Chaotic and complex systems

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We discuss the concepts of chaotic and complex systems, much in vogue these days, due to the potential of a wide spectrum of applications to natural phenomena in terms of mathematical modelling and a better prediction of future behaviour. Actually, the theory of chaotic systems is in an increasingly more solid basis and we can now present a possible global and universal scenario for them, in terms of conjectures that have been partially fulfilled. The situation concerning complex systems is much more delicate. They seem to be dynamical systems that are on the border of the chaotic systems, to depend on a large number of parameters and to be in constant evolution. Some of the possible features of complex system are presented, as well as some of the basic references on applications.

THE theory of chaotic systems is now quite well-developed and at this point we can even offer a global scenario for it: uncertainty of the long-range behaviour of solutions of the systems is very common, which does not mean total uncertainty. This is often quoted in rather non-scientific terms as 'there is order in chaos'.

The concept of a chaotic system and the fact that this is commonly the rule, and not the exception, for nonlinear and non-gradient deterministic systems came as a big surprise just about 25 years ago. Actually, the famous Lorenz[1] example dates back to 1963, but became well known to mathematicians working in dynamics only in 1975.

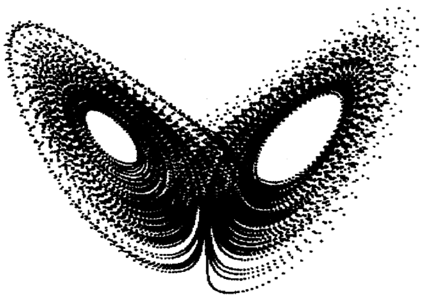
Due to the wide success of the concept of chaos, there was for some time an animated controversy: To whom should we attribute the original idea? In fact, uncertainty and randomness are more than a century old concepts in science and can be traced back to the great mathematicians and physicists Maxwell[2], Boltzmann[3], Poincaré[4], during the second half of the 19th and early 20th centuries. So, should the idea of chaos be attributed to one of these previous scientific giants or, jumping over to much more recent times, to Smale[5], the inventor of the 'horseshoe map', who was also involved in developing the so-called hyperbolic theory in dynamics in the sixties, or to Lorenz[1], with his famous 1963 *butterfly*-sensitive attractor, as briefly described in the sequel, or to others? To avoid such a controversy, perhaps we can honour Edgar Alan Poe, who prior to all of them, wrote the following beautiful paragraph in a short-story published in 1850:

> 'For, in respect to the latter branch of the supposition, it should be considered that the most trifling variation in the facts of the two cases might give rise to the most important miscalculations, by diverting thoroughly the two courses of events; very much as in arithmetic an error which, in its own individuality, may be inappreciable, produces, at length, by dint of multiplication at all points, a result enormously at variance with truth.'
>
> *The Mystery of Marie Roget*
> A Sequel to *The Murder in the Rue Morgue*
> by Edgar Alan Poe (1850)

Lorenz attractor – A paradigm for chaotic behaviour

Lorenz exhibited a relatively simple flow in 3-dimensions with quadratic equations, and consequently a deterministic system, whose future behaviour of solutions (positive limit set) blends together a singularity of the flow (rest point) with periodic motions, actually infinitely many of them (Figure 1):

The butterfly corresponds to the limit points of all future orbits starting near the rest point (0, 0, 0): it is an attractor since all nearby orbits will tend to it in the future. Often we call the points 'events' and the space, 'space of events': the butterfly and the flow on it represents how the flow behaves after a long period of time, that is, in its 'horizon'. In general, the key question in dynamics is the understanding of its future or past horizon, when we can invert the direction of time.



$$\dot{x} = -10x + 10y$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = -\frac{8}{3}y + xy$$

Figure 1. Lorenz attractor. A model for weather prediction.

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Moreover, Lorenz attractor is robust: it persists even when we change slightly the coefficients of the quadratic equations of the system. It is also sensitive with respect to the initial conditions. When we consider two very nearby points (also called events), their orbits after a long time in the future may be very distant apart, as much as the diameter of the *butterfly*. This *uncertainty* occurs with total probability for pairs of initial events and with the long time behaviour of a deterministic system. It is interesting to observe that the original paper by Lorenz in 1963 had no mathematical proofs in the usual sense, but only computational approximations of the behaviour of the trajectories of the flow above, from which he inferred with superb intuition all the facts that we have just mentioned. Only very recently were all these 'facts' completely confirmed in mathematical terms by Tucker[6].

Other attractors

First, let us define dynamical systems just as transformations in a space of events. Usually such a space has a nice geometric shape and it is covered by a set of local differentiable charts and change of charts is made through differentiable invertible maps. The space of events is what we call *differentiable manifolds*. So, dynamical systems are transformations defined on a differentiable manifold. This case corresponds to *discrete* dynamics and the flow case to *continuous* dynamics. In both situations, we are always interested in the long time behaviour of trajectories, starting at initial points like p (Figure 2).

An attractor is a subset of the space of events that attracts all future orbits starting at nearby points or at least most of them (a set of positive probability) (see Figure 3).

Important examples of attractors

As we have mentioned before in the case proposed by Lorenz, an attractor for a given dynamical system is a piece (subset) of the space of events which is invariant by the dynamics (i.e. made of trajectories) and attracts all future trajectories starting near it.

Hyperbolic attractors: Established in the sixties, here distances along trajectories increase and decrease exponentially in complementary dimensions in the ambient space. In the case of flows, we have to add to the expansion and contraction, the direction of the flow as a neutral one (at each non-singular point or non-zero value of the flow), as shown in the Figure 4. See refs 5, 7 and 8 for further discussions, achievements as well as other references.

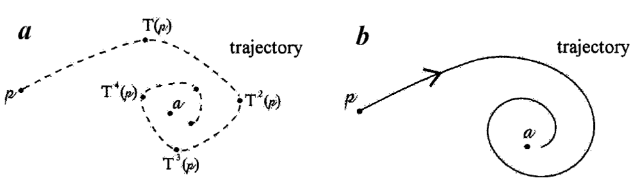

Figure 2. *a*, Transformation T; and *b*, Flow.

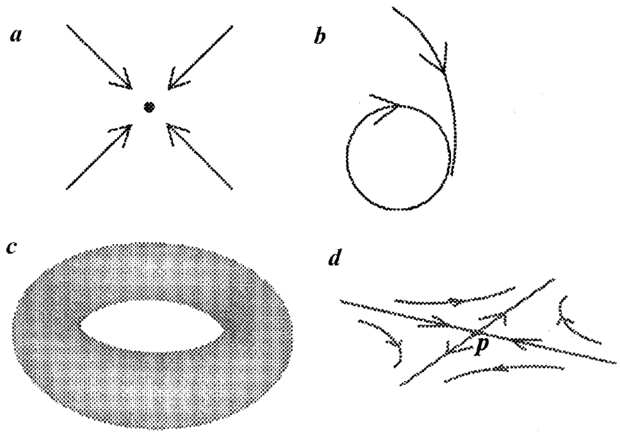

Figure 3. Different types of attractors. *a*, Point attractor; *b*, Attractive circle; *c*, Torus as an attractor. It consists of all points of the space of events. The transformation considered on the torus is induced by the linear map $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$; *d*, p is NOT an attractor.

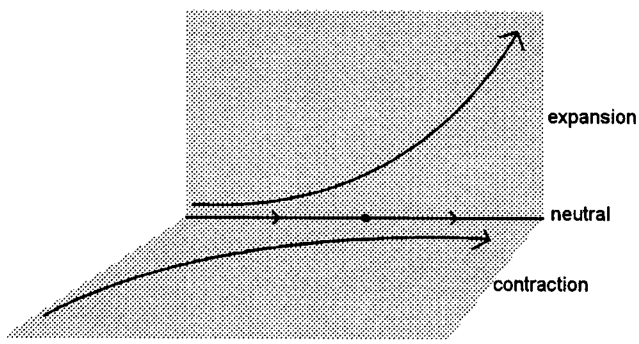

Figure 4. Hyperbolic attractors.

Chaotic (sensitive) non-hyperbolic attractors: These are of two types.

1. Lorenz 'butterfly' attractor, proposed in 1963, but better known only in the middle seventies. Actually, the fact that Lorenz equations, shown earlier, yield an attractor with the properties predicted by him, was only mathematically proved to be true about 35 years later[6]. That shows the remarkable ingenuity of Lorenz, but it is also worthwhile mentioning that he was inspired by the work of Saltzmann[9]. Lorenz was interested in the long-range weather forecasting, that certainly we can suggest as a first example of a complex system. Indeed, he was convinced that sta-

tistical methods common at the time, like linear regression, were essentially with flaw since evolution equations are certainly not linear. It is to be noticed that Lorenz attractor is robust: if we change slightly the coefficients of the equations presented earlier, we still get an attractor with the same properties.

2. Hénon[10] attractor, proposed in the mid-seventies, but mathematical proofs of its existence were provided in the late eighties (refs 11 and 12). Hénon's initial idea was to provide an attractor with similar properties to the Lorenz attractor, but in two dimensions. Since this is clearly not possible for flows, he thought about the simplest possible nonlinear transformation in two dimensions, namely a quadratic diffeomorphism of the plane (see Figure 5).

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx), \text{ for } a \cong 1.4 \text{ and } b \cong 0.3.$$

This transformation contains pieces of curves, where we see expansion followed by folds (Figure 5). Locally, the attractor consists of the product of two different elements: a segment and a fractal set, i.e. a discontinuous set having a fractional dimension smaller than one (see ref. 7). A beautiful aspect of this example is the fact that it exists for positive probability in the parameters a, b above, but is not robust in the sense that it does not persist for all small perturbations of the initial parameters for which we have a Hénon attractor. This is quite a subtle feature, but a very important one for the present new perspective on dynamical systems that we are discussing here. It is also remarkable that near the Hénon attractor all future orbits converge to it: we say that there are no 'holes' in its basin of attraction, again a much more profound fact than in the case of the Lorenz attractor. This has been recently proved by Benedicks and Viana[13], answering a couple of decades-old question by Ruelle and Sinai. The same authors have also proved that Hénon attractors are stochastically stable, i.e. they are stable in the average or, equivalently, in a probabilistic way (see ref. 14).

Hénon's ingenuous work was done numerically, like the case of the Lorenz attractor constructed a decade or so before. In a sense, Hénon was perhaps even more audacious, because his 'attractor' was not robust, but it can only exist for positive probability in the coefficients of the system: in this case, just a quadratic transformation of the plane with an inverse (a diffeomorphism of the plane) and thus, again, it is a deterministic system. It is important to notice that the Hénon attractor also corresponds to a complex system in terms of population growth, as shown recently by Yoccoz in a still unpublished work, obtained from experimental data from a certain species of small animals in the mountains of Norway, provided by local biologists.

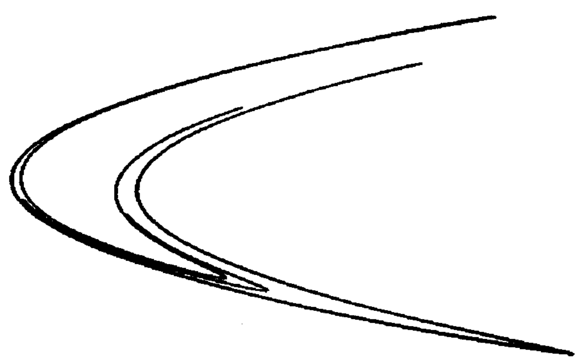

Figure 5. Hénon attractor.

So, from Lorenz to Hénon, we went from a 3-dimensional to a 2-dimensional attractor. There is also the 1-dimensional case, that appeared subsequently in the late seventies. Feigenbaum[15], Coullet and Tresser[16], independently discovered the period-doubling phenomenon for periodic orbits when the parameter a in the equation below varies:

$$f_a(x) = 1 - ax^2.$$

Also, the rate of approach of such a period-doubling phenomenon to a limiting point in the parameter line is a 'universal constant' for all quadratic families. This has been verified in experiments at the onset of turbulence of fluids, again a complex system for which we have some good insights. They also obtained chaotic sensitive attractors: they are fractal sets in the line.

We notice that in more global conceptual terms, since Poincaré, the following central question has been much discussed: Can we have a global scenario for dynamics? That is, can we describe the future behaviour of solutions of most or almost all systems? Can we approximate any system by one with a reasonably predictable future behaviour?

Several such scenarios were tried in the past, especially in the sixties. They were basically concerned with systems displaying an exponential increase (decrease) of distances along trajectories: they are called hyperbolic systems. Among them, we have the gradient-like systems, for which there is no uncertainty in the future behaviour of trajectories.

All such possible scenarios collapsed by the end of the sixties and early seventies. After a decade or so of bewilderment, caused by Lorenz, Hénon, Feigenbaum, Coullet and Tresser, and others, the dynamicists moved forward to obtain several remarkable results in the last twenty years, that allowed us to propose in 1995, the following global scenario:

Conjecture on the denseness of finitude of attractors: There is a dense set D of dynamics such that any element of D has finitely many attractors, whose union of basins of attraction has total probability. The attractors carry nice invariant physical probability measures and are stochastically stable at their basins of attraction (ref. 14).

Such attractors will be simple, like points and periodic orbits, hyperbolic ones or the more complicated ones like Lorenz and Hénon. So, most points in the space of events have only finitely many choices to go in the future (finitely many attractors), with some degree of uncertainty in each choice, unless the attractor is just a point or a trajectory.

This programme has been very successful for transformations of the interval with one critical point. There are also important related results in higher dimensions, but the conjecture is, in general, quite open and likely to remain so for some time.

Complex systems

Complex systems are much less understood and not even well-defined mathematically. Roughly speaking, they are in the frontier between simple and chaotic systems: A complex system is a dynamical system depending on many parameters, in constant evolution and distances along trajectories increase (decrease) polynomially and not exponentially! One considers that the brain's neural network is one such system.

Some of the properties assumed to be features of complex systems are:

1. It is a dynamical system in constant evolution, formed by a great number of unities.
2. Each unit interacts with other units (a much smaller number than the total).
3. Each unit responds to signals received from the others in a nonlinear manner.
4. The system in its evolution is adaptative – memory (hard to treat mathematically).
5. Some characteristics of the systems are randomly distributed.
6. In its evolution, sometimes the system develops a self-organization structure from a very disordered state – emerging order.
7. The system is hierarchical: a sign may be treated in several different levels before reaching the 'centre of action'.
8. The system may have several attractors.
9. Local interactions may have global effect: they may produce considerable global change in the system.
10. In many complex systems, there appear fractal structures.

The possible notion and applications of complex systems are not yet conclusive, like the case of the brain, origin of life and economics. Nevertheless, it is worth mentioning some literature in this direction: Bak and Chen[17] concerning evolution and self-organizing criticalities, Kauffman[18] on the origin of life, Hopfield[19] for the introduction of a basic model of neural networks, Holland[20] for a good discussion on adaptative complex systems and genetic algorithms, and finally Arthur[21] on complexity and economics. There is a fine collection of articles on complexity and chaos, corresponding to a series of interdisciplinary lectures, edited by Nussenzveig[22], still waiting to be translated from Portuguese.

Finally, as we have mentioned before, the theory of dynamical systems is in much solid grounds and the applications to celestial mechanics, population growth, weather prediction and turbulence, electric circuits and earthquake prediction, among others, are increasingly more reliable. And finally, we must call the reader's attention to the fact that in some of these cases the two words 'chaotic' and 'complex' are often loosely used as having the same meaning.

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