken as an 'infinitesimal increment' of a d-dimensional Brownian motion. The idea behind the above is: as Brownian motion represents diffusion in a homogeneous isotropic medium (without any external forces acting), a diffusion can be thought of 'locally' as having two components, one predominantly due to fluctuations governed by a d-dimensional Brownian motion with a 'covariance matrix' $\sigma(t, Z(t))\, \sigma^*(t, Z(t))$, and the other a 'drift' (may be due to external forces, temperature gradient, etc.) given by $b(t, Z(t))\, dt$. So $\sigma(\ldots)\, \sigma^*(\ldots)$ will be $a(\ldots) = ((a_{ij}(\ldots)))$ of eq. (1). The problem here is, of course, to give a meaning to the first term on the r.h.s. of eq. (2), as it is well known that the Brownian motion process has infinite variation over any finite interval. This was overcome in an ingenious way by Ito by defining stochastic integrals and interpreting eq. (2) as a stochastic integral equation

$$Z(t) = Z(0) + \int_0^t \sigma(s, Z(s))\, dB(s)$$

$$+ \int_0^t b(s, Z(s))\, ds. \qquad (3)$$

When the above equation (also referred to as a stochastic differential equation) has a unique solution, eq. (3) gives a 'pathwise' representation to a diffusion process and can be shown to be the continuous Markov process associated with the Fokker–Planck eq. (1), provided further that the fundamental solution exists. However this approach is severely inhibited by the requirement that $a(t, z) = ((a_{ij}(t, z)))$ have a Lipschitz continuous non-negative definite square root $\sigma(t, z) = ((\sigma_{ij}(t, z)))$.

In either approach it can be shown that for any nice function f,

$$f(t, Z(t)) - f(s, x)$$

$$-\int_s^t \left(\frac{\partial}{\partial r} + L_x\right) f(r, Z(r))\, dr, \qquad (4)$$

is a martingale, where L_x is the differential operator

$$L_x f(r, x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(r, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(r, x)$$

$$+ \sum_{i=1}^{d} b_i(r, x) \frac{\partial f}{\partial x_i}(r, x). \qquad (5)$$

(Martingale: Roughly speaking a process $\{\xi(t)\}$ is a martingale if $\xi(t) - \xi(s)$ and $\xi(r)$ are uncorrelated for any $r \le s \le t$; that is, the increments are uncorrelated and in particular $E(\xi(t)) = E(\xi(0))$ for all t; as many objects/concepts in probability theory, 'martingale' has its humble origins in gambling; however, martingale theory is now a very versatile tool for not only probabilists but also analysts.)

The above can also be put slightly differently. Let $\{Z(t): t \ge s\}$ be the diffusion process associated with eq. (1) or eq. (2) starting from x at time s. It will induce a probability measure $P_{s,x}$ on the function space $\Omega = C([0, \infty) : \mathbb{R}^d) = \{w: [0, \infty) \to \mathbb{R}^d : w \text{ continuous}\}$. Let $X(t, w) = w(t)$, $t \ge 0$, $w \in \Omega$ denote the t-th coordinate projection. Then eq. (4) can be rephrased as

$$f(t, X(t)) - f(s, x)$$

$$-\int_s^t \left(\frac{\partial}{\partial r} + L_x\right) f(r, X(r))\, dr$$

$$= P_{s,x} - \text{martingale}, \qquad (6)$$

for any $s \geq 0$, $x \in \mathbb{R}^d$ for any nice function f. Here note that

$$\left(\frac{\partial}{\partial t} + L_x\right) \text{ and } \left(-\frac{\partial}{\partial t} + L_x^*\right)$$

are formal adjoints of each other whenever the operators make sense. Note also that, for L_x to make sense one needs less stringent assumptions on a_{ij}, b_i. Under $P_{s,x}$ the process $\{X(t) : t \geq s\}$ has all the properties of a diffusion starting from (s, x) governed by eq. (1).

As $\{P_{s,x}: s \geq 0, x \in \mathbb{R}^d\}$ contains all the information about the diffusion, this family of probability measures is the focus in the Stroock–Varadhan theory. The question is: Does the above martingale property eq. (6) characterize $\{P_{s,x}\}$? In two fundamental papers Stroock and Varadhan[1] have answered the question in the affirmative; and, in fact, the martingale property eq. (6) determines uniquely a family $\{P_{s,x}\}$ of probability measures even if a_{ij}, b_i are just bounded and continuous (even these requirements can be relaxed in lower dimensions and with suitable growth conditions), and $((a_{ij}(\ldots)))$ nondegenerate. Moreover under $P_{s,x}$, the process $\{X(t) : t \geq s\}$ is a strong Markov process. The Stroock–Varadhan characterization of diffusions through a family of martingales is called 'Martingale problem' in the probabilistic literature. Anyone familiar with probability theory will recall that all the relevant information concerning a random variable/random characteristic under consideration is contained in its distribution, which is a probability measure on $\mathbb{R}$ or $\mathbb{R}^d$. The point of view here is similar, viz. the family $\{P_{s,x}\}$ of probability measures on the function space Ω are the relevant objects. And the Stroock–Varadhan characterization is akin to the characterization of probability measures on Euclidean spaces by their Fourier/Laplace transforms (i.e. expectations of certain test functions).

The Stroock–Varadhan theory enlarges in a substantial manner the class of coefficients a_{ij}, b_i for which diffusion process can be defined in a meaningful way; for example the class of differentiable or Lipschitz continuous functions forms a 'meagre' set in the class of continuous functions. However, the Stroock–Varadhan theory has provided not just a technical improvement, but has resulted in a conceptual change in the way Markov processes are viewed upon.

The martingale problem has now become a standard technique for characterizing many Markov processes (not just diffusion processes); see the monograph of Ethier and Kurtz[5] for a detailed discussion of this aspect. As a consequence, martingale problem provides a very useful way of establishing convergence of many Markov chains and processes to diffusion-type processes. A detailed account of this aspect is given in Ethier and Kurtz[5], and, of course, in Stroock and Varadhan[4].

Traditionally probability theory has borrowed heavily from PDE theory; the advent of the martingale problem, however, has marked a turning point. Since then probabilistic methods have become an important tool for studying PDEs. In two other important papers, Stroock and Varadhan[2,6] have applied these ideas to obtain their famous Stroock–Varadhan support theorem (characterizing the support of the measures $\{P_{s,x}\}$ even when the diffusion coefficients are degenerate), strong maximum principles for the corresponding differential operators, and studied Dirichlet problems for degenerate parabolic and elliptic operators. Freidlin[7], Hsu[8] and others have used the probabilistic notion of solution to study various boundary value problems.

The martingale problem has also given rise to the concept of the so-called 'weak solutions' of stochastic differential equations: this has become quite indispensable in applied fields like control theory and filtering theory. A nice account is given in the monograph of Karatzas and Shreve[9].

Diffusion processes with reflecting boundary conditions have similarly been characterized in terms of the 'submartingale problem' by Stroock and Varadhan[3] in the case of smooth domains, and later by Varadhan and Williams[10] in the context of reflecting Brownian motion in a wedge; (in the latter case there is an interesting phenomenon: the corner of the wedge can become an absorbing boundary under certain conditions!), see Ramasubramanian[11] for an exposition.

1. Stroock, D. W. and Varadhan, S. R. S., *Commun. Pure Appl. Math.*, 1969, **22**, 345–400, 479–530.
2. Stroock, D. W. and Varadhan, S. R. S., Proc. Sixth Berkeley Symp. Math. Stat. Probability, 1970, vol. III, pp. 333–360.
3. Stroock, D. W. and Varadhan, S. R. S., *Commun. Pure Appl. Math.*, 1971, **24**, 147–225.
4. Stroock, D. W. and Varadhan, S. R. S., *Multidimensional Diffusion Processes*, Springer–Verlag, New York, 1979.
5. Ethier, S. N. and Kurtz, T. G., *Markov Processes: Characterization and Convergence*, Wiley, New York, 1986.
6. Stroock, D. W. and Varadhan, S. R. S., *Commun. Pure Appl. Math.*, 1972, **25**, 651–714.
7. Freidlin, M., *Functional Integration and Partial Differential Equations*, Princeton University Press, 1985.
8. Hsu, P., *Commun. Pure Appl. Math.*, 1985, **38**, 445–472.
9. Karatzas, I. and Shreve, S. E., *Brownian Motion and Stochastic Calculus*, Springer, New York, 1991.
10. Varadhan, S. R. S. and Williams, R. J., *Commun. Pure Appl. Math.*, 1985, **38**, 405–443.
11. Ramasubramanian, S., *Curr. Sci.*, 1995, **69**, 231–236.

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