Number – From Ahmes to Cantor. Midhat Gazalé. Princeton University Press, 41 William Street, Princeton, New Jersey 08540, USA. 2000. 297 pp. $29.95/£18.95.

In the preface the author writes that as a student he often asked himself ‘fundamental questions about the notion of number, whose answers were not provided within the confines of the engineering curriculum’; questions about number systems, the possibility of non-uniform number systems, irrational numbers, infinity, and so on. This spurred him to write a book on these topics, aimed at similarly placed students, which would present the main ideas in an historical context and show the effort that led to their elaboration. The author quotes at the start (to set the tone, no doubt) one of those pithy maxims for which the French are famous: *That which is well-conceived is clearly stated*, and adds that he has adopted an approach using the ‘simplest possible language’. My reading suggests, however, that this noble aim may not quite have been met.

Chapter 1 is historical in nature; material is presented relating to the Mayan and Mesopotamian positional number systems, the Hindu–Arabic decimal system, the binary system, ... The facts would be familiar to most readers, but the exposition is good, with lots of nice quotes thrown in; the reader gets a good sense of the history of positional number systems.

In Chapter 2 the author discusses non-uniform positional systems. The chapter is dense in symbolism and elaborate in detail. Lots of results are presented, but there is no clear gain in the end. For instance, there is no clear statement about the point of a mixed-base positional number system (the preface states that it helps in the understanding of fractals). Though the material is interesting, its significance is not brought out clearly. Chapter 3 introduces number theory; once again there are symbols in good supply! Pascal’s divisibility test for mixed positional systems is described. (The author is obviously an electrical engineer; relay devices keep appearing all through the book.) Chapter 4, about the real numbers, discusses the axiomatic construction of the integers and the rationals, the Pythagoras theorem, Pythagorean triples (the ‘Plimton 322’ historical connection), the notion of irrationality, and so on. There is a discussion on the integral solutions of the equation $x^2 - 2y^2 = 1$, a nice proof (given in base 3) of the irrationality of $\sqrt{2}$, then a discussion on Dedekind cuts; finally, a proof of the irrationality of *e*. The chapter contains standard stuff, but it is well presented. Chapter 5, on simple continued fractions, is short and succinct; the main results needed for Chapter 6 are presented here.

In Chapter 6, which is one of the most interesting, we see a nice discussion of a striking property possessed by any irrational number: its partitioning of the lattice points of the coordinate plane, with a cleavage line dividing them. Yet again one sees a profusion of symbols, but the material holds intrinsic interest, so we can afford to be indulgent. The language is graphic; e.g. *Imagine that the number lattice is shaken by a powerful earthquake that cleaves its nodes into two distinct sets.* (The author actually uses ‘two distinct compact sets’, which is unfortunate, as it conflicts with standard mathematical usage.) For the number $\sqrt{2}$, the sets in question are $\{(x, y) \in \mathbf{N}^2\colon\ y^2/2x^2 < 1\}$ and $\{(x, y) \in \mathbf{N}^2\colon\ y^2/2x^2 > 1\}$. Each set is bounded by an irregular staircase-shaped border, and the author calls the lower border the *cleavage line* of $\sqrt{2}$. This idea allows for a Dedekind-cut type definition of the irrationals. For a given number $\mu > 0$, let $h_\mu(x) = [\mu x]$; then $h_\mu(x)/x$ is the *lower slope* of μ at x and $(h_\mu(x) + 1)/x$ is the *upper slope* of μ at x. The integer pair $(x, h_\mu(x))$ is said to *belong* to μ. Two pairs (b, a) and (d, c) in $\mathbf{N}^2$ are said to be *coherent* if they both belong to some number. (Example: (5, 7) and (17, 23) both belong to $\sqrt{2}$ and are therefore coherent.) The author shows that (b, a) and (d, c) are coherent iff $(a + 1)/b > c/d$ and $(c + 1)/d > a/b$. Other related results are presented. As said earlier, this chapter has a lot of very interesting material.

Finally, Chapter 7 is about the notion of infinity. The famous paradoxes associated with the names of Zeno, Hilbert, etc. are discussed; also well-known material like Cantor’s proof relating to the power of the continuum. Once again the symbols threaten to overwhelm! Here is an example. In Cantor’s diagonalization proof, we assume – in order to set up the contradiction – that the real numbers may be placed in a sequence $r_0, r_1, r_3, \ldots$; then we write the numbers in base-10, say $r_0 = .a_0b_0c_0d_0 \ldots$, $r_1 = .a_1b_1c_1d_1\ldots$, and so on. This is the symbolism that most of us would use. Not so the author!; he plunges ahead with $r_0 = .\delta^{r_0}_{-1}\delta^{r_0}_{-2}\delta^{r_0}_{-3}\delta^{r_0}_{-4}\ldots$, $r_1 = .\delta^{r_1}_{-1}\delta^{r_1}_{-2}\delta^{r_1}_{-3}\delta^{r_1}_{-4}\ldots$, and so on. Whew! No essential loss; only, the symbols tend to obscure rather than help. However, this apart, the chapter is well written, and the reader comes away with a good sense of the counter-intuitive nature of infinity; e.g. the debate in earlier times on potential versus actual infinity.

I have mixed feelings about the book. There are nuggets to be found in good supply, including some great quotes (‘It takes a scientist of established reputation to dare the axiomatization of the obvious...’; this is the author’s, but he also gives a number of insightful quotations from well-known mathematicians and philosophers). The ‘mixed’ part concerns the writing. It is surely any author’s duty to present his ideas in the simplest language possible. Symbolism should be transparent, kept to a minimum. In this regard I feel that the author’s writing has been less transparent than need be, principally because of his usage of symbols. His choice of topics is excellent, but his language could, I feel, have been more reader friendly.

The following remarks too could be kept in mind. In some places the word ‘evanescent’ is used, with no definition; page 93 has: ‘a fractional number is the sum of the terms of a convergent infinite series, with each term finite, though evanescent.’ A definition does turn up later – but on page 260! On page 130, he defines the *conjugate* of a number relative to a given base, then makes no further reference to the concept anywhere in the book; not much point served by the definition! On page 164 he writes that the logical union of the Pythagoras theorem and its converse is ‘more exact’ than the theorem itself. Stronger, yes; but more exact? On page 168, he describes (119, 120, 169) as the ‘second’ Pythagorean triple in which the smaller numbers differ by 1, the first being (3, 4, 5); he appears to have missed (20, 21, 29). On page 170, he deals with ‘Pell’s equation’, now known to be incorrectly attributed to Pell. As the author has throughout been particular about inclusion of historical detail, he could have said more about the work by Bhāskarā; and similarly for the series for $\pi/4$ due to Mādhavā, Gregory and Leibnitz.

Is a verdict needed? Then, here it is: the book has a lot of excellent stuff–but take a deep breath before plunging in!

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